A threshold-type algorithm to fourth-order geometric motions

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1 Introduction

This is a brief report of Ishii–Kohsaka–Miyake–Sakakibara [14]. In this article we propose a thresholding algorithm to a family $\{\Gamma(t)\}_{t\geq 0}$ of hyprsurfaces moving by the following equation:

$$(1.1)\qquad V = \begin{cases} -\Delta_{\Gamma(t)}H - H|A|^2 + \dfrac{1}{2}H^3 - \lambda H & (N=2,3), \\ -\Delta_{\Gamma(t)}H - H|A|^2 + \dfrac{1}{2}H^3 - 2\displaystyle\sum_{\substack{i,j,k=1\\ i<j<k}}^{N-1} \kappa_i\kappa_j\kappa_k - \lambda H & (N\geq 4), \end{cases}$$

on $\Gamma(t)$, where V is the outward normal velocity, κ_i $(i=1,2,\dots,N-1)$ is the principal curvature of $\Gamma(t)$ with respect to the outer unit normal, H is the $((N-1)$-times) mean curvature of $\Gamma(t)$, $|A|^2$ denotes the sum of the squares of all the principal curvatures of $\Gamma(t)$ and $\lambda\in\mathbb{R}$. This is the gradient flow equation of the following functional:

$$\mathcal{E}_\lambda^N(\Gamma) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Gamma \kappa^2\,ds + \lambda\int_\Gamma ds & \text{if } N=2, \\ \dfrac{1}{2}\displaystyle\int_\Gamma H^2\,dS_\Gamma - \dfrac{2}{3}\int_\Gamma \sum_{\substack{i,j=1\\ i<j}}^{N-1} \kappa_i\kappa_j\,dS_\Gamma + \lambda\int_\Gamma dS_\Gamma & \text{if } N\geq 3. \end{cases}$$

For $N=2$, κ is the curvature of a planar curve Γ and s is the arc-length parameter. For $N\geq 3$, dS_Γ is the surface element on Γ. This can be regarded as a generalization of the Willmore energy and note that in the case $N\geq 3$ it appears in the asymptotic expansion of the heat content (cf. Angiuli–Massari–Miranda[1]). Notice that the term $\sum_{i,j,k=1,i<j<k}^{N-1} \kappa_i\kappa_j\kappa_k$ for $N\geq 4$ in (1.1) is derived as the first variation of the integral of $-\dfrac{1}{3}\displaystyle\sum_{i,j=1,i<j}^{N-1}\kappa_i\kappa_j$ on Γ (see e.g. Reilly[19]). Also notice that if $N=3$ and the topology of Γ is fixed, the integral of $\kappa_1\kappa_2$ on Γ is constant by virtue of the Gauss-Bonnet theorem, so that its first variation is zero.

In the case $N=2,3$ and $\lambda=0$ a family $\{\Gamma(t)\}_{t\geq 0}$ moving by (1.1) is called the Willmore flow (WF for short). As for the approximation schemes and the methods of numerical computations of the flow by (1.1), there are many results taking account of various applications. Mayer–Simonett [17] is one of the first numerical approaches for the WF in $\mathbb{R}^3$. They used a finite difference scheme to the WF and numerically observed that the WF can develop singularities in finite time. Rusu [20] presented an algorithm to the WF in $\mathbb{R}^3$ based on the variational method and studied a semi- and a fully discrete scheme in space and a semi-implicit method in time. Dziuk [7] introduced a parametric finite element method to the WF in general space dimensions. In [3, 4] etc., Barrett, Garcke and Nürnberg studied parametric finite element methods for fourth order geometric evolution problems such as surface diffusion flow and the WF and numerically observed the development of singularities.

The purpose of this article is to introduce a thresholding algorithm by using the following Cauchy problem for the fourth order linear parabolic equation:

$$\begin{cases} u_t = -\Delta^2 u + \lambda \Delta u \ \text{ in } \mathbb{R}^N \times (0,\infty), \\ u(\boldsymbol{x},0) = \chi_{\Omega_0}(\boldsymbol{x}) := \begin{cases} 1 \ \text{ in } \Omega_0, \\ 0 \ \text{ in } \mathbb{R}^N \setminus \Omega_0. \end{cases} \end{cases} \tag{1.2}$$

Here $N \geq 2$, $\lambda \in \mathbb{R}$, and $\Omega_0 \subset \mathbb{R}^N$ is a compact set. By the use of the above equation, we define a thresholding algorithm for the motion of $\Gamma(t)$ as follows: Set $h > 0$ as a time step. For a given compact set Ω_0 in $\mathbb{R}^N$, let u^0 be the solution to (1.2) and set $u^0_a(\boldsymbol{x},t) := u^0(\boldsymbol{x}, a^4 t)$ for $a > 0$. Then, define a threshold function U^0 as

$$U^0(\boldsymbol{x},t) := u^0_{3a}(\boldsymbol{x},t) - 3u^0_{2a}(\boldsymbol{x},t) + 3u^0_a(\boldsymbol{x},t)$$

and give a new set Ω_1 by

$$\Omega_1 := \left\{ \boldsymbol{x} \in \mathbb{R}^N \;\middle|\; U^0(\boldsymbol{x},h) \geq \frac{1}{2} \right\}.$$

Repeating this procedure inductively, we obtain a sequence $\{\Omega_k\}_{k=0}^\infty$ of compact subsets of $\mathbb{R}^N$. Set

$$\Omega^h(t) := \Omega_k \quad \text{for } kh \leq t < (k+1)h, \ k = 0,1,2,\ldots.$$

Then letting $h \to 0$, we observe at least formally that $\Omega^h(t)$ converges to a compact set $\Omega(t) (\subset \mathbb{R}^N)$ and that $\partial\Omega(t)$ moves by (1.1) if we choose a suitable constant a. In order to justify the thresholding algorithm explained above, we derive the asymptotic expansion of the solution to (1.2) near $\partial\Omega_0$.

Threshold-type algorithm to the geometric motions was firstly introduced by Bence–Merriman–Osher [5] to compute the mean curvature flows. Based on the level set approach for geometric motions, the convergence and generalizations of their algorithm were studied by Mascarenhas [16], Evans [10], Barles–Georgelin [2], Ishii [12], Ishii–Pires–Souganidis [13], Vivier [21] and Leoni [15], and so on. On the thresholding algorithm to the WF, there are results by Grzhibovskis–Heintz [11] in $\mathbb{R}^3$ and Esedoḡlu–Ruuth–Tsai [9] in $\mathbb{R}^2$. In [11] and [9] the asymptotic expansion of the convolution $(t^{-N/4}\rho(|\cdot|/t^{1/4}) * \chi_{\Omega_0})(\boldsymbol{x})$ is used to define a threshold function. Here $\rho(|\cdot|)$ is the Gauss kernel or some similar ones. For the difference between their thresholding algorithm and ours, see Remark 4.1 below. Metivet–Sengers–Ismaïl–Maitre [18] treated the diffusion-redistance scheme in $\mathbb{R}^2$ or $\mathbb{R}^3$, which is a variant of the algorithm by [9].

Referring to [11] and [9], the space-time scale $|\boldsymbol{x}|/t^{1/4}$ plays a key role to obtain the WF from the formal asymptotic expansions of their convolutions. Indeed, in [11] and [9], they used a modified heat kernel whose space-time scale is $|\boldsymbol{x}|/t^{1/4}$ instead of a natural space-time scale $|\boldsymbol{x}|/t^{1/2}$ of the usual heat kernel (see Remark 4.1 on the details of the calculation). Based on this fact, we arrive at the idea of using the fundamental solution to the fourth order linear parabolic equation in (1.2) to construct a thresholding algorithm to the flow by (1.1) since a natural space-time scale of its solution is $|\boldsymbol{x}|/t^{1/4}$.

This paper is organized in the following way. In section 2 we derive some formulae and pointwise estimates of the fundamental solution $G_{N,\lambda}$ to the operator $\partial_t + (-\Delta)^2 + \lambda(-\Delta)$ and of its derivatives. In section 3 we discuss the formal asymptotic expansion of the solution to (1.2). Section 4 is devoted to the justification of our algorithm. In subsection 4.1 we recall the algorithm and show some properties of evolving ses in subsection 4.2.

See [14] for some numerical experiments to our algorithm.

2 Preliminaries

In this section we derive some properties and estimates of the fundamental solution $G_{N,\lambda}$ to the operator $\partial_t + (-\Delta)^2 + \lambda(-\Delta)$ on $\mathbb{R}^N \times (0,\infty)$ where Δ is the Laplace operator on $\mathbb{R}^N$ and $\lambda \in \mathbb{R}$.

Define the Fourier transform as

$$\mathcal{F}_N[\psi](\boldsymbol{\xi}) := \int_{\mathbb{R}^N} \psi(\boldsymbol{x}) e^{-\mathbf{i}\langle \boldsymbol{\xi},\boldsymbol{x}\rangle_N}\, d\boldsymbol{x}, \quad \mathcal{F}_N^{-1}[\psi](\boldsymbol{x}) := c_N \int_{\mathbb{R}^N} \psi(\boldsymbol{\xi}) e^{\mathbf{i}\langle \boldsymbol{x},\boldsymbol{\xi}\rangle_N}\, d\boldsymbol{\xi}$$

for $\psi \in L^1(\mathbb{R}^N)$ where $\mathbf{i} := \sqrt{-1}$, $c_N := (2\pi)^{-N}$ and $\langle \cdot, \cdot \rangle_N$ is the inner product on $\mathbb{R}^N$. Then $G_{N,\lambda}$ is given by

$$G_{N,\lambda}(\boldsymbol{x},t) = \mathcal{F}_N^{-1}[e^{-(|\cdot|^4+\lambda|\cdot|^2)t}](\boldsymbol{x}) = c_N \int_{\mathbb{R}^N} e^{-(|\boldsymbol{\xi}|^4+\lambda|\boldsymbol{\xi}|^2)t+\mathbf{i}\langle \boldsymbol{x},\boldsymbol{\xi}\rangle_N}\, d\boldsymbol{\xi},$$

and we readily see that

$$\int_{\mathbb{R}^N} G_{N,\lambda}(\boldsymbol{x},t)\, d\boldsymbol{x} = \mathcal{F}_N[\mathcal{F}_N^{-1}[e^{-(|\cdot|^4+\lambda|\cdot|^2)t}]](\mathbf{0}) = e^{-(|\boldsymbol{\xi}|^4+\lambda|\boldsymbol{\xi}|^2)t}\Big|_{\boldsymbol{\xi}=\mathbf{0}} = 1.$$

Set $g_N(\boldsymbol{x}) := G_{N,0}(\boldsymbol{x},1)$. That is,

$$g_N(\boldsymbol{x}) = \mathcal{F}_N^{-1}[e^{-|\cdot|^4}](\boldsymbol{x}) = c_N \int_{\mathbb{R}^N} e^{-|\boldsymbol{\xi}|^4+\mathbf{i}\langle \boldsymbol{x},\boldsymbol{\xi}\rangle_N}\, d\boldsymbol{\xi}. \tag{2.1}$$

We derive the expansion of $G_{N,\lambda}$ by use of g_N and its derivatives.

Proposition 2.1 *$G_{N,\lambda}$ is represented as*

$$G_{N,\lambda}(\boldsymbol{x},t) = \frac{1}{t^{N/4}} \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} (-\Delta_{\boldsymbol{z}})^m g_N\Big(\frac{\boldsymbol{x}}{t^{1/4}}\Big),$$

where $\Delta_{\boldsymbol{z}} g_N$ is the Laplacian of $g_N(\boldsymbol{z})$ with respect to $\boldsymbol{z} \in \mathbb{R}^N$.

Proof. Applying the change of variable $\boldsymbol{\xi} = \boldsymbol{\eta}/t^{1/4}$, we have

$$G_{N,\lambda}(\boldsymbol{x},t) = \frac{c_N}{t^{N/4}} \int_{\mathbb{R}^N} \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} |\boldsymbol{\eta}|^{2m} e^{-|\boldsymbol{\eta}|^4+\mathbf{i}\langle \frac{\boldsymbol{x}}{t^{1/4}},\boldsymbol{\eta}\rangle_N}\, d\boldsymbol{\eta}.$$

By the Lebesgue convergence theorem, we are able to interchange of integration and summation. We obtain the desired result by some properties of the Fourier transform. $\square$

Set $\mathbb{Z}_+^N := \{\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{Z}^N \,|\, \alpha_i \geq 0\,(i = 1, \cdots, N)\}$ and $\mathbb{Z}_+ := \mathbb{Z}_+^1$. Hereafter, $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{Z}_+^N$ is a multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $D_{\boldsymbol{x}}^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}$.

Theorem 2.2 *There exist C, ν, μ, $K > 0$ such that for all $\alpha \in \mathbb{Z}_+^N$, $m \in \mathbb{Z}_+$, $\lambda \in \mathbb{R}$, and $(\boldsymbol{x}, t) \in \mathbb{R}^N \times (0, \infty)$*

$$\begin{aligned}
&|D_{\boldsymbol{x}}^\alpha(-\Delta_{\boldsymbol{x}})^m G_{N,\lambda}(\boldsymbol{x}, t)| \\
&\leq C\nu^{|\alpha|+2m}\Gamma\Big(\frac{|\alpha| + 2m + N}{4}\Big)t^{-(N+|\alpha|+2m)/4}\Big(1 + \frac{|\boldsymbol{x}|}{t^{1/4}}\Big)^{(|\alpha|+2m)/3} e^{-\mu(|\boldsymbol{x}|^4/t)^{1/3}+K|\lambda|^2 t}.
\end{aligned}$$

In the case $\lambda = 0$ this estimate is originally obtained in Eidel'man [8, Section 3 in Chapter I] and Cui [6, Theorem 3.2]. However, in these references, the precise dependence of the constant on $\alpha \in \mathbb{Z}_+^N$ and $m \in \mathbb{Z}_+$ is not stated.

Corollary 2.3 *There exist C, ν, $\mu > 0$ such that for all $\alpha \in \mathbb{Z}_+^N$, $m \in \mathbb{Z}_+$, and $\boldsymbol{x} \in \mathbb{R}^N$*

$$|D_{\boldsymbol{x}}^\alpha(-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x})| \leq C\nu^{|\alpha|+2m}\Gamma\Big(\frac{|\alpha| + 2m + N}{4}\Big)(1 + |\boldsymbol{x}|)^{(|\alpha|+2m)/3}e^{-\mu|\boldsymbol{x}|^{4/3}}.$$

We need the following lemma to estimate some integrations of $G_{N,\lambda}$ in the next section.

Lemma 2.4 *Let $\alpha \in \mathbb{Z}_+^N$ with $|\alpha| \leq 2$. Then there exists $\gamma > 0$ such that for $|\lambda|t^{1/2} \leq \gamma$*

$$\sum_{m=0}^\infty \frac{|\lambda|^m t^{m/2}}{m!}\int_{\mathbb{R}^N} \big|D_{\boldsymbol{x}}^\alpha(-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x})\big|\, d\boldsymbol{x} < \infty.$$

Proof. By Corollary 2.3, there exist C, ν, $\mu > 0$ independent of m, such that

$$\begin{aligned}
&\frac{|\lambda|^m t^{m/2}}{m!}\int_{\mathbb{R}^N} \big|D_{\boldsymbol{x}}^\alpha(-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x})\big|\, d\boldsymbol{x} \\
&\leq \frac{C(|\lambda|\nu^2 t^{1/2})^m}{m!}\Gamma\Big(\frac{m}{2} + \frac{|\alpha| + N}{4}\Big)\int_{\mathbb{R}^N}(1 + |\boldsymbol{x}|)^{(|\alpha|+2m)/3}e^{-\mu|\boldsymbol{x}|^{4/3}}\, d\boldsymbol{x}.
\end{aligned}$$

We use the polar coordinate to obtain

$$\int_{\mathbb{R}^N}(1 + |\boldsymbol{x}|)^{(|\alpha|+2m)/3}e^{-\mu|\boldsymbol{x}|^{4/3}}\, d\boldsymbol{x} \leq 2^{(|\alpha|+2m)/3}C_1\left\{1 + \Big(\frac{1}{\mu}\Big)^{(|\alpha|+2m)/4}\Gamma\Big(\frac{m}{2} + \frac{|\alpha| + 3N}{4}\Big)\right\}$$

for a constant $C_1 > 0$ independent of m. It follows from the Schwarz inequality that

$$\Gamma\Big(\frac{m}{2} + \frac{|\alpha| + kN}{4}\Big) \leq (m!)^{1/2}\Big\{\Gamma\Big(\frac{|\alpha| + kN}{2}\Big)\Big\}^{1/2}$$

for $m \in \mathbb{N}$ and $k = 1, 3$. Therefore, we see that

$$\frac{|\lambda|^m t^{m/2}}{m!}\int_{\mathbb{R}^N}\big|D_{\boldsymbol{x}}^\alpha(-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x})\big|\, d\boldsymbol{x} \leq C_2\frac{(4^{1/3}|\lambda|\nu^2 t^{1/2})^m}{(m!)^{1/2}} + C_3\left(\frac{4^{1/3}|\lambda|\nu^2 t^{1/2}}{\mu^{1/2}}\right)^m$$

for constants $C_2, C_3 > 0$ independent of m. Choose $\gamma > 0$ satisfying $\gamma < \dfrac{\mu^{1/2}}{4^{1/3}\nu^2}$. Then, with the help of d'Alembert test, we can judge that the series

$$\sum_{m=1}^{\infty} \frac{(4^{1/3}|\lambda|\nu^2 t^{1/2})^m}{(m!)^{1/2}}, \quad \sum_{m=1}^{\infty} \Big(\frac{4^{1/3}|\lambda|\nu^2 t^{1/2}}{\mu^{1/2}}\Big)^m$$

converge uniformly for $|\lambda| t^{1/2} \le \gamma$, so that we obtain the desired result. $\square$

Lemma 2.4 and the Lebesgue convergence theorem lead to the following lemma.

Lemma 2.5 *Let $\alpha \in \mathbb{Z}_+^N$ with $|\alpha| \le 2$ and $h \in L^\infty(\mathbb{R}^N)$. Then there exists $\gamma > 0$ such that for $|\lambda| t^{1/2} \le \gamma$*

$$\begin{aligned}
&\int_{\mathbb{R}^N} \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} D_{\boldsymbol{x}}^\alpha (-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}) h(\boldsymbol{x})\, d\boldsymbol{x} \\
&= \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} \int_{\mathbb{R}^N} D_{\boldsymbol{x}}^\alpha (-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}) h(\boldsymbol{x})\, d\boldsymbol{x}.
\end{aligned}$$

With regard to $(-\partial_{x_N}^2)^\ell (-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}', 0)$, we have the following representation.

Lemma 2.6 *Let g_N be given by* (2.1)*. Then*

$$(-\partial_{x_N}^2)^\ell (-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}', 0) = c_1 \sum_{j=0}^{m} \binom{m}{j} \mathcal{F}_{N-1}^{-1}\Big[\sum_{k=0}^{\infty} \frac{(-2)^k}{k!} L_{2(k+j+\ell)} |\cdot|^{2(k+m-j)} e^{-|\cdot|^4}\Big](\boldsymbol{x}')$$

for $\ell, m \in \mathbb{Z}_+$, where

$$L_\sigma := 2\int_0^\infty \xi^\sigma e^{-\xi^4}\, d\xi = \frac{1}{2}\Gamma\Big(\frac{\sigma+1}{4}\Big) \tag{2.2}$$

for $\sigma \ge 0$.

Outline of Proof. Fix any $\boldsymbol{x} \in \mathbb{R}^N$. Taking account of $|\boldsymbol{\xi}|^4 = |\boldsymbol{\xi}'|^4 + \xi_N^4 + 2|\boldsymbol{\xi}'|^2 \xi_N^2$ for $\boldsymbol{\xi}' = (\xi_1, \cdots, \xi_{N-1})$, we see that for $\boldsymbol{x}' = (x_1, \cdots, x_{N-1})$

$$\begin{aligned}
&(-\partial_{x_N}^2)^\ell (-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}) \\
&= c_N \sum_{j=0}^{m} \binom{m}{j} \int_{\mathbb{R}^N} \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} |\boldsymbol{\xi}'|^{2(k+m-j)} \xi_N^{2(k+j+\ell)} e^{-|\boldsymbol{\xi}'|^4 + \mathbf{i}\langle \boldsymbol{x}', \boldsymbol{\xi}'\rangle_{N-1}} e^{-\xi_N^4 + \mathbf{i} x_N \xi_N}\, d\boldsymbol{\xi}.
\end{aligned}$$

Using Fubini's theorem and Lebesgue convergence theorem, we get

$$\begin{aligned}
(-\partial_{x_N}^2)^\ell (-\Delta_{\boldsymbol{x}})^m g_N(\boldsymbol{x}) = & c_N \sum_{j=0}^{m} \binom{m}{j} \int_{\mathbb{R}^{N-1}} |\boldsymbol{\xi}'|^{2(m-j)} e^{-|\boldsymbol{\xi}'|^4 + \mathbf{i}\langle \boldsymbol{x}', \boldsymbol{\xi}'\rangle_{N-1}} \\
& \Big\{ \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} |\boldsymbol{\xi}'|^{2k} \int_{\mathbb{R}} \xi_N^{2(k+j+\ell)} e^{-\xi_N^4 + \mathbf{i} x_N \xi_N}\, d\xi_N \Big\}\, d\boldsymbol{\xi}'.
\end{aligned}$$

Substituting $x_N = 0$, we have

$$\sum_{k=0}^{\infty} \frac{(-2)^k}{k!} |\boldsymbol{\xi}'|^{2k} \int_{\mathbb{R}} \xi_N^{2(k+j+\ell)} e^{-\xi_N^4} \, d\xi_N = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} |\boldsymbol{\xi}'|^{2k} L_{2(k+j+\ell)}.$$

Since

$$|\cdot|^{2(m-j)} e^{-|\cdot|^4 + \mathbf{i}\langle \boldsymbol{x}', \cdot \rangle_{N-1}} \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} L_{2(k+j+\ell)} |\cdot|^{2k} \in L^1(\mathbb{R}^{N-1})$$

by Fubini's Theorem, we obtain the desired result. $\square$

From Lemma 2.6, it is seen that

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}} g_N(\boldsymbol{z}', 0) d\boldsymbol{z}' &= c_1 \mathcal{F}_{N-1} \left[\mathcal{F}_{N-1}^{-1} \left[\sum_{k=0}^{+\infty} \frac{(-2)^k}{k!} L_{2k} |\cdot|^{2k} e^{-|\cdot|^4} \right] (\boldsymbol{z}') \right] (\mathbf{0}') \\
&= c_1 \sum_{k=0}^{+\infty} \frac{(-2)^k}{k!} |\mathbf{0}'|^{2k} e^{-|\mathbf{0}'|^4} = c_1 L_0, \\
\int_{\mathbb{R}^{N-1}} g_N(\boldsymbol{z}', 0) z_j^2 d\boldsymbol{z}' &= c_1 \mathcal{F}_{N-1} \left[\mathcal{F}_{N-1}^{-1} \left[\sum_{k=0}^{+\infty} \frac{(-2)^k}{k!} L_{2k} |\cdot|^{2k} e^{-|\cdot|^4} \right] (\boldsymbol{z}') z_j^2 \right] (\mathbf{0}') \\
&= (-\mathbf{i})^2 c_1 \mathcal{F}_{N-1} \left[\mathcal{F}_{N-1}^{-1} \left[\sum_{k=0}^{+\infty} \frac{(-2)^k}{k!} L_{2k} |\cdot|^{2k} e^{-|\cdot|^4} \right] (\boldsymbol{z}') (-\mathbf{i} z_j)^2 \right] (\mathbf{0}') \\
&= -c_1 \sum_{k=0}^{+\infty} \frac{(-2)^k}{k!} L_{2k} \partial_{\xi_j}^2 (|\xi'|^{2k} e^{-|\xi'|^4}) \Big|_{\xi' = \mathbf{0}'} = 4 c_1 L_2 \quad (j = 1, \ldots, N-1).
\end{aligned}$$

Similar calculations yield that for $j_1,\ j_2,\ j_3 = 1, \ldots, N-1$,

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}} g_N(\boldsymbol{z}', 0) z_{j_1}^2 z_{j_2}^2 d\boldsymbol{z}' &= \begin{cases} -12 c_1 L_0 & (j_1 = j_2), \\ -4 c_1 L_0 & (j_1 \neq j_2), \end{cases} \\
\int_{\mathbb{R}^{N-1}} g_N(\boldsymbol{z}', 0) z_{j_1}^2 z_{j_2}^2 z_{j_3}^2 d\boldsymbol{z}' &= \begin{cases} -60 c_1 L_0 & (j_1 = j_2 = j_3), \\ -12 c_1 L_0 & (j_1 = j_2 \neq j_3), \\ -4 c_1 L_0 & (j_1 \neq j_2 \neq j_3 \neq j_1). \end{cases}
\end{aligned}$$

Here we have used $L_4 = \dfrac{1}{4} L_0$, $L_8 = \dfrac{5}{16} L_0$ (cf. (2.2)).

3 Asymptotic expansion of solutions to linear parabolic equations

The purpose of section is to derive the asymptotic expansion of the solution $u(\boldsymbol{x}, t)$ to (1.2) as $t \to +0$. Originally $\Omega_0 \subset \mathbb{R}^N$ is supposed to be compact. However, to simplify the argument, we assume in this section that Ω_0 is defined by

$$\Omega_0 := \{ (\boldsymbol{x}', x_N) \mid \boldsymbol{x}' \in \mathbb{R}^{N-1}, x_N \leq f(\boldsymbol{x}') \}.$$

Here $f \in C^5(\mathbb{R}^{N-1})$ satisfies $\nabla^5_{\boldsymbol{x}'} f \in L^\infty(\mathbb{R}^{N-1})$ and

$$f(\boldsymbol{x}') = \frac{1}{2}\sum_{j=1}^{N-1} \kappa_j x_j^2 + \frac{1}{3!}\langle \boldsymbol{x}', \nabla_{\boldsymbol{x}'}\rangle^3_{N-1} f(\boldsymbol{0}') + \frac{1}{4!}\langle \boldsymbol{x}', \nabla_{\boldsymbol{x}'}\rangle^4_{N-1} f(\boldsymbol{0}') + \frac{1}{5!}\langle \boldsymbol{x}', \nabla_{\boldsymbol{x}'}\rangle^5_{N-1} f(\theta \boldsymbol{x}') \quad \text{for some } \theta \in (0,1) \tag{3.1}$$

for $\boldsymbol{x}' \in \mathbb{R}^{N-1}$ by choosing a suitable orthonormal coordinate in $\mathbb{R}^{N-1}$, $\langle \boldsymbol{x}', \nabla_{\boldsymbol{x}'}\rangle^n_{N-1} f$ is defined as follows:

$$\begin{aligned}\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle^n_{N-1} f &= \left(\sum_{i\in\Lambda} z_i \partial_{z_i}\right)^n f \\ &= \sum_{d_1+\cdots+d_{N-1}=n} \frac{n!}{d_1!\cdots d_{N-1}!}(z_1\partial_{z_1})^{d_1}\cdots(z_{N-1}\partial_{z_{N-1}})^{d_{N-1}} f.\end{aligned}$$

Besides κ_j $(j = 1, 2, \ldots, N-1)$ is the principal curvature of $\partial\Omega_0$ at $\boldsymbol{0}'$.

Define the signed distance function to $\partial\Omega_0$ as

$$d(\boldsymbol{y}, \partial\Omega_0) := \begin{cases} \inf_{\boldsymbol{x}\in\partial\Omega_0} |\boldsymbol{y}-\boldsymbol{x}| & (\boldsymbol{y} \in \Omega_0), \\ -\inf_{\boldsymbol{x}\in\partial\Omega_0} |\boldsymbol{y}-\boldsymbol{x}| & (\boldsymbol{y} \in \mathbb{R}^N \setminus \Omega_0). \end{cases} \tag{3.2}$$

Furthermore, in the following, we often use the notation $F(\boldsymbol{x}, t) = O_D(b(t))$. This means that there exists $C > 0$ uniformly for $\boldsymbol{x} \in D$, such that $|F(\boldsymbol{x}, t)| \le C|b(t)|$.

Set $\boldsymbol{r}(t) := (\boldsymbol{0}', Vt)$ and consider the behavior of $u(\boldsymbol{r}(t), t)$ as $t \to +0$. We assume that V is the outward normal velocity of $\partial\Omega_0$ at $\boldsymbol{0}'$.

Recalling Proposition 2.1 and Lemma 2.5, we see that

$$\begin{aligned} u(\boldsymbol{r}(t), t) &= \frac{1}{2} + \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} \int_{\mathbb{R}^{N-1}} \int_0^{\zeta} (-\Delta_{\boldsymbol{z}})^m g_N(\boldsymbol{z})\, d\boldsymbol{z}, \\ \partial_{x_j} u(\boldsymbol{r}(t), t) &= -\frac{1}{t^{1/4}} \sum_{m=0}^{\infty} \frac{(-\lambda)^m t^{m/2}}{m!} \int_{\mathbb{R}^{N-1}} \int_{-\infty}^{\zeta} \partial_{z_j} (-\Delta_{\boldsymbol{z}})^m g_N(\boldsymbol{z})\, d\boldsymbol{z}, \end{aligned}$$

where $\zeta := -Vt^{3/4} + t^{-1/4} f(t^{1/4}\boldsymbol{z}')$ and $j = 1, 2, \ldots, N$.

By (3.1), we see that

$$\zeta = -Vt^{3/4} + \frac{t^{1/4}}{2}\sum_{j=1}^{N-1} \kappa_j z_j^2 + \frac{t^{1/2}}{3!}\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle^3_{N-1} f(\boldsymbol{0}') + \frac{t^{3/4}}{4!}\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle^4_{N-1} f(\boldsymbol{0}') + \frac{t}{5!}\langle \boldsymbol{z}', \nabla_{\boldsymbol{z}'}\rangle^5_{N-1} f(\theta t^{1/4}\boldsymbol{z}') \tag{3.3}$$

for some $\theta \in (0, 1)$.

We obtain the following precise asymptotic expansion of the solution to (1.2).

Theorem 3.1 *Let γ be a constant obtained in Lemma 2.4. Then, an asymptotic expansion of a solution to (1.2) is given by*

$$
\begin{aligned}
(3.4)\quad & u(\boldsymbol{r}(t),t) \\
&= \frac{1}{2} + c_1\Gamma\left(\frac{3}{4}\right) H t^{1/4} \\
&\quad - \frac{c_1}{2}\Gamma\left(\frac{1}{4}\right)\left\{ V + \frac{1}{2}\left(\Delta_g H + H|A|^2 - \frac{1}{2}H^3 + 2\sum_{\substack{i_1,i_2,i_3=1 \\ i_1<i_2<i_3}}^{N-1} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3} - \lambda H \right)\right\} t^{3/4} \\
&\quad + O_{\partial\Omega_0}(t)
\end{aligned}
$$

for any $\boldsymbol{x} \in \partial\Omega_0$, $\lambda \in \mathbb{R}$ and $t > 0$ satisfying $|\lambda| t^{1/2} \le \gamma$.

Outline of Proof. Set

$$
\Xi^{(0)} := \int_{\mathbb{R}^{N-1}} \int_0^{\zeta} g_N(\boldsymbol{z})\, dz_N d\boldsymbol{z}', \ \Xi^{(1)} := \int_{\mathbb{R}^{N-1}} \int_0^{\zeta} (-\Delta_{\boldsymbol{z}}) g_N(\boldsymbol{z})\, dz_N d\boldsymbol{z}',
$$

where $\zeta = -Vt^{3/4} + t^{-1/4} f(t^{1/4}\boldsymbol{z}')$. Then we are able to prove that

$$
u(\boldsymbol{r}(t),t) = \frac{1}{2} + \Xi^{(0)} - \lambda t^{1/2} \Xi^{(1)} + O_{\partial\Omega_0}(t). \tag{3.5}
$$

We derive the precise asymptotics for $\Xi^{(0)}$ and $\Xi^{(1)}$. Applying the Taylor's expansion of $g_N(\boldsymbol{z})$ and $(-\Delta_{\boldsymbol{z}})g_N(\boldsymbol{z})$ in the variable z_N and taking account of

$$
\partial_{z_N} g_N(\boldsymbol{z}',0) = \partial_{z_N}^3 g_N(\boldsymbol{z}',0) = 0, \quad \partial_{z_N}(-\Delta_{\boldsymbol{z}}) g_N(\boldsymbol{z}',0) = 0,
$$

which follows from $g_N'(\rho) = -\rho g_{N+2}(\rho)$ for $\rho \ge 0$, we see that

$$
\begin{aligned}
\Xi^{(0)} &= \int_{\mathbb{R}^{N-1}} g_N(\boldsymbol{z}',0)\zeta\, d\boldsymbol{z}' + \frac{1}{3!}\int_{\mathbb{R}^{N-1}} \partial_{z_N}^2 g_N(\boldsymbol{z}',0)\zeta^3\, d\boldsymbol{z}' + \frac{1}{5!}\int_{\mathbb{R}^{N-1}} \partial_{z_N}^4 g_N(\boldsymbol{z}',\theta_0\zeta)\zeta^5\, dz_N d\boldsymbol{z}', \\
\Xi^{(1)} &= \int_{\mathbb{R}^{N-1}} (-\Delta_{\boldsymbol{z}}) g_N(\boldsymbol{z}',0)\zeta d\boldsymbol{z}' + \frac{1}{2}\int_{\mathbb{R}^{N-1}} \partial_{z_N}^2 (-\Delta_{\boldsymbol{z}}) g_N(\boldsymbol{z}',\theta_1\zeta)\zeta^3\, dz_N d\boldsymbol{z}',
\end{aligned}
$$

for some θ_0, $\theta_1 \in (0,1)$. Since ζ is a polynomial of at most fifth-degree, we observe by Corollary 2.3 and this fact that

$$
\left| \int_{\mathbb{R}^{N-1}} \partial_{z_N}^4 g_N(\boldsymbol{z}',\theta_0\zeta)\zeta^5 d\boldsymbol{z}' \right| = O_{\partial\Omega_0}(t^{5/4}), \ \left| \int_{\mathbb{R}^{N-1}} \partial_{z_N}^2 (-\Delta_{\boldsymbol{z}}) g_N(\boldsymbol{z}',\theta_1\zeta)\zeta^3 d\boldsymbol{z}' \right| = O_{\partial\Omega_0}(t^{3/4}).
$$

Then it follows from (3.3) and the formulae in the end of the previous subsection that

$$
\begin{aligned}
(3.6)\qquad \Xi^{(0)} =& 2c_1 L_2 \sum_{i_0=1}^{N-1} \kappa_{i_0} t^{1/4} \\
& - c_1 L_0 \left\{ V + \frac{1}{2}\left(\sum_{i_0=1}^{N-1} \partial_{z_{i_0}}^4 f + \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{N-1} \partial_{z_{i_1}}^2 \partial_{z_{i_2}}^2 f - \frac{5}{2}\sum_{i_0=1}^{N-1} \kappa_{i_0}^3 - \frac{3}{2}\sum_{\substack{i_1,i_2=1 \\ i_1\ne i_2}}^{N-1} \kappa_{i_1}^2\kappa_{i_2} \right.\right. \\
& \left.\left. - \sum_{\substack{i_1,i_2,i_3=1 \\ i_1<i_2<i_3}}^{N-1} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3} \right)\right\} t^{3/4} + O_{\partial\Omega_0}(t).
\end{aligned}
$$

Similar arguments yilds that

$$\Xi^{(1)} = -\frac{1}{2}t^{1/4}c_1L_0\sum_{j=1}^{N-1}\kappa_j + O_{\partial\Omega_0}(t^{3/4}). \tag{3.7}$$

From (3.5), (3.6) and (3.7), we prove (3.4). Taking account of $L_0 = \Gamma(1/4)/2$ and $L_2 = \Gamma(3/4)/2$, we obtain

$$\begin{aligned} u(\boldsymbol{r}(t),t) =& \frac{1}{2} + c_1\Gamma\left(\frac{3}{4}\right)\sum_{i_0=1}^{N-1}\kappa_{i_0}t^{1/4} \\ &- \frac{c_1}{2}\Gamma\left(\frac{1}{4}\right)\Bigg\{V + \frac{1}{2}\Bigg(\sum_{i_0=1}^{N-1}\partial_{z_{i_0}}^4 f + 2\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{N-1}\partial_{z_{i_1}}^2\partial_{z_{i_2}}^2 f - \frac{5}{2}\sum_{i_0=1}^{N-1}\kappa_{i_0}^3 - \frac{3}{2}\sum_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{N-1}\kappa_{i_1}^2\kappa_{i_2} \\ &\qquad - \sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{N-1}\kappa_{i_1}\kappa_{i_2}\kappa_{i_3} - \lambda\sum_{i_0=1}^{N-1}\kappa_{i_0}\Bigg)\Bigg\}t^{3/4} + O_{\partial\Omega_0}(t). \end{aligned}$$

Since we observe that

$$\begin{aligned} &H = \sum_{i=1}^{N-1}\kappa_i, \quad |A|^2 = \sum_{i=1}^{N-1}\kappa_i^2, \\ &\Delta_g H + H|A|^2 - \frac{1}{2}H^3 \\ &= \sum_{i_0=1}^{N-1}\partial_{z_{i_0}}^4 f + 2\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{N-1}\partial_{z_{i_1}}^2\partial_{z_{i_2}}^2 f - \frac{5}{2}\sum_{i=1}^{N-1}\kappa_i^3 - \frac{3}{2}\sum_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{N-1}\kappa_{i_1}^2\kappa_{i_2} - 3\sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{N-1}\kappa_{i_1}\kappa_{i_2}\kappa_{i_3}, \end{aligned}$$

we have (3.4). □

4 A thresholding algorithm to the flow by (1.1)

In this section, we assume that $\Omega_0 \subset \mathbb{R}^N$ is a compact set and $\partial\Omega_0$ is of class C^5.

4.1 A thresholding algorithm

Let u be a solution to (1.2). Set $u_a(\boldsymbol{x},t) := u(\boldsymbol{x},a^4t)$ for $\boldsymbol{x} \in \mathbb{R}^N$ and $a > 0$. Then, it follows from Theorem 3.1 that for $V \in \mathbb{R}$ and small $t > 0$

$$\begin{aligned} &u_a(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}),t) \\ &= \frac{1}{2} + c_1\Gamma\left(\frac{3}{4}\right)aHt^{1/4} \\ &\quad - \frac{c_1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\frac{V}{a} + \frac{a^3}{2}\left(\Delta_g H + H|A|^2 - \frac{1}{2}H^3 + 2\sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{N-1}\kappa_{i_1}\kappa_{i_2}\kappa_{i_3} - \lambda H\right)\right\}t^{3/4} \\ &\quad + O_{\partial\Omega_0}(t). \end{aligned}$$

Choosing $a > 0$ such that $18a^4/11 = 1$, we see that

$$u_{3a}(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) - 3u_{2a}(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) + 3u_a(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) - \frac{1}{2} = -\frac{11c_1}{12a}\Gamma\left(\frac{1}{4}\right)\left(V + \Delta_g H + H|A|^2 - \frac{1}{2}H^3 + 2\sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{N-1} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3} - \lambda H\right)t^{3/4} + O_{\partial\Omega_0}(t). \tag{4.1}$$

From the above observation, let us introduce a threshold function and a new set generated by it. Define a threshold function $U(\boldsymbol{x}, t)$ as

$$U(\boldsymbol{x}, t) := u_{3a}(\boldsymbol{x}, t) - 3u_{2a}(\boldsymbol{x}, t) + 3u_a(\boldsymbol{x}, t) \tag{4.2}$$

for $a > 0$ satifying $18a^4/11 = 1$ and set

$$\Omega(t) := \left\{\boldsymbol{x} \in \mathbb{R}^N \,\middle|\, U(\boldsymbol{x}, t) \geq \frac{1}{2}\right\}. \tag{4.3}$$

For any $\boldsymbol{x} \in \partial\Omega_0$ and small $t > 0$, we define $V = V(\boldsymbol{x}, t)$ by

$$\boldsymbol{x} + V(\boldsymbol{x}, t)t\boldsymbol{n}(\boldsymbol{x}) \in \partial\Omega(t). \tag{4.4}$$

Then, setting $\boldsymbol{y}(\boldsymbol{x}, t) := \boldsymbol{x} + V(\boldsymbol{x}, t)t\boldsymbol{n}(\boldsymbol{x})$, we notice that

$$V(\boldsymbol{x}, t) = -\frac{d(\boldsymbol{y}(\boldsymbol{x}, t), \partial\Omega_0)}{t}, \quad |d(\boldsymbol{y}(\boldsymbol{x}, t), \partial\Omega_0)| = |\boldsymbol{y}(\boldsymbol{x}, t) - \boldsymbol{x}|$$

for all $\boldsymbol{x} \in \partial\Omega_0$ and small $t > 0$. Hence we are able to regard that V is the outward normal velocity from $\partial\Omega_0$ to $\partial\Omega(t)$. Here $d(\boldsymbol{y}, \partial\Omega_0)$ is defined by (3.2).

We assume that

$$\|V\|_{L^\infty(\partial\Omega_0\times(0,t_0))} < \infty \quad \text{for some } t_0 > 0, \tag{4.5}$$

and $U(\boldsymbol{x} + V(\boldsymbol{x}, t)t\boldsymbol{n}(\boldsymbol{x}), t) = 1/2$. Then, by (4.1)

$$V + \Delta_g H + H|A|^2 - \frac{1}{2}H^3 + 2\sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{N-1} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3} - \lambda H = O_{\partial\Omega_0}(t^{1/4}),$$

where $g = g(\boldsymbol{x})$ and $\kappa_i = \kappa_i(\boldsymbol{x})$ for $\boldsymbol{x} \in \partial\Omega_0$. We emphasize that the assumption (4.5) is actually valid as we show in the next subsection.

Based on the above argument, let us derive a thresholding algorithm to (1.1). First we solve the initial value problem (1.2) for the initial function $\chi_{\Omega_0}(\boldsymbol{x})$ and let u^0 be the corresponding solution. Define a threshold function $U^0(\boldsymbol{x}, t)$ as (4.2) and a set $\Omega^0(t)$ as (4.3). Fix a time step $h > 0$ and define $\Omega_1 := \Omega^0(h)$. Repeat this procedure to obtain a sequence $\{\Omega_k\}_{k\in\mathbb{Z}_+}$ of compact sets in $\mathbb{R}^N$. Then, setting

$$\widehat{\Omega}^h(t) = \Omega_k \text{ for } t \in [kh, (k+1)h) \ (k \in \mathbb{Z}_+)$$

and letting $h \to 0$, we can expect that at least formally there is a limit flow $\{\partial\widehat{\Omega}(t)\}_{t\geq 0}$ of $\{\partial\widehat{\Omega}^h(t)\}_{t\geq 0}$ as $h \to 0$, which moves by (1.1) with $\widehat{\Gamma}(t) = \partial\widehat{\Omega}(t)$.

Remark 4.1 *Set*

$$E_\alpha(\boldsymbol{x},t) := \frac{1}{(t^\alpha)^N}\rho\left(\frac{|\boldsymbol{x}|}{t^\alpha}\right), \quad \rho(s) := \frac{1}{(4\pi)^{N/2}}\exp\left(-\frac{s^2}{4}\right),$$

Let $\nabla_{L^2}\mathcal{E}_0^N(\Gamma)$ be the L^2-gradient of $\mathcal{E}_0^N(\Gamma)$ given by

$$\nabla_{L^2}\mathcal{E}_0^N(\Gamma) = \Delta_g H + H|A|^2 - \frac{1}{2}H^3 + 2\sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{N-1} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3},$$

and set

$$w_\alpha(\boldsymbol{x},t) := (E_\alpha(\cdot,t) * \chi_{\Omega_0})(\boldsymbol{x}).$$

Since $E_{1/2}(\boldsymbol{x},t)$ is the usual heat kernel, $w_{1/2}$ is a solution to

$$\begin{cases} w_t = \Delta w \ \ \text{in } \mathbb{R}^N \times (0,\infty), \\ w(\boldsymbol{x},0) = \chi_{\Omega_0}(\boldsymbol{x}) \ \ \text{in } \mathbb{R}^N. \end{cases}$$

Here the precise asymptotic expansion of $w_{1/2}(\boldsymbol{x}+Vt\boldsymbol{n}(\boldsymbol{x}),t)$ up to the order $t^{3/2}$ is represented as

$$\begin{aligned} &w_{1/2}(\boldsymbol{x}+Vt\boldsymbol{n}(\boldsymbol{x}),t) \\ =&\frac{1}{2} - c_1\sqrt{\pi}(V-H)t^{1/2} \\ &+ c_1\sqrt{\pi}\left\{\frac{1}{2}\nabla_{L^2}\mathcal{E}_0^N(\Gamma) + \frac{1}{4}V(2|A|^2+H^2) - \frac{1}{4}V^2H + \frac{1}{12}V^3\right\}t^{3/2} + O_{\partial\Omega_0}(t^2). \end{aligned}$$

We remark that the precise asymptotic expansion up tp the order $t^{1/2}$ is obtained in Evans[10] and its expansion is the basis of the BMO algorithm for the mean curvature flow (cf. Bence–Merriman–Osher [5]). On the other hand, considering the asymptotic expansion of $w_{1/4}(\boldsymbol{x}+Vt\boldsymbol{n}(\boldsymbol{x}),t)$, we have

$$\begin{aligned} &w_{1/4}(\boldsymbol{x}+Vt\boldsymbol{n}(\boldsymbol{x}),t) \\ =&\frac{1}{2} + c_1\sqrt{\pi}Ht^{1/4} - c_1\sqrt{\pi}\left(V - \frac{1}{2}\nabla_{L^2}\mathcal{E}_0^N(\Gamma)\right)t^{3/4} + O_{\partial\Omega_0}(t). \end{aligned} \tag{4.6}$$

The difference between (4.6) and our expansion

$$\begin{aligned} &u(\boldsymbol{x}+Vt\boldsymbol{n}(\boldsymbol{x}),t) \\ =&\frac{1}{2} + c_1\Gamma\left(\frac{3}{4}\right)Ht^{1/4} - \frac{c_1}{2}\Gamma\left(\frac{1}{4}\right)\left(V + \frac{1}{2}\nabla_{L^2}\mathcal{E}_0^N(\Gamma)\right)t^{3/4} + O_{\partial\Omega_0}(t), \end{aligned}$$

which is given by (3.4) with $\lambda = 0$, is the sign before $\nabla_{L^2}\mathcal{E}_0^N(\Gamma)$. This difference is due to the fact that $w_{1/4}$ satisfies the second order parabolic equation $w_t = (2\sqrt{t})^{-1}\Delta w$, whereas u fulfills the fourth order parabolic equation $u_t = -\Delta^2 u$. The thresholding algorithms of the Willmore flow based on (4.6) are derived for $N = 2$ in Esedoḡlu–Ruuth–Tsai[9] and for $N = 3$ in Grzhibovskis–Heintz[11]. Since the difference explained above is essentially related to the parabolicity, the same threshold function as ours, that is, U given by (4.2) can not be chosen if $w_{1/4}$ is used.

4.2 Properties of evolving sets

We give a justification of the argument in the previous subsection, more precisely, we prove that the assumption (4.5) is actually valid. To do so, we prepare several propositions and lemmas. In the following, we use the notation defined as in (4.2), (4.3), and (4.4).

Proposition 4.2 *There exists $K_* > 0$ and $t_* > 0$ such that*

$$\partial\Omega(t) \subset \{\boldsymbol{x} \in \mathbb{R}^N \,|\, |d(\boldsymbol{x}, \partial\Omega_0)| \leq K_* t^{1/4}\}$$

for $t \in (0, t_*)$, where $d(\cdot, \partial\Omega_0)$ is the signed distance function to $\partial\Omega_0$ given by (3.2).

This is proved by directly calculating $U(\boldsymbol{x}, t)$.

Furthermore, we are able to derive the following refinement of Proposition 4.2.

Proposition 4.3 *For any $\varepsilon \in (0,1)$, there exists $t_{*,\varepsilon} > 0$ such that*

$$\partial\Omega(t) \subset \{\boldsymbol{x} \in \mathbb{R}^N \,|\, |d(\boldsymbol{x}, \partial\Omega_0)| \leq \varepsilon t^{1/4}\}$$

for $t \in (0, t_{*,\varepsilon})$.

In order to prove this proposition, we need several preparations. Let us consider the case where $\lambda = 0$ and $\Omega_0 = \{\boldsymbol{x} \in \mathbb{R}^N \mid x_N \leq 0\}$. Moreover, let $\widetilde{u} = \widetilde{u}(x_N, t)$ be a solution of

$$\left\{ \begin{array}{l} \widetilde{u}_t = -\partial_{x_N}^4 \widetilde{u} \ \text{ in } \mathbb{R} \times (0, \infty), \\ \widetilde{u}(x_N, 0) = \chi_{(-\infty,0]}(x_N) = \left\{ \begin{array}{l} 1 \ \text{ in } (-\infty, 0], \\ 0 \ \text{ in } \mathbb{R}^N \setminus (-\infty, 0]. \end{array} \right. \end{array} \right. \tag{4.7}$$

and it is represented as

$$\widetilde{u}(x_N, t) = \frac{1}{2} + \int_0^{-x_N t^{-1/4}} g_1(z) dz.$$

Since $\widetilde{u}$ is also a solution of (1.2) with $\lambda = 0$ and $\Omega_0 = \{\boldsymbol{x} \in \mathbb{R}^N \mid x_N \leq 0\}$, we have $u(\boldsymbol{x}, t) = \widetilde{u}(x_N, t)$ by the uniqueness. As a result, it follows that for $\boldsymbol{x} \in \partial\Omega_0 = \{\boldsymbol{x} \in \mathbb{R}^N \mid x_N = 0\}$

$$u(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t) = \frac{1}{2} + \int_0^{-vt^{-1/4}} g_1(r) dr. \tag{4.8}$$

For $r \geq 0$, set

$$I(r) := \int_0^{r/(3a)} g_1(z_N) dz_N - 3\int_0^{r/(2a)} g_1(z_N) dz_N + 3\int_0^{r/a} g_1(z_N) dz_N.$$

Using this formula, we can prove the following lemma.

Lemma 4.4 *$I(r) > 0$ (resp., < 0) for $r > 0$ (resp. < 0).*

We omit the proof of this lemma because it is very complicated.

Now we are ready to prove the Proposition 4.3.

Outline of Proof of Proposition 4.3. In this proof we assume that Ω_0 is given by

$$\Omega_0 := \left\{ (\boldsymbol{x}', x_N) \;\middle|\; x_N \leq \frac{1}{2}(A'\boldsymbol{x}', \boldsymbol{x}')_{N-1} \right\}$$

and consider the motion of $\partial\Omega_0$ at the origin. Here $(\cdot,\cdot)_{N-1}$ denotes the inner product in $\mathbb{R}^{N-1}$ and A' is a symmetric matrix of order $(N-1)$. Set $\boldsymbol{x} := (\boldsymbol{0}', x_N) \in \partial\Omega(t)$. Then x_N/t is the normal velocity of $\partial\Omega_0$ at the origin.

It is seen that

$$u(\boldsymbol{x}, a^4 t) = \frac{1}{2} + \int_{\mathbb{R}^{N-1}} \int_0^{\zeta_a} g_N(\boldsymbol{z})\, dz_N d\boldsymbol{z}',$$

where

$$\zeta_a := -\frac{x_N}{at^{1/4}} + \frac{at^{1/4}}{2}(A'\boldsymbol{z}', \boldsymbol{z}')_{N-1}$$

for $a > 0$. Then we have

$$U(\boldsymbol{x}, t) = \frac{1}{2} + \left(\int_{\mathbb{R}^{N-1}} \int_0^{\zeta_{3a}} -3 \int_{\mathbb{R}^{N-1}} \int_0^{\zeta_{2a}} +3 \int_{\mathbb{R}^{N-1}} \int_0^{\zeta_a} \right) g_N(\boldsymbol{z})\, dz_N d\boldsymbol{z}'.$$

Note that $U(\boldsymbol{x}, t) = 1/2$ and hence $|x_N| \leq K_* t^{1/4}$ for $t \in (0, t_*)$ by Proposition 4.2.

Let us prove $x_N = o_{\partial\Omega_0}(t^{1/4})$ as $t \to +0$. In the proof by contradiction, suppose that there exists $\{(t_k, r_k)\}_{k=1}^{+\infty} \subset (0, +\infty) \times \mathbb{R}$ such that

$$U(\boldsymbol{x}_k, t_k) = \frac{1}{2}, \;\; \boldsymbol{x}_k = (0, -r_k t_k) \in \partial\Omega(t_k), \;\; \lim_{k \to +\infty} (t_k, r_k) = (0, r_*) \quad (r_* \neq 0)$$

Then

$$\zeta_a = \frac{r_k}{a} + \frac{at_k^{1/4}}{2}(A'\boldsymbol{z}', \boldsymbol{z}')_{N-1}.$$

It follows from (4.8) and the exponential decay of g_N that

$$\begin{aligned} \frac{1}{2} = \lim_{k \to \infty} U(\boldsymbol{x}_k, t_k) &= \frac{1}{2} + \left(\int_{\mathbb{R}^{N-1}} \int_0^{r_*/(3a)} -3 \int_{\mathbb{R}^{N-1}} \int_0^{r_*/(2a)} +3 \int_{\mathbb{R}^{N-1}} \int_0^{r_*/a} \right) g_N(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \frac{1}{2} + \left(\int_0^{r_*/(3a)} -3 \int_0^{r_*/(2a)} +3 \int_0^{r_*/a} \right) g_1(r)\, dr. \end{aligned}$$

By Lemma 4.4 and $r_* \neq 0$, we see that

$$\left(\int_0^{r_*/(3a)} -3 \int_0^{r_*/(2a)} +3 \int_0^{r_*/a} \right) g_1(r)\, dr \neq 0.$$

This leads to a contradiction. Therefore, $x_N = o_{\partial\Omega_0}(t^{1/4})$ as $t \to +0$. $\square$

We next prove an estimate of the derivative of U, which guarantees that $\partial\Omega(t)$ is a smooth hypersurface. Set $M_{0,a} := 11 c_1 L_0/(6a)$.

Proposition 4.5 *There exist $\varepsilon_0 \in (0,1)$ and $C > 0$ such that for $t > 0$ small enough, $\boldsymbol{x} \in \partial\Omega_0$ and $|v| \leq \varepsilon_0 t^{1/4}$*

$$
\begin{aligned}
&-\frac{3M_{0,a}}{2}t^{-1/4} - C(\varepsilon_0 + t^{1/4}) \\
&\leq \langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + v\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{n}(\boldsymbol{x})\rangle_N \leq -\frac{M_{0,a}}{2}t^{-1/4} + C(\varepsilon_0 + t^{1/4}).
\end{aligned} \tag{4.9}
$$

Furthermore, $\partial\Omega(t)$ is a smooth hypersurface for $t > 0$ small enough.

We omit the proof of this proposition because it is proved by direct calculations.

Theorem 4.6 *There exists a $C_0 > 0$ such that*

$$\sup_{\boldsymbol{x}\in\partial\Omega(t)} |d(\boldsymbol{x}, \partial\Omega_0)| \leq C_0 t$$

for $t > 0$ small enough.

Proof. Applying (3.4) with $V = 0$, there exists a $C_1 > 0$ such that

$$\left| U(\boldsymbol{x}, t) - \frac{1}{2} \right| \leq C_1 t^{3/4} \tag{4.10}$$

for $\boldsymbol{x} \in \partial\Omega_0$ and $t > 0$ small enough. For any $\boldsymbol{y} \in \partial\Omega(t)$, let $\boldsymbol{x} \in \partial\Omega_0$ be a point satisfying $|d(\boldsymbol{y}, \partial\Omega_0)| = |\boldsymbol{y} - \boldsymbol{x}|$. Then $\boldsymbol{y}$ can be represented as $\boldsymbol{y} = \boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x})$ where $Vt = d(\boldsymbol{y}, \partial\Omega_0)$. This implies that

$$U(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t) - U(\boldsymbol{x}, t) = \langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + \theta Vt\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{n}(\boldsymbol{x})\rangle_N Vt$$

for some $\theta \in (0,1)$. Taking account of $U(\boldsymbol{y}, t) = 1/2$ for $\boldsymbol{y} \in \partial\Omega(t)$ and using Proposition 4.5 and (4.10), we see that

$$
\begin{aligned}
&|\langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + \theta Vt\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{n}(\boldsymbol{x})\rangle_N Vt| \leq C_1 t^{3/4}, \\
&-C_2 t^{-1/4} \leq \langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + \theta Vt\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{n}(\boldsymbol{x})\rangle_N \leq -C_3 t^{-1/4}
\end{aligned}
$$

for $t > 0$ small enough where C_2, C_3 are positive constants independent of $\boldsymbol{x} \in \partial\Omega_0$ and t. As a result, it follows that there exists $C_4 > 0$ such that $|V| \leq C_4$ for $\boldsymbol{x} \in \partial\Omega_0$ and small $t > 0$. This is the desired result. $\square$

From this theorem, it follows that V is bounded on $\partial\Omega_0 \times (0, t_0)$ for some small $t_0 > 0$ and hence the argument in subsection 4.1 is justified, that is:

Theorem 4.7 *There exist $C > 0$ and $t_0 > 0$ such that*

$$\left| V + \Delta_g H + H|A|^2 - \frac{1}{2}H^3 + 2\sum_{\substack{i_1,i_2,i_3\in\Lambda \\ i_1<i_2<i_3}} \kappa_{i_1}\kappa_{i_2}\kappa_{i_3} - \lambda H \right| \leq Ct^{1/4}$$

for all $t \in (0, t_0)$ and $\boldsymbol{x} \in \partial\Omega_0$. Especially, this estimate turns to

$$\left| V + \Delta_g H + H|A|^2 - \frac{1}{2}H^3 - \lambda H \right| \leq Ct^{1/4} \quad \text{if } N = 2, 3$$

for all $t \in (0, t_0)$ and $\boldsymbol{x} \in \partial\Omega_0$.

Remark 4.8 *(i) If $\partial\Omega_0$ is of class $C^n\,(n \geq 6)$, we can replace $t^{1/4}$ with $t^{1/2}$ in Theorem 4.7. But we omit the detail because the actual calculations are more complicated.*

(ii) Based on the boundedness of V, we can improve (4.9) as follows:

$$|\langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{n}(\boldsymbol{x})\rangle_N + M_{0,a} t^{-1/4}| \leq Ct^{3/4}.$$

Moreover, we can also estimate

$$|\langle \nabla_{\boldsymbol{x}} U(\boldsymbol{x} + Vt\boldsymbol{n}(\boldsymbol{x}), t), \boldsymbol{\tau}(\boldsymbol{x})\rangle| \leq Ct^{3/4}$$

where $\boldsymbol{\tau}(\boldsymbol{x})$ is any unit tangential vector of $\partial\Omega_0$ at $\boldsymbol{x} \in \partial\Omega_0$. Thus the outer unit normal $-\nabla_{\boldsymbol{x}} U(\cdot, t)/|\nabla_{\boldsymbol{x}} U(\cdot, t)|$ of $\partial\Omega(t)$ is nearly equal to $\boldsymbol{n}(\cdot)$, that of $\partial\Omega_0$, for any small $t > 0$.

References

[1] L. Angiuli, U. Massari, M. Miranda Jr, Geometric properties of the heat content, *manuscripta math.*, **140** (2013), 497–529.

[2] G. Barles and C. Georgelin, A simple proof of convergence for an approximation scheme for computing motion by mean curvature, *SIAM J. Numer. Anal.*, **32** (1995), pp. 484–500.

[3] J. W. Barrett, H. Garcke and R. Nürmberg, A parametric finite element method for fourth order geometric evolution equations, *J. Comput. Phys.*, **222** (2007), 441–467.

[4] J. W. Barrett, H. Garcke and R. Nürmberg, Parametric approximation of Willmore flow and related geometric evolution equations, *SIAM J.. Sci. Comput.*, **31** (2008), 225–253.

[5] J. Bence, B. Merriman and S. Osher, Diffusion generated motion by mean curvature, Computational Crystal Growers Workshop, *J. Taylored., Selected Lectures in Math.*, AMS, Province,RI, 1992.

[6] S. Cui, Local and global existence of solutions to semilinear parabolic initial value problems, *Nonlinear Anal.*, **43** (2001), 293–323.

[7] G. Dziuk, Computational parametric Willmore flow, *Numer. Math.*, **111** (2008), 55–80.

[8] S. D. Eidel'man, Parabolic Systems, North-Holland Publ. Comp., Amsterdam, London, 1969.

[9] S. Esedoḡlu, S. J. Ruuth and R. Tsai, Threshold dynamics for higher order geometric motions, *Interfaces and Free Bound.*, **10** (2008), 263–282.

[10] L. C. Evans, Convergence of an algorithm for mean curvature motion, *Indiana Univ. Math. J.*, **42** (1993), 533–557. *Discrete Contin. Dyn. Syst.*, **21** (2008), no. 4, 1129–1157. *Nonlinear Anal.*, **70** (2009), 2965–2973.

[11] R. Grzhibovskis and A. Heintz, A convolution thresholding scheme for the Willmore flow, *Interfaces and Free Bound.*, **10** (2008), 139–153.

[12] H. Ishii, A generalization of the Bence, Merryman and Osheralgorithm for motion by mean curvature, Curvature flows and related topics, A. Damlamian, J. Spruck, A. Visintin eds., Gakuto Intermat. Ser. Math. Sci. Appl., Gakkōtosho, Tokyo, 1995, 111–127.

[13] H. Ishii, G. E. Pires and P. E. Souganidis, Threshold dynamics type approximation schemes for propagating fronts, *J. Math. Soc. Japan*, **50** (1999), 267–308.

[14] K. Ishii, Y. Kohsaka, N. Miyake and K. Sakakibara, A thresholding algorithm for Willmore-type flows via fourth order linear parabolic equations, *Interfaces Free Bound.*, **27** (2025), 235–283.

[15] F. Leoni, Convergence of an apprximation scheme for curvature-dependent motion of sets, *SIAM J. Numer. Anal.*, **39** (2001), 1115–1131.

[16] P. Mascarenhas, Diffusion generated motion by mean curvature, Campus Report, Math. Dept., University of California, Los Angeles, 1992,

[17] U. F. Mayer and G. Simonett, A numerical scheme for axisymmetric solutions of curvature-driven free boundary problems, with applications to the Willmore flow, *Interfaces and Free bound.*, **4** (2002), 89–109.

[18] T. Metivet, A. Sengers, M. Ismaïl and E. Maitre, Diffusion-redistanciation schemes for 2D and 3D constrained Willmore flow: Application to the equilibrium shapes of vesicles, *J. Comput. Phys.*, **436** (2021), 110288.

[19] R. C. Reilly, Variational properties of functions of the mean curvatures for hypersurfaces in space forms, *J. Differential Geometry*, **8** (1973), 465–477.

[20] R. E. Rusu, An algorithmfor the elastic flow of surfaces, *Interfaces Free Bound.*, **7** (2005), 229–239.

[21] L. Vivier, Convergence of an approximation scheme for computing motions with curvature dependent velocities, *Differential Integral Equations*, **13** (2000), 1263–1288.

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