TIME-FRACTIONAL NONLINEAR EVOLUTION EQUATIONS WITH TIME-DEPENDENT CONSTRAINTS

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1. Introduction

This paper is an expository account of [13]. It is devoted to developing an abstract theory of time-fractional gradient flow equations for time-dependent convex functionals in real Hilbert spaces.

Throughout this paper, let $T \in (0, \infty)$ be fixed, and let H be a real Hilbert space with an inner product $(\bullet, \bullet)_H$ and the norm $\| \bullet \|_H := \sqrt{(\bullet, \bullet)_H}$. For each $t \in [0, T]$, let $\varphi^t : H \to (-\infty, \infty]$ be a proper (i.e., $\varphi^t \not\equiv \infty$) lower-semicontinuous convex functional. For $k \in L^1(0, T)$ and $w \in L^1(0, T; H)$, the convolution $k * w$ is given by

$$(k * w)(t) := \int_0^t k(t-s)w(s)\,\mathrm{d}s \quad \text{for } t \in (0, T).$$

We consider the following abstract Cauchy problem:

$$\partial_t[k * (u - u_0)](t) + \partial\varphi^t\big(u(t)\big) \ni f(t) \quad \text{in } H \quad \text{for } t \in (0, T), \tag{P}$$

where $u_0 \in H$ and $f : (0, T) \to H$ are prescribed, $\partial\varphi^t$ is the subdifferential operator of φ^t, and k is a kernel satisfying the following condition:

(PC) The kernel $k \in L^1_{\mathrm{loc}}([0, \infty))$ is nonnegative and nonincreasing. Moreover, there exists a nonnegative and nonincreasing kernel $\ell \in L^1_{\mathrm{loc}}([0, \infty))$ such that

$$(k * \ell)(t) = \int_0^t k(t-s)\ell(s)\,\mathrm{d}s = 1 \quad \text{for all } t \in (0, \infty).$$

Under this assumption, we write $(k, \ell) \in PC$. A typical example satisfying (PC) is the *Riemann–Liouville kernel*,

$$k_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \quad \text{for } t \in (0, \infty) \text{ and } \alpha \in (0, 1).$$

It is known that $(k_{1-\alpha}, k_\alpha) \in PC$ for $\alpha \in (0, 1)$. In this case, $\partial_t[k_{1-\alpha} * (u - u_0)](t)$ coincides with the α-th order Riemann–Liouville fractional derivative of $u - u_0$,

The study of classical gradient flows dates back to the early work of Haïm Brézis (see, e.g., [5]) on the evolution equation,

$$\partial_t u(t) + \partial\varphi\big(u(t)\big) \ni f(t) \quad \text{in } H \quad \text{for } t \in (0, T), \quad u(0) = u_0,$$

for a proper lower-semicontinuous convex functional φ on a real Hilbert space H. This fundamental framework is nowadays referred to as the *Brézis–Kōmura theory* (see also [12]). Building upon this theory, Kenmochi [10] subsequently extended the Brézis–Kōmura theory to cover evolution equations governed by *time-dependent* subdifferential operators,

$$\partial_t u(t) + \partial\varphi^t\big(u(t)\big) \ni f(t) \quad \text{in } H \quad \text{for } t \in (0, T), \quad u(0) = u_0.$$

This generalization makes it possible to treat nonlinear parabolic equations on moving domains and pave the way for applications to free boundary problems such as Stefan problems (see [11]).

In recent years, there have been several extensions of these classical results on gradient flows to *time-fractional* variants. In [1], the well-posedness of the abstract Cauchy problem,

$$\partial_t[k * (u - u_0)](t) + \partial\varphi\big(u(t)\big) \ni f(t) \quad \text{in } H \quad \text{for } t \in (0, T),$$

was established for proper lower-semicontinuous convex functionals $\varphi\colon H \to (-\infty,\infty]$. Moreover, it is also extended to Lipschitz perturbation problems. Furthermore, the abstract theory has been applied to time-fractional variants of nonlinear diffusion equations as well as Allen–Cahn equations.

The main results in [13] is to establish an abstract theory concerning the existence of strong solutions to the Cauchy problem (P). We immediately face a major difficulty due to the lack of a valid chain-rule formula for time-fractional derivatives: while the chain-rule formula is a crucial tool in the study of gradient flow equations, the usual chain-rule formula is no longer valid for time-fractional derivatives (see, e.g., [17]). Nevertheless, an alternative formula was established in [1], which yields a fractional chain-rule formula for subdifferential operators in a practical form instead of the usual identities. However, this formula is restricted to time-*independent* subdifferential operators, and hence, in order to handle the Cauchy problem (P), where the subdifferential operator explicitly depends on time, we shall develop a time-dependent version of the fractional chain-rule formula. Moreover, we also develop Gronwall-type lemmas for nonlinear Volterra integral inequalities. This paper provides a streamlined exposition of these arguments.

2. Main setting and results

In this section, we present main results concerning existence of strong solutions to the abstract Cauchy problem (P). For each $t \in [0,T]$, let $\varphi^t\colon H \to (-\infty,\infty]$ be a proper lower-semicontinuous convex functional with the *effective domain*,

$$D(\varphi^t) := \{w \in H\colon \varphi^t(w) < \infty\}.$$

We define a notion of strong solution for the Cauchy problem (P) in the following sense:

Definition 2.1 (Strong solutions to (P)). *A function $u \in L^2(0,T;H)$ is called a* strong solution *on $[0,T]$ to (P) for $(u_0,f) \in H \times L^2(0,T;H)$, if the following conditions are satisfied:*

(i) *It holds that $k*(u-u_0) \in W^{1,2}(0,T;H)$, $[k*(u-u_0)](0) = 0$, and $u(t) \in D(\partial\varphi^t)$ for a.e. $t \in (0,T)$.*

(ii) *There exists $\xi \in L^2(0,T;H)$ such that*

$$\xi(t) \in \partial\varphi^t\big(u(t)\big), \quad \partial_t[k*(u-u_0)](t) + \xi(t) = f(t)$$

for a.e. $t \in (0,T)$.

We write $(u,\xi) \in (\mathrm{P})_{u_0,f}$ if $u,\xi \in L^2(0,T;H)$ satisfy the above conditions. The following theorem concerns the uniqueness and continuous dependence on initial data of strong solutions to (P).

Theorem 2.2 (Uniqueness and continuous dependence on initial data of strong solutions to (P)). *Let $(k,\ell) \in PC$, and let $(u_i,\xi_i) \in (\mathrm{P})_{u_{0,i},f_i}$ for $i = 1,2$, where $u_{0,i} \in H$ and $f_i \in L^2(0,T;H)$. Then there exists a constant $c_T \in [0,\infty)$ independent of $u_{0,1}$, $u_{0,2}$, f_1 and f_2 such that*

$$\|u_1 - u_2\|^2_{L^2(0,T;H)} \le c_T \left(\|u_{0,1} - u_{0,2}\|_H^2 + \|f_1 - f_2\|^2_{L^2(0,T;H)} \right).$$

In particular, if $u_{0,1} = u_{0,2}$ and $f_1 = f_2$, then $u_1 = u_2$.

In order to state existence results, we employ the so-called Kenmochi condition (cf. [10, 11, 14, 19]):

(A1) There is a constant $c_1 \in [0,\infty)$ with the following property: for each $s,t \in [0,T]$ with $s \le t$ and each $z_s \in D(\varphi^s)$, there exists $z_{s,t} \in D(\varphi^t)$ such that

$$\|z_{s,t} - z_s\|_H \le c_1 |t-s| \big(1 + |\varphi^s(z_s)|\big)^{1/2},$$

$$\varphi^t(z_{s,t}) \le \varphi^s(z_s) + c_1|t-s|\big(1 + |\varphi^s(z_s)|\big).$$

The following theorem is concerned with the existence of strong solutions to (P) for $f \in W^{1,2}(0,T;H)$:

Theorem 2.3 (Existence of strong solutions for $f \in W^{1,2}(0,T;H)$). *Let $(k,\ell) \in PC$ and assume that* (A1) *holds. Then, for every $u_0 \in D(\varphi^0)$ and $f \in W^{1,2}(0,T;H)$, the Cauchy problem* (P) *admits a unique strong solution $u \in L^2(0,T;H)$ on $[0,T]$ such that*

$$\varphi^\bullet\big(u(\bullet)\big) \in L^\infty(0,T), \quad \ell * \|\partial_t[k*(u-u_0)]\|_H^2 \in L^\infty(0,T). \tag{2.1}$$

Furthermore, it holds that u belongs to $C([0,T];H)$ and $u(0) = u_0$.

The above existence result can be extended to more general external forces under the following additional assumption:

(A2) There exists a constant $c_2 \in [0,\infty)$ such that the following property holds: for each $u \in L^2(0,T;H)$ satisfying $\varphi^\bullet(u(\bullet)) \in L^1(0,T)$ and for each $t \in (0,T)$, there exists a function $w_t \in L^2(0,t;H)$ such that

$$\|w_t(s) - u(s)\|_H \le c_2\,|t-s|\big(1 + \big|\varphi^s\big(u(s)\big)\big|\big)^{1/2},$$
$$\varphi^t\big(w_t(s)\big) \le \varphi^s\big(u(s)\big) + c_2\,|t-s|\big(1 + \big|\varphi^s\big(u(s)\big)\big|\big)$$

for a.e. $s \in (0,t)$.

Remark 2.4 (A sufficient condition for (A1) and (A2)). Let Δ be defined by

$$\Delta := \{(t,s) \in [0,T]\times[0,T] \colon t \ge s\}.$$

We introduce a nonlocal version of the Kenmochi condition:

$(A\varphi^t)_{NL}$ There exist a constant $C \in [0,\infty)$ and a Carathéodory function $\Psi\colon \Delta \times H \to H$, i.e., for each $w \in H$, the mapping $(t,s) \mapsto \Psi(\bullet,\bullet,w)$ is strongly measurable, and for a.e. $(t,s) \in \Delta$, $\Psi(t,s,\bullet)$ is continuous in H, such that

$$\|\Psi(t,s,w) - w\|_H \le C\,|t-s|\big(1 + |\varphi^s(w)|\big)^{1/2},$$
$$\varphi^t\big(\Psi(t,s,w)\big) \le \varphi^s(w) + C\,|t-s|\big(1 + |\varphi^s(w)|\big)$$

for all $(t,s) \in \Delta$ and all $w \in H$.

Then $(A\varphi^t)_{NL}$ implies both (A1) and (A2) with $c_1 = c_2 = C$.

Under these assumptions, we obtain the following existence result for the case where $f \in L^2(0,T;H)$:

Theorem 2.5 (Existence of strong solutions for $f \in L^2(0,T;H)$). *Let $(k,\ell) \in PC$ and assume that* (A1) *and* (A2) *hold. Then, for every $u_0 \in D(\varphi^0)$ and $f \in L^2(0,T;H)$, the Cauchy problem* (P) *admits a unique strong solution $u \in L^2(0,T;H)$ on $[0,T]$.*

3. Auxiliary facts

3.1. **Maximal monotone operators and subdifferentials.** We briefly recall several basic facts on subdifferential operators; see, e.g., [3–5, 8, 16]. Throughout this paper, let $I\colon H \to H$ be the identity mapping on H. For a set-valued operator $A\colon H \to 2^H$, the domain $D(A)$, the range $R(A)$, and the graph $G(A)$ are defined by

$$D(A) := \{x \in H : Ax \ne \emptyset\}, \quad R(A) := \bigcup_{x\in D(A)} Ax, \quad G(A) := \{(x,y) \in H\times H : y \in Ax\},$$

respectively. The inverse $A^{-1}\colon H \to 2^H$ is defined by

$$A^{-1}x := \{z \in H : x \in Az\} \quad \text{for } x \in H.$$

Let $\varphi\colon H \to (-\infty, \infty]$ be a proper lower-semicontinuous convex functional. The *subdifferential operator* $\partial\varphi\colon H \to 2^H$ is defined by

$$\partial\varphi(z) := \{\xi \in H \colon \varphi(v) - \varphi(z) \geq (\xi, v - z)_H \text{ for all } v \in D(\varphi)\}$$

for $z \in H$, where $D(\varphi) := \{w \in H \colon \varphi(w) < \infty\}$. It is well known that $\partial\varphi$ is a maximal monotone operator. Here, an operator $A\colon H \to 2^H$ is said to be *maximal monotone* (or m-accretive) if the following two conditions hold:

(i) $(\xi_1 - \xi_2, u_1 - u_2)_H \geq 0$ for all $(u_1, \xi_1), (u_2, \xi_2) \in G(A)$,
(ii) $R(I + \lambda A) = H$ for every $\lambda \in (0, \infty)$.

For each maximal monotone operator $A\colon H \to 2^H$, the *resolvent* $J_\lambda^A\colon H \to H$ and the *Yosida approximation* $A_\lambda\colon H \to H$ are defined by

$$J_\lambda^A := (I + \lambda A)^{-1}, \quad A_\lambda := \frac{I - J_\lambda^A}{\lambda}$$

for $\lambda \in (0, \infty)$, respectively. We also write $\mathring{A}(w)$ for the minimal section of A at $w \in D(A)$. The following properties are well known:

Proposition 3.1 (see, e.g., [3–5, 16]). *Let $A\colon H \to 2^H$ be a maximal monotone operator. Then the following properties hold*:

(i) *For each $\lambda \in (0, \infty)$, the mapping $J_\lambda^A\colon H \to H$ is non-expansive (i.e., 1-Lipschitz continuous). Moreover, $J_\lambda^A w \to w$ strongly in H as $\lambda \to 0_+$ for all $w \in \overline{D(A)}^H$.*
(ii) *$A_\lambda\colon H \to H$ is maximal monotone. Moreover, A_λ is Lipschitz continuous in H with Lipschitz constant $1/\lambda$.*
(iii) *$A_\lambda(w) \in A(J_\lambda^A w)$ for every $w \in H$ and $\lambda \in (0, \infty)$.*
(iv) *$\|A_\lambda(w)\|_H \leq \|\mathring{A}(w)\|_H$ for every $w \in D(A)$ and $\lambda \in (0, \infty)$. Moreover, $A_\lambda(w) \to \mathring{A}(w)$ strongly in H as $\lambda \to 0_+$ for all $w \in D(A)$.*

We next define the *Moreau–Yosida regularization* $\varphi_\lambda\colon H \to \mathbb{R}$ of φ for $\lambda \in (0, \infty)$ by

$$\varphi_\lambda(w) := \inf_{z \in H} \left(\frac{1}{2\lambda} \|w - z\|_H^2 + \varphi(z) \right)$$

for $w \in H$. We recall the following well known result:

Proposition 3.2 (see, e.g., [3–5, 15]). *Let $\varphi\colon H \to (-\infty, \infty]$ be a proper lower-semicontinuous convex functional, and let φ_λ be its Moreau–Yosida regularization for $\lambda \in (0, \infty)$. Then the following assertions hold*:

(i) *φ_λ is convex and Fréchet differentiable in H, and its derivative satisfies $\partial\varphi_\lambda = (\partial\varphi)_\lambda$, that is, the derivative $\partial(\varphi_\lambda)$ of φ_λ coincides with the Yosida approximation $(\partial\varphi)_\lambda$ of $\partial\varphi$. Hence, for simplicity, the notation $\partial\varphi_\lambda$ is used instead of both $\partial(\varphi_\lambda)$ and $(\partial\varphi)_\lambda$ in what follows.*
(ii) *It holds that $\varphi(J_\lambda^{\partial\varphi} w) \leq \varphi_\lambda(w) \leq \varphi(w)$ for every $w \in H$ and $\lambda \in (0, \infty)$.*
(iii) *$\varphi_\lambda(w) \to \varphi(w)$ as $\lambda \to 0_+$ for every $w \in H$.*

3.2. **Time-dependent subdifferential operators.** We briefly collect some properties of time-dependent subdifferential operators under the assumption (A1). For details, we refer the reader to [10, 11, 14, 19].

For each $t \in [0, T]$, let $\varphi^t\colon H \to (-\infty, \infty]$ be a proper lower-semicontinuous convex functional, and assume that (A1) holds. In what follows, for each $\lambda \in (0, \infty)$ and $t \in [0, T]$, the resolvent $J_\lambda^{\partial\varphi^t}$ of $\partial\varphi^t$ is denoted by $J_\lambda^t := (I + \lambda\,\partial\varphi^t)^{-1}$. We collect below several basic properties.

Proposition 3.3 (see [11]). *Assume that* (A1) *holds. Then there exists a constant* $C \in [0, \infty)$ *such that for every* $\lambda \in (0,1)$, $z \in H$ *and* $t \in [0, T]$, *the following inequalities hold*:

$$\varphi^t(z) \geq -C\big(\|z\|_H + 1\big), \quad \|J^t_\lambda z\|_H \leq C\big(1 + \|z\|_H\big),$$
$$\|\partial\varphi^t_\lambda(z)\|_H \leq \frac{C}{\lambda}\big(1 + \|z\|_H\big).$$

From Propositions 3.2 and 3.3, there exists a constant $C \in [0, \infty)$ such that

$$\varphi^t(z) \geq \varphi^t_\lambda(z) \geq \varphi^t\big(J^t_\lambda z\big) \geq -C\big(\|z\|_H + 1\big)$$

for all $\lambda \in (0,1)$, $z \in H$ and $t \in [0,T]$. Hence there exists a constant $D \in (0, \infty)$ such that

$$|\varphi^t(z)| \leq \varphi^t(z) + D\big(\|z\|_H + 1\big), \tag{3.1}$$
$$|\varphi^t_\lambda(z)| \leq \varphi^t_\lambda(z) + D\big(\|z\|_H + 1\big) \tag{3.2}$$

for all $\lambda \in (0,1)$, $z \in H$ and $t \in [0,T]$.

Proposition 3.4 (see [11]). *Assume that* (A1) *holds. Then for each* $\lambda \in (0,1)$ *and* $v \in L^1(0,T;H)$, *the maps* $t \mapsto \varphi^t_\lambda\big(v(t)\big)$ *and* $t \mapsto \varphi^t\big(v(t)\big)$ *are measurable in* $(0,T)$. *Furthermore, the maps* $t \mapsto J^t_\lambda\big(v(t)\big)$ *and* $t \mapsto \partial\varphi^t_\lambda\big(v(t)\big)$ *are strongly measurable in* $(0,T)$.

Proposition 3.5 (see [10, 11]). *Assume that* (A1) *holds. Let* $\Phi\colon L^2(0,T;H) \to (-\infty, \infty]$ *be a functional defined by*

$$\Phi(u) := \begin{cases} \displaystyle\int_0^T \varphi^t\big(u(t)\big)\,\mathrm{d}t & \text{if } \varphi^\bullet\big(u(\bullet)\big) \in L^1(0,T), \\ \infty & \text{otherwise} \end{cases}$$

for $u \in L^2(0,T;H)$. *Then* Φ *is a proper, lower-semicontinuous, and convex functional with the effective domain,*

$$D(\Phi) = \left\{w \in L^2(0,T;H)\colon \varphi^\bullet\big(w(\bullet)\big) \in L^1(0,T)\right\}.$$

Moreover, for each $\lambda \in (0,1)$, *the Moreau–Yosida regularization* Φ_λ *of* Φ *is given by*

$$\Phi_\lambda(u) = \int_0^T \varphi^t_\lambda\big(u(t)\big)\,\mathrm{d}t \quad \text{for } u \in L^2(0,T;H).$$

Furthermore, for each $u \in L^2(0,T;H)$, *the subdifferential operator* $\partial\Phi_\lambda$ *satisfies*

$$[\partial\Phi_\lambda(u)](t) = \partial\varphi^t_\lambda\big(u(t)\big) \quad \text{for a.e. } t \in (0,T),$$

and the corresponding resolvent $J^{\partial\Phi}_\lambda$ *is given by*

$$[J^{\partial\Phi}_\lambda(u)](t) = J^t_\lambda\big(u(t)\big) \quad \text{for a.e. } t \in (0,T).$$

The subdifferential operator $\partial\Phi$ *can be characterized as*

$$\partial\Phi(u) = \Big\{w \in L^2(0,T;H)\colon w(t) \in \partial\varphi^t\big(u(t)\big) \quad \text{for a.e. } t \in (0,T)\Big\}$$

for $u \in L^2(0,T;H)$.

Proposition 3.6 (see [11]). *Assume that* (A1) *holds. Let* $\lambda \in (0,1)$ *and* $u \in W^{1,2}(0,T;H)$. *Then the mapping* $t \mapsto \varphi^t_\lambda\big(u(t)\big)$ *is differentiable a.e. in* $(0,T)$, *and for every* $s, t \in [0,T]$ *with* $s \leq t$, *it holds that*

$$\varphi^t_\lambda\big(u(t)\big) - \varphi^s_\lambda\big(u(s)\big) \leq \int_s^t \frac{\mathrm{d}}{\mathrm{d}\tau}\varphi^\tau_\lambda\big(u(\tau)\big)\,\mathrm{d}\tau.$$

Moreover, there exist a constant $c_0 \in [0,1)$ *and nonnegative functions* $\eta_1, \eta_2 \in L^1(0,T)$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\lambda^t\big(u(t)\big) \le \big(\partial\varphi_\lambda^t\big(u(t)\big), u'(t)\big)_H + c_0\big\|\partial\varphi_\lambda^t\big(u(t)\big)\big\|_H^2 + \eta_1(t)\big|\varphi_\lambda^t\big(u(t)\big)\big| + \eta_2(t)$$

for a.e. $t \in (0,T)$. *Here the constant* c_0 *and the functions* η_1, η_2 *are independent of both* λ *and* u. *In particular, it holds that*

$$\begin{aligned}\varphi_\lambda^t\big(u(t)\big) - \varphi_\lambda^0\big(u(0)\big) &\le \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\varphi_\lambda^\tau\big(u(\tau)\big)\,\mathrm{d}\tau \\ &\le \int_0^t \big(\partial\varphi_\lambda^\tau\big(u(\tau)\big), u'(\tau)\big)_H\,\mathrm{d}\tau \\ &\quad + \int_0^t \Big(c_0\big\|\partial\varphi_\lambda^\tau\big(u(\tau)\big)\big\|_H^2 + \eta_1(\tau)\big|\varphi_\lambda^\tau\big(u(\tau)\big)\big| + \eta_2(\tau)\Big)\,\mathrm{d}\tau\end{aligned}$$

for all $t \in (0,T]$.

3.3. **Nonlocal and local time-differential operators.** We introduce several nonlocal and local time-differential operators. For more general settings, see [6, 7, 9, 18, 20].

Let $(k,\ell) \in PC$. We define the nonlocal operator $\mathcal{B}\colon D(\mathcal{B}) \subset L^2(0,T;H) \to L^2(0,T;H)$ by

$$D(\mathcal{B}) := \big\{w \in L^2(0,T;H)\colon k * w \in W^{1,2}(0,T;H) \text{ and } (k*w)(0) = 0\big\},$$
$$\mathcal{B}(w) := \partial_t(k*w) \quad \text{for } w \in D(\mathcal{B}).$$

Then $\mathcal{B}$ is a linear maximal monotone operator in $L^2(0,T;H)$ (see [6, 9, 18]), and for each $\lambda \in (0,\infty)$, the Yosida approximation $\mathcal{B}_\lambda$ is given by

$$\mathcal{B}_\lambda(w) = \partial_t(k_\lambda * w) \quad \text{for } w \in L^2(0,T;H),$$

where $k_\lambda \in W^{1,1}_{\mathrm{loc}}([0,\infty))$ solves the Volterra equation,

$$\lambda k_\lambda(t) + (\ell * k_\lambda)(t) = 1 \quad \text{for } t \in (0,\infty), \quad k_\lambda(0) = 1/\lambda.$$

It has been shown that k_λ is a nonnegative and nonincreasing function. Furthermore, it is known that $k_\lambda \to k$ strongly in $L^1(0,T)$ as $\lambda \to 0_+$. Moreover, the following chain-rule formula holds.

Proposition 3.7 (see [1, 2]). *Let* $\varphi : H \to (-\infty,\infty]$ *be a proper lower-semicontinuous convex functional. Let* $u \in L^2(0,T;H)$ *and* $u_0 \in D(\varphi)$ *be such that* $\varphi\big(u(\bullet)\big) \in L^1(0,T)$, $u - u_0 \in D(\mathcal{B})$ *and* $u(t) \in D(\partial\varphi)$ *for a.e.* $t \in (0,T)$. *Assume that there exists* $g \in L^2(0,T;H)$ *satisfying* $g(t) \in \partial\varphi(u(t))$ *for a.e.* $t \in (0,T)$. *Then it holds that*

$$\int_0^t \big(\mathcal{B}(u-u_0)(\tau), g(\tau)\big)_H\,\mathrm{d}\tau \ge \big[k * \big(\varphi(u) - \varphi(u_0)\big)\big](t)$$

for a.e. $t \in (0,T)$.

Let $\mathcal{A}\colon D(\mathcal{A}) \subset L^2(0,T;H) \to L^2(0,T;H)$ be the local time-differential operator defined by

$$D(\mathcal{A}) := \{w \in W^{1,2}(0,T;H)\colon w(0) = 0\}, \quad \mathcal{A}(w) := \partial_t w \quad \text{for } w \in D(\mathcal{A}).$$

Then we observe that $D(\mathcal{A}) \subset D(\mathcal{B})$. We recall that for each $u \in D(\mathcal{A})$, the following inequality holds:

$$\int_0^t \big(\mathcal{B}(u)(\tau), \mathcal{A}(u)(\tau)\big)_H\,\mathrm{d}\tau \ge \frac{1}{2}\big(\ell * \|\mathcal{B}(u)\|_H^2\big)(t) \tag{3.3}$$

for a.e. $t \in (0,T)$ (see [1]). Therefore, for every $u \in D(\mathcal{A})$, Hölder's inequality yields

$$\frac{1}{2}\big(\ell * \|\mathcal{B}(u)\|_H^2\big)(t) \le \big(\mathcal{B}(u), \mathcal{A}(u)\big)_{L^2(0,t;H)}$$

$$\leq \|\mathcal{B}(u)\|_{L^2(0,T;H)}\|\mathcal{A}(u)\|_{L^2(0,T;H)} \tag{3.4}$$

for a.e. $t \in (0,T)$.

4. Key estimates

For the proof of the main results, several auxiliary results were established in [13]. We state these results without proofs. First, we consider the following Gronwall-type inequality.

Proposition 4.1. *Let $g_1 \in L^\infty(0,T)$, and let $g_2, g_3 \in L^1(0,T)$ be nonnegative functions. Suppose that $f \in L^\infty(0,T)$ satisfies*

$$f(t) \leq g_1(t) + \int_0^t g_2(s)f(s)\,\mathrm{d}s + (g_3 * f)(t) \quad \text{for a.e. } t \in (0,T).$$

Then $f(t) \leq G(t)$ for a.e. $t \in (0,T)$, where $G \in L^\infty(0,T)$ denotes the unique solution of the Volterra integral equation,

$$G(t) = g_1(t) + \int_0^t g_2(s)G(s)\,\mathrm{d}s + (g_3 * G)(t) \quad \text{for } t \in (0,T).$$

Next, we state two fractional chain-rule formulae for time-dependent subdifferential operators. We first consider the case of Sobolev-regular kernels.

Lemma 4.2 (Nonlocal chain-rule formula for regular kernels). *Let $k \in W^{1,1}(0,T)$ be a nonnegative and nonincreasing function. For each $t \in [0,T]$, let $\varphi^t \colon H \to (-\infty,\infty]$ be a proper lower-semicontinuous convex functional, and suppose that* (A1) *and* (A2) *hold. Let $u_0 \in D(\varphi^0)$ and let $u, g \in L^2(0,T;H)$ be such that $\varphi^\bullet(u(\bullet)) \in L^1(0,T)$, $u(t) \in D(\partial\varphi^t)$ and $g(t) \in \partial\varphi^t\big(u(t)\big)$ for a.e. $t \in (0,T)$. Then there exists a constant $c \in [0,\infty)$ independent of u, g, and k such that, for all $\varepsilon \in (0,1)$, the following inequality holds*:

$$\begin{aligned}
&\big(\partial_t[k*(u-u_0)](t),\, g(t)\big)_H \\
&\quad \geq \partial_t\big[k*\varphi^\bullet(u(\bullet))\big](t) - k(t)\varphi^0(u_0) - \varepsilon\, c\,\|k\|_{L^1(0,T)}\,\|g(t)\|_H^2 \\
&\qquad - \frac{c}{\varepsilon}\|k\|_{L^1(0,T)}\big(1+|\varphi^0(u_0)|\big) \\
&\qquad - \frac{c}{\varepsilon}\int_0^t \big(-sk'(s)\big)\,|\varphi^{t-s}(u(t-s))|\,\mathrm{d}s
\end{aligned}$$

for a.e. $t \in (0,T)$, and moreover, the following inequality holds:

$$\begin{aligned}
&\int_0^t \big(\partial_t[k*(u-u_0)](s),\, g(s)\big)_H\,\mathrm{d}s \\
&\quad \geq \big[k*\varphi^\bullet(u(\bullet))\big](t) - \varphi^0(u_0)\int_0^t k(s)\,\mathrm{d}s - \varepsilon\, c\,\|k\|_{L^1(0,T)}\int_0^t \|g(s)\|_H^2\,\mathrm{d}s \\
&\qquad - \frac{c}{\varepsilon}t\,\|k\|_{L^1(0,T)}\big(1+|\varphi^0(u_0)|\big) \\
&\qquad - \frac{c}{\varepsilon}\|k\|_{L^1(0,T)}\int_0^t |\varphi^s(u(s))|\,\mathrm{d}s
\end{aligned} \tag{4.1}$$

for all $t \in (0,T]$.

Here, combining Lemma 4.2 with the results stated in Section 3.3, we can obtain the following lemma.

Lemma 4.3 (Nonlocal chain-rule formula for $(k,\ell)\in PC$). *For each $t\in[0,T]$, let $\varphi^t\colon H\to(-\infty,\infty]$ be a proper lower-semicontinuous convex functional, and suppose that both assumptions* (A1) *and* (A2) *are satisfied. Let $(k,\ell)\in PC$, $u_0\in D(\varphi^0)$, and let $u,g\in L^2(0,T;H)$ be such that $u-u_0\in D(\mathcal{B})$ (see Section 3.3), $\varphi^\bullet(u(\bullet))\in L^1(0,T)$, $u(t)\in D(\partial\varphi^t)$ and $g(t)\in\partial\varphi^t(u(t))$ for a.e. $t\in(0,T)$. Then, for all $\varepsilon\in(0,1)$, the following inequality holds*:

$$\begin{aligned}&\int_0^t \big(\partial_t[k*(u-u_0)](s),\, g(s)\big)_H\,\mathrm{d}s\\&\quad\geq \big[k*\varphi^\bullet(u(\bullet))\big](t)-\varphi^0(u_0)\int_0^t k(s)\,\mathrm{d}s-\varepsilon\, c\,\|k\|_{L^1(0,T)}\int_0^t\|g(s)\|_H^2\,\mathrm{d}s\\&\qquad-\frac{c}{\varepsilon}\,t\,\|k\|_{L^1(0,T)}\big(1+|\varphi^0(u_0)|\big)\\&\qquad-\frac{c}{\varepsilon}\,\|k\|_{L^1(0,T)}\int_0^t\big|\varphi^s(u(s))\big|\,\mathrm{d}s\end{aligned}$$

for all $t\in(0,T]$, where c is the constant of (4.1) *in Lemma 4.2 and is independent of u, g, k and ε. In particular, there exists a constant $C\in[0,\infty)$ independent of u and g such that, for all $\varepsilon\in(0,1)$, it holds that*

$$\begin{aligned}&\int_0^t \big(\partial_t[k*(u-u_0)](s),\, g(s)\big)_H\,\mathrm{d}s\\&\quad\geq \big[k*\varphi^\bullet(u(\bullet))\big](t)-\varphi^0(u_0)\int_0^t k(s)\,\mathrm{d}s-\varepsilon\, C\int_0^t\|g(s)\|_H^2\,\mathrm{d}s\\&\qquad-\frac{C}{\varepsilon}\left[T\big(1+|\varphi^0(u_0)|\big)+\int_0^t\big|\varphi^s(u(s))\big|\,\mathrm{d}s\right]\end{aligned}\tag{4.2}$$

for all $t\in(0,T]$.

5. Outline of proof for Theorem 2.2

In this section, we give an outline of proof for the uniqueness and continuous dependence on initial data of strong solutions to (P). Let $(u_i,\xi_i)\in(\mathrm{P})_{u_{0,i},\,f_i}$ for $i=1,2$, where $u_{0,i}\in H$ and $f_i\in L^2(0,T;H)$. Then we have $(u_1-u_2)-(u_{0,1}-u_{0,2})\in D(\mathcal{B})$ (see Section 3.3). Furthermore, since $\xi_1(t)\in\partial\varphi^t\big(u_1(t)\big)$ and $\xi_2(t)\in\partial\varphi^t\big(u_2(t)\big)$ for a.e. $t\in(0,T)$, it follows from the monotonicity of $\partial\varphi^t$ (see Section 3.1) that

$$\int_0^t\big(\xi_1(s)-\xi_2(s),\,u_1(s)-u_2(s)\big)_H\,\mathrm{d}s\geq 0\tag{5.1}$$

for a.e. $t\in(0,T)$. Moreover, we observe that

$$\mathcal{B}\big[(u_1-u_2)-(u_{0,1}-u_{0,2})\big](t)=-\big(\xi_1(t)-\xi_2(t)\big)+f_1(t)-f_2(t)$$

for a.e. $t\in(0,T)$. By multiplying both sides of the above identity by $u_1-u_2\in L^2(0,T;H)$ and integrating it over $(0,t)$, we deduce from (5.1), Proposition 3.7 and Hölder's inequality that

$$\frac{1}{2}\big[k*\big(\|u_1-u_2\|_H^2-\|u_{0,1}-u_{0,2}\|_H^2\big)\big](t)\leq\|f_1-f_2\|_{L^2(0,T;H)}\,\|u_1-u_2\|_{L^2(0,T;H)}\tag{5.2}$$

for a.e. $t\in(0,T)$. Since $k*\ell\equiv 1$ on $(0,T)$ and $(1*\ell)(t)\leq\|\ell\|_{L^1(0,T)}$ for a.e. $t\in(0,T)$, by convolving both sides of (5.2) with ℓ and using Young's inequality, we obtain

$$\frac{1}{2}\int_0^t\|u_1(s)-u_2(s)\|_H^2\,\mathrm{d}s$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^t \left(\|u_1(s)-u_2(s)\|_H^2 - \|u_{0,1}-u_{0,2}\|_H^2\right)\,\mathrm{d}s + \frac{t}{2}\|u_{0,1}-u_{0,2}\|_H^2 \\
&= \frac{1}{2}\Big[\ell * \big[k * \left(\|u_1-u_2\|_H^2 - \|u_{0,1}-u_{0,2}\|_H^2\right)\big]\Big](t) + \frac{t}{2}\|u_{0,1}-u_{0,2}\|_H^2 \\
&\le \|\ell\|_{L^1(0,T)}\,\|f_1-f_2\|_{L^2(0,T;H)}\,\|u_1-u_2\|_{L^2(0,T;H)} + \frac{t}{2}\|u_{0,1}-u_{0,2}\|_H^2 \\
&\le \|\ell\|_{L^1(0,T)}^2\,\|f_1-f_2\|_{L^2(0,T;H)}^2 + \frac{1}{4}\,\|u_1-u_2\|_{L^2(0,T;H)}^2 \\
&\quad + \frac{T}{2}\|u_{0,1}-u_{0,2}\|_H^2
\end{aligned}
$$

for a.e. $t \in (0,T)$. Hence it follows that

$$
\begin{aligned}
&\|u_1-u_2\|_{L^2(0,T;H)}^2 \\
&\quad \le 4\,\|\ell\|_{L^1(0,T)}^2\,\|f_1-f_2\|_{L^2(0,T;H)}^2 + 2T\,\|u_{0,1}-u_{0,2}\|_H^2 \\
&\quad \le \max\left\{4\,\|\ell\|_{L^1(0,T)}^2,\, 2T\right\}\Big(\|u_{0,1}-u_{0,2}\|_H^2 + \|f_1-f_2\|_{L^2(0,T;H)}^2\Big).
\end{aligned}
$$

This completes the outline of the proof.

6. Outline of proof for Theorem 2.3

In this section, we give an outline of proof for Theorem 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be the operators introduced in Section 3.3, and let Φ be the proper lower-semicontinuous convex functional on $L^2(0,T;H)$ defined in Proposition 3.5.

6.1. **Approximate problem.** For $\nu \in (0,\infty)$ and $\lambda \in (0,1)$, we show that the following approximate problem admits a unique strong solution:

$$
(\nu\mathcal{A}+\mathcal{B})(u_{\nu,\lambda}-u_0) + \partial\Phi_\lambda(u_{\nu,\lambda}) = f \quad \text{in } L^2(0,T;H). \tag{6.1}
$$

It suffices to show that there exists a unique function $u_{\nu,\lambda} \in W^{1,2}(0,T;H)$ with $u_{\nu,\lambda}(0) = u_0$ such that

$$
\nu\,\partial_t u_{\nu,\lambda}(t) + \partial_t\big[k*(u_{\nu,\lambda}-u_0)\big](t) + \partial\varphi_\lambda^t\big(u_{\nu,\lambda}(t)\big) = f(t) \quad \text{in } H
$$

for a.e. $t \in (0,T)$. As already mentioned in Sections 3.1 and 3.2, the mapping $(t,w) \mapsto \partial\varphi_\lambda^t(w)$ is a Carathéodory function, that is, for each $w \in H$, the map $t \mapsto \partial\varphi_\lambda^t(w)$ is strongly measurable in $(0,T)$, and moreover, for each $t \in [0,T]$, the map $w \mapsto \partial\varphi_\lambda^t(w)$ is continuous in H (see Propositions 3.1, 3.2 and 3.4). The following lemma guarantees that the approximate problem (6.1) admits a unique strong solution. We omit the proof for brevity.

Lemma 6.1. *Let X be a real Banach space. Let $k \in L^1(0,T)$, $p \in [1,\infty]$, and let $F\colon [0,T]\times X \to X$ be a Carathéodory function such that there exist a constant $C_0 \in [0,\infty)$ and a function $\rho \in L^p(0,T)$ satisfying the following*:

$$
\|F(t,w)\|_X \le C_0\|w\|_X + |\rho(t)|,
$$
$$
\|F(t,x)-F(t,y)\|_X \le C_0\|x-y\|_X
$$

for all $w,x,y \in X$ and for a.e. $t \in (0,T)$. Then, for each $\nu \in (0,\infty)$, $v_0 \in X$ and $f \in L^p(0,T;X)$, there exists a unique function $v \in W^{1,p}(0,T;X)$ with $v(0) = v_0$ such that

$$
\nu\,\partial_t v(t) + \partial_t\big[k*(v-v_0)\big](t) + F(t,v(t)) = f(t) \quad \text{in } X \quad \text{for } t \in (0,T).
$$

6.2. **A priori estimate.** We next derive a priori estimates. For each $\nu \in (0,1)$, let $u_\nu \in W^{1,2}(0,T;H)$ be such that $u_\nu(0) = u_0$ (i.e., $u_\nu - u_0 \in D(\mathcal{A}) = D(\nu\mathcal{A}+\mathcal{B})$, see Section 3.3) and

$$(\nu\mathcal{A}+\mathcal{B})\big(u_\nu - u_0\big) + \partial\Phi_\nu(u_\nu) = f \quad \text{in } L^2(0,T;H). \tag{6.2}$$

Testing (6.2) by $\mathcal{A}(u_\nu - u_0) = \partial_t u_\nu = \partial_t(u_\nu - u_0)$, we get

$$\begin{aligned}
&\nu\big\|\mathcal{A}(u_\nu - u_0)(s)\big\|_H^2 + \big(\mathcal{B}(u_\nu - u_0)(s),\, \mathcal{A}(u_\nu - u_0)(s)\big)_H \\
&\quad + \big(\partial\varphi_\nu^s(u_\nu(s)),\, \partial_s u_\nu(s)\big)_H \\
&= \big(f(s),\, \partial_s(u_\nu - u_0)(s)\big)_H \\
&= \partial_s\big(f(s),\, u_\nu(s) - u_0\big)_H - \big(\partial_s f(s),\, u_\nu(s) - u_0\big)_H
\end{aligned}$$

for a.e. $s \in (0,T)$. Integrating both sides over $(0,t)$ and using (3.3), (6.2), Propositions 3.2 and 3.6 together with Young's inequality, there exist nonnegative functions $\eta_1, \eta_2 \in L^1(0,T)$ independent of ν such that

$$\begin{aligned}
&\int_0^t \nu\big\|\mathcal{A}(u_\nu - u_0)(s)\big\|_H^2 \,\mathrm{d}s + \frac{1}{2}\big[\ell * \|\mathcal{B}(u_\nu - u_0)\|_H^2\big](t) + \varphi_\nu^t\big(u_\nu(t)\big) \\
&\le \varphi^0(u_0) + \int_0^t \eta_2(s)\,\mathrm{d}s + \|f\|_{L^\infty(0,T;H)}\,\|u_\nu(t) - u_0\|_H \\
&\quad + 4\int_0^t \|f(s)\|_H^2\,\mathrm{d}s + 4\nu^2\int_0^t \big\|\mathcal{A}(u_\nu - u_0)(s)\big\|_H^2\,\mathrm{d}s \\
&\quad + 4\int_0^t \big\|\mathcal{B}(u_\nu - u_0)(s)\big\|_H^2\,\mathrm{d}s + \int_0^t \eta_1(s)\big|\varphi_\nu^s(u_\nu(s))\big|\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \|\partial_s f(s)\|_H^2\,\mathrm{d}s + \frac{1}{2}\int_0^t \|u_\nu(s) - u_0\|_H^2\,\mathrm{d}s
\end{aligned}$$

for a.e. $t \in (0,T)$. Consequently, there exists a constant $C_0 \in [0,\infty)$ independent of ν such that

$$\begin{aligned}
&(\nu - 4\nu^2)\int_0^t \big\|\mathcal{A}(u_\nu - u_0)(s)\big\|_H^2\,\mathrm{d}s + \frac{1}{2}\big[\ell * \|\mathcal{B}(u_\nu - u_0)\|_H^2\big](t) \\
&\quad + \varphi_\nu^t\big(u_\nu(t)\big) - \|f\|_{L^\infty(0,T;H)}\,\|u_\nu(t) - u_0\|_H \\
&\le C_0 + 4\int_0^t \big\|\mathcal{B}(u_\nu - u_0)(s)\big\|_H^2\,\mathrm{d}s + \int_0^t \eta_1(s)\big|\varphi_\nu^s(u_\nu(s))\big|\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \|u_\nu(s) - u_0\|_H^2\,\mathrm{d}s
\end{aligned} \tag{6.3}$$

for a.e. $t \in (0,T)$. Since $(k,\ell) \in PC$ and $u_\nu - u_0 \in D(\mathcal{A}) \subset D(\mathcal{B})$, it follows from Hölder's inequality that

$$\begin{aligned}
\|u_\nu(t) - u_0\|_H^2 &= \big\|[\ell * \mathcal{B}(u_\nu - u_0)](t)\big\|_H^2 \\
&= \left\|\int_0^t \ell^{\frac{1}{2}}(t-s)\,\ell^{\frac{1}{2}}(t-s)\,\mathcal{B}(u_\nu - u_0)(s)\,\mathrm{d}s\right\|_H^2 \\
&\le \big(\|\ell\|_{L^1(0,T)} + 1\big)\,\big[\ell * \|\mathcal{B}(u_\nu - u_0)\|_H^2\big](t)
\end{aligned} \tag{6.4}$$

for a.e. $t \in (0,T)$. From (3.2), (6.4) and Young's inequality, for every $\varepsilon \in (0,1)$, we observe that

$$\begin{aligned}
&\frac{1}{2}\big[\ell * \|\mathcal{B}(u_\nu - u_0)\|_H^2\big](t) + \varphi_\nu^t\big(u_\nu(t)\big) - \|f\|_{L^\infty(0,T;H)}\,\|u_\nu(t) - u_0\|_H \\
&\ge \frac{1}{4}\big[\ell * \|\mathcal{B}(u_\nu - u_0)\|_H^2\big](t) + \frac{1}{4}\big(1 + \|\ell\|_{L^1(0,T)}\big)^{-1}\|u_\nu(t) - u_0\|_H^2
\end{aligned}$$

$$+\left|\varphi_\nu^t(u_\nu(t))\right| - \frac{\varepsilon}{2}\|u_\nu(t)-u_0\|_H^2 - \frac{1}{2\varepsilon}D^2 - D\|u_0\|_H - D$$
$$-\frac{\varepsilon}{2}\|u_\nu(t)-u_0\|_H^2 - \frac{1}{2\varepsilon}\|f\|_{L^\infty(0,T;H)}^2$$

for a.e. $t \in (0,T)$, where D is a constant satisfying (3.2). Hence there exists a constant $\varepsilon_0 \in (0,1/4)$ independent of ν such that

$$\frac{1}{2}\left[\ell * \|\mathcal{B}(u_\nu - u_0)\|_H^2\right](t) + \varphi_\nu^t\big(u_\nu(t)\big) - \|f\|_{L^\infty(0,T;H)}\|u_\nu(t)-u_0\|_H$$
$$\geq \frac{1}{4}\left[\ell * \|\mathcal{B}(u_\nu - u_0)\|_H^2\right](t) + \varepsilon_0\|u_\nu(t)-u_0\|_H^2$$
$$+\left|\varphi_\nu^t(u_\nu(t))\right| - \frac{1}{\varepsilon_0} \tag{6.5}$$

for a.e. $t \in (0,T)$. Using (6.3) and (6.5) together with $k * \ell \equiv 1$ on $(0,T)$, there exist a constant $C_3 \in (0,\infty)$ and a nonnegative function $\eta_3 \in L^1(0,T)$ independent of ν such that

$$(\nu - 4\nu^2)\int_0^t \left\|\mathcal{A}(u_\nu - u_0)(s)\right\|_H^2 \,\mathrm{d}s + \frac{1}{C_3}F_\nu(t)$$
$$\leq C_3 + C_3\big(k * F_\nu\big)(t) + \int_0^t \eta_3(s)\, F_\nu(s)\,\mathrm{d}s \tag{6.6}$$

for a.e. $t \in (0,T)$, where

$$F_\nu(\bullet) := \|u_\nu(\bullet) - u_0\|_H^2 + \left|\varphi_\nu^\bullet(u_\nu(\bullet))\right| + \left[\ell * \|\mathcal{B}(u_\nu - u_0)\|_H^2\right](\bullet).$$

From the embedding $u_\nu \in W^{1,2}(0,T;H) \subset L^\infty(0,T;H)$ together with (3.2), (3.4) and Propositions 3.1 to 3.3, we can deduce that $F_\nu \in L^\infty(0,T)$. Since $\nu - 4\nu^2 > 0$ for all $\nu \in (0,1/4)$, it follows from Proposition 4.1 that

$$\operatorname*{ess\,sup}_{\nu\in(0,1/4)} \|F_\nu\|_{L^\infty(0,T)} < \infty. \tag{6.7}$$

In particular, recalling $k * \ell \equiv 1$ on $(0,T)$ and using Young's convolution inequality, we obtain

$$\operatorname*{ess\,sup}_{\nu\in(0,1/4)} \|\mathcal{B}(u_\nu - u_0)\|_{L^2(0,T;H)}^2 = \operatorname*{ess\,sup}_{\nu\in(0,1/4)} \sup_{t\in(0,T)} \int_0^t \|\mathcal{B}(u_\nu - u_0)(s)\|_H^2 \,\mathrm{d}s$$
$$= \operatorname*{ess\,sup}_{\nu\in(0,1/4)} \|\,(k*\ell) * \|\mathcal{B}(u_\nu - u_0)\|_H^2\,\|_{L^\infty(0,T)}$$
$$\leq \|k\|_{L^1(0,T)} \operatorname*{ess\,sup}_{\nu\in(0,1/4)} \|\ell * \|\mathcal{B}(u_\nu - u_0)\|_H^2\|_{L^\infty(0,T)} < \infty. \tag{6.8}$$

Moreover, from (6.6), (6.7) and the fact that $\nu - 4\nu^2 \geq (1/2)\nu$ for all $\nu \in (0,1/8)$, we also deduce that

$$\operatorname*{ess\,sup}_{\nu\in(0,1/8)} \nu^{-1}\|\nu\mathcal{A}(u_\nu - u_0)\|_{L^2(0,T;H)}^2$$
$$= \operatorname*{ess\,sup}_{\nu\in(0,1/8)} \sup_{t\in(0,T)} \int_0^t \nu\|[\mathcal{A}(u_\nu - u_0)](s)\|_H^2 \,\mathrm{d}s < \infty. \tag{6.9}$$

Combining (6.2), (6.8) and (6.9), we can deduce that

$$\operatorname*{ess\,sup}_{\nu\in(0,1/8)} \|\partial\Phi_\nu(u_\nu)\|_{L^2(0,T;H)}^2 = \operatorname*{ess\,sup}_{\nu\in(0,1/8)} \|\partial\varphi_\nu^\bullet(u_\nu(\bullet))\|_{L^2(0,T;H)}^2 < \infty. \tag{6.10}$$

6.3. **Convergence of approximate solutions.** We first show that

$$\lim_{\mu,\nu\to 0+} \|u_\mu - u_\nu\|_{L^2(0,T;H)} = 0,$$

that is, the family $(u_\nu)_{\nu\in(0,1/8)}$ forms a Cauchy sequence in $L^2(0,T;H)$. Fix arbitrary $\mu,\nu \in (0,1/8)$, and let u_μ and u_ν denote the solutions to (6.2) corresponding to the parameters μ and ν, respectively. From (6.2) and $u_\mu - u_\nu \in D(\mathcal{A}) \subset D(\mathcal{B})$, we get

$$\begin{aligned}\mathcal{B}(u_\mu - u_\nu)(t) = &-\mu\,\mathcal{A}(u_\mu - u_0)(t) + \nu\,\mathcal{A}(u_\nu - u_0)(t)\\ &- \big(\partial\varphi_\mu^t(u_\mu(t)) - \partial\varphi_\nu^t(u_\nu(t))\big)\end{aligned}$$

for a.e. $t\in(0,T)$. Multiplying both sides by $u_\mu - u_\nu \in L^2(0,T;H)$ and integrating it over $(0,t)$, we deduce from Proposition 3.7 and Hölder's inequality that

$$\begin{aligned}&\frac{1}{2}\big[k * \|u_\mu - u_\nu\|_H^2\big](t)\\ &\quad\le \|\mu\mathcal{A}(u_\mu - u_0)\|_{L^2(0,T;H)}\,\|u_\mu - u_\nu\|_{L^2(0,T;H)}\\ &\qquad + \|\nu\mathcal{A}(u_\nu - u_0)\|_{L^2(0,T;H)}\,\|u_\mu - u_\nu\|_{L^2(0,T;H)}\\ &\qquad + \frac{\mu+\nu}{4}\Big(\|\partial\Phi_\mu(u_\mu)\|_{L^2(0,T;H)}^2 + \|\partial\Phi_\nu(u_\nu)\|_{L^2(0,T;H)}^2\Big)\end{aligned}\tag{6.11}$$

for a.e. $t\in(0,T)$, where we used Kōmura's trick (see, e.g., [5,16]),

$$\begin{aligned}&\big(\partial\varphi_\mu^t(a) - \partial\varphi_\nu^t(b),\, a-b\big)_H\\ &\quad\ge -\frac{\mu+\nu}{4}\Big(\|\partial\varphi_\mu^t(a)\|_H^2 + \|\partial\varphi_\nu^t(b)\|_H^2\Big)\end{aligned}$$

for all $a,b\in H$ and all $t\in(0,T)$. Since $k*\ell \equiv 1$ on $(0,T)$ and $(1*\ell)(t) \le \|\ell\|_{L^1(0,T)}$ for a.e. $t\in(0,T)$, convolving both sides of (6.11) with ℓ and applying Young's inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\int_0^t \|u_\mu(\tau) - u_\nu(\tau)\|_H^2\,\mathrm{d}\tau = \frac{1}{2}[(\ell * k) * \|u_\mu - u_\nu\|_H^2](t)\\ &\le 2\big(\|\mu\mathcal{A}(u_\mu - u_0)\|_{L^2(0,T;H)}\,\|\ell\|_{L^1(0,T)}\big)^2 + \frac{1}{8}\|u_\mu - u_\nu\|_{L^2(0,T;H)}^2\\ &\quad + 2\big(\|\nu\mathcal{A}(u_\nu - u_0)\|_{L^2(0,T;H)}\,\|\ell\|_{L^1(0,T)}\big)^2 + \frac{1}{8}\|u_\mu - u_\nu\|_{L^2(0,T;H)}^2\\ &\quad + \left(\frac{\mu+\nu}{4}\|\partial\Phi_\mu(u_\mu)\|_{L^2(0,T;H)}^2 + \frac{\mu+\nu}{4}\|\partial\Phi_\nu(u_\nu)\|_{L^2(0,T;H)}^2\right)\|\ell\|_{L^1(0,T)}\end{aligned}$$

for a.e. $t\in(0,T)$. Taking the supremum of both sides over $t\in(0,T)$, one can take a constant $c_0\in[0,\infty)$ independent of μ and ν such that

$$\begin{aligned}&\|u_\mu - u_\nu\|_{L^2(0,T;H)}^2\\ &\quad\le c_0\Big(\|\mu\mathcal{A}(u_\mu - u_0)\|_{L^2(0,T;H)}^2 + \|\nu\mathcal{A}(u_\nu - u_0)\|_{L^2(0,T;H)}^2\Big)\\ &\qquad + c_0(\mu+\nu)\Big(\|\partial\Phi_\mu(u_\mu)\|_{L^2(0,T;H)}^2 + \|\partial\Phi_\nu(u_\nu)\|_{L^2(0,T;H)}^2\Big).\end{aligned}$$

From (6.9) and (6.10), we obtain $\lim_{\mu,\nu\to 0_+}\|u_\mu - u_\nu\|_{L^2(0,T;H)} = 0$. Hence there exists a function $u\in L^2(0,T;H)$ such that

$$u_\nu \to u \quad \text{in } L^2(0,T;H) \text{ as } \nu\to 0_+.\tag{6.12}$$

Furthermore, combining (6.10) with (6.12) and using the identity $u_\nu(\bullet) - J_\nu^\bullet(u_\nu(\bullet)) = \nu\,\partial\varphi_\nu^\bullet(u_\nu(\bullet))$ (which follows from the definition of the Yosida approximation), we can deduce that

$$J_\nu^\bullet(u_\nu(\bullet)) \to u(\bullet) \quad \text{in } L^2(0,T;H) \text{ as } \nu\to 0_+.\tag{6.13}$$

By virtue of (6.8), (6.10) and the reflexivity of $L^2(0,T;H)$, there exist functions $\zeta_1, \zeta_2 \in L^2(0,T;H)$ and a subsequence (not relabeled) such that

$$\mathcal{B}(u_\nu - u_0) \rightharpoonup \zeta_1 \quad \text{weakly in } L^2(0,T;H) \quad \text{as } \nu \to 0_+, \tag{6.14}$$

$$\partial\Phi_\nu(u_\nu) \rightharpoonup \zeta_2 \quad \text{weakly in } L^2(0,T;H) \quad \text{as } \nu \to 0_+. \tag{6.15}$$

Passing to the limit in (6.2) and using (6.9), (6.14) and (6.15), we obtain

$$\zeta_1(t) + \zeta_2(t) = f(t) \quad \text{in } H \quad \text{for a.e. } t \in (0,T).$$

Since $\mathcal{B}$ is maximal monotone and demiclosed (see Section 3.1), it follows from (6.12) and (6.14) that $u - u_0 \in D(\mathcal{B})$ and $\zeta_1 = \mathcal{B}(u - u_0)$. Furthermore, by virtue of Propositions 3.1 and 3.5, we have $\partial\varphi^\bullet_\nu(u_\nu(\bullet)) \in \partial\Phi\big(J^\bullet_\nu(u_\nu(\bullet))\big)$. Since $\partial\Phi$ is also maximal monotone and demiclosed (see Section 3.1), we can deduce from (6.13) and (6.15) that $\zeta_2 \in \partial\Phi(u)$. From Proposition 3.5, we obtain $\zeta_2(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in (0,T)$.

Therefore, u is a strong solution to (P). Moreover, due to Theorem 2.2, the strong solution is unique. Since $\left|\varphi^t\big(J^t_\nu(u_\nu(t))\big)\right| \le |\varphi^t_\nu(u_\nu(t))|$ for a.e. $t \in (0,T)$ (see Proposition 3.2), it follows from (6.7) and (6.13) that (2.1) holds. Furthermore, from [2], we can deduce that u belongs to $C([0,T];H)$ and $u(0) = u_0$. This completes the outline of proof.

7. Proof of Theorem 2.5

In this section, we give an outline of proof for Theorem 2.5. We define $\mathcal{B}$ as in Section 3.3, and let Φ denote the proper lower-semicontinuous convex functional on $L^2(0,T;H)$ defined as in Proposition 3.5.

Let (f_n) be a sequence in $W^{1,2}(0,T;H)$ such that $f_n \to f$ strongly in $L^2(0,T;H)$. Then, from Theorem 2.3, there exist functions $u_n, \xi_n \in L^2(0,T;H)$ such that $\varphi^\bullet(u_n(\bullet)) \in L^\infty(0,T)$ and $(u_n, \xi_n) \in (\mathrm{P})_{u_0,f_n}$, that is, $u_n - u_0 \in D(\mathcal{B})$, $\xi_n \in \partial\Phi(u_n)$, and moreover, u_n and ξ_n satisfy the equation,

$$\mathcal{B}(u_n - u_0)(s) + \xi_n(s) = f_n(s) \quad \text{for a.e. } s \in (0,T). \tag{7.1}$$

We first consider Theorem 2.5 under the additional assumption,

$$\varphi^t(z) \ge 0 \quad \text{for all } t \in [0,T] \text{ and all } z \in H. \tag{7.2}$$

Multiplying both sides of (7.1) by $\xi_n \in L^2(0,T;H)$, integrating it over $(0,t)$, and employing the fractional chain-rule formula obtained in Lemma 4.3, for every $\varepsilon \in (0,1)$, we deduce from (7.2) and Young's inequality that

$$\begin{aligned}
&\big[k * \varphi^\bullet(u_n(\bullet))\big](t) - \varphi^0(u_0)\int_0^t k(s)\,\mathrm{d}s - \varepsilon C\int_0^t \|\xi_n(s)\|_H^2\,\mathrm{d}s \\
&\quad - \frac{C}{\varepsilon}\left[T\big(1+\varphi^0(u_0)\big) + \int_0^t \varphi^s(u_n(s))\,\mathrm{d}s\right] + \int_0^t \|\xi_n(s)\|_H^2\,\mathrm{d}s \\
&\le \int_0^t \Big(\mathcal{B}(u_n - u_0)(s) + \xi_n(s),\, \xi_n(s)\Big)_H \mathrm{d}s \\
&= \int_0^t \big(f_n(s),\, \xi_n(s)\big)_H \mathrm{d}s \le \frac{1}{2\varepsilon}\int_0^t \|f_n(s)\|_H^2\,\mathrm{d}s + \frac{\varepsilon}{2}\int_0^t \|\xi_n(s)\|_H^2\,\mathrm{d}s \\
&\le \frac{1}{2\varepsilon}\sup_{n\in\mathbb{N}} \|f_n\|^2_{L^2(0,T;H)} + \frac{\varepsilon}{2}\int_0^t \|\xi_n(s)\|_H^2\,\mathrm{d}s
\end{aligned}$$

for a.e. $t \in (0,T)$. Here, $C \in [0,\infty)$ denotes the constant satisfying (4.2) and independent of ε and n. Hence, choosing ε sufficiently small and using $k * \ell \equiv 1$ on $(0,T)$, we can take a constant

$C_0 \in [0,\infty)$ independent of n such that

$$\begin{aligned}&\big[k * \varphi^\bullet(u_n(\bullet))\big](t) + \int_0^t \|\xi_n(s)\|_H^2 \,\mathrm{d}s\\&\quad \le C_0\left(1+\int_0^t \varphi^s(u_n(s))\,\mathrm{d}s\right) = C_0 + C_0\left[\ell * \big(k * \varphi^\bullet(u_n(\bullet))\big)\right](t)\end{aligned} \tag{7.3}$$

for a.e. $t \in (0,T)$. Since $\varphi^\bullet(u_n(\bullet)) \in L^\infty(0,T)$ and $k \in L^1(0,T)$, we have $k * \varphi^\bullet(u_n(\bullet)) \in L^\infty(0,T)$. Therefore, applying Proposition 4.1, we obtain

$$\sup_{n\in\mathbb{N}} \|k * \varphi^\bullet(u_n(\bullet))\|_{L^\infty(0,T)} < \infty. \tag{7.4}$$

Moreover, combining (7.3) with (7.4), we get

$$\sup_{n\in\mathbb{N}} \|\xi_n\|_{L^2(0,T;H)}^2 = \sup_{n\in\mathbb{N}} \left\| \int_0^\bullet \|\xi_n(s)\|_H^2\,\mathrm{d}s \right\|_{L^\infty(0,T)} < \infty. \tag{7.5}$$

Furthermore, since (7.1) holds and $\sup_{n\in\mathbb{N}} \|f_n\|_{L^2(0,T;H)} < \infty$, we also have

$$\sup_{n\in\mathbb{N}} \|\mathcal{B}(u_n - u_0)\|_{L^2(0,T;H)} < \infty. \tag{7.6}$$

Next, we prove that (u_n) forms a Cauchy sequence in $L^2(0,T;H)$. Fix arbitrary $n, m \in \mathbb{N}$. Since $(u_n,\xi_n) \in (\mathrm{P})_{u_0,f_n}$ and $(u_m,\xi_m) \in (\mathrm{P})_{u_0,f_m}$, it follows from Theorem 2.2 that there exists a constant $C_1 \in [0,\infty)$ independent of n and m such that

$$\|u_n - u_m\|_{L^2(0,T;H)}^2 \le C_1 \|f_n - f_m\|_{L^2(0,T;H)}^2.$$

Since $f_n \to f$ strongly in $L^2(0,T;H)$, we conclude that $\lim_{n,m\to\infty} \|u_n - u_m\|_{L^2(0,T;H)} = 0$. Hence there exists a function $u \in L^2(0,T;H)$ such that

$$u_n \to u \quad \text{in } L^2(0,T;H) \text{ as } n \to \infty.$$

From (7.5), (7.6) and the reflexivity of $L^2(0,T;H)$, there exist functions $\zeta_1, \zeta_2 \in L^2(0,T;H)$ and a subsequence (not relabeled) such that

$$\begin{aligned}\mathcal{B}(u_n - u_0) &\rightharpoonup \zeta_1 \quad \text{weakly in } L^2(0,T;H) \quad \text{as } n\to\infty,\\ \xi_n &\rightharpoonup \zeta_2 \quad \text{weakly in } L^2(0,T;H) \quad \text{as } n\to\infty.\end{aligned}$$

Passing to the limit in (7.1), we obtain

$$\zeta_1(t) + \zeta_2(t) = f(t) \quad \text{in } H \quad \text{for a.e. } t \in (0,T).$$

Repeating the same argument as in Section 6.3, we conclude that u is a strong solution to (P). Hence the proof is complete under the assumption (7.2).

Finally, we treat the general case. From (3.1), there exists a constant $D_0 \in [0,\infty)$ such that

$$\varphi^t(z) + \frac{D_0}{2}\|z\|_H^2 + D_0 \ge |\varphi^t(z)| + 1 \ge 0$$

for all $t \in [0,T]$ and all $z \in H$. For each $t \in [0,T]$, we define a functional $\psi^t\colon H \to (-\infty,\infty]$ by

$$\psi^t(z) := \varphi^t(z) + \frac{D_0}{2}\|z\|_H^2 + D_0 \quad \text{for } z \in H.$$

Then ψ^t is the proper lower-semicontinuous convex functional, and the condition (7.2) holds when φ^t is replaced by ψ^t. Furthermore, the family $\{\psi^t\}_{t\in[0,T]}$ satisfies assumptions (A1) and (A2). Thus, for every $g \in L^2(0,T;H)$, there exist unique functions $v, \zeta \in L^2(0,T;H)$ such that $v - u_0 \in D(\mathcal{B})$ (see Section 3.3) and

$$\zeta(t) \in \partial\psi^t\big(v(t)\big), \qquad \mathcal{B}(v-u_0)(t) + \zeta(t) = g(t) \tag{PP}$$

for a.e. $t \in (0,T)$. We write $(v,\zeta) \in (\mathrm{PP})_{u_0,\,g}$ if $v,\zeta \in L^2(0,T;H)$ satisfy the above conditions. We next define $\Lambda\colon L^2(0,T;H) \to L^2(0,T;H)$ as follows. For each $h \in L^2(0,T;H)$, denote by (v,ζ) the unique pair in $L^2(0,T;H) \times L^2(0,T;H)$ such that $(v,\zeta) \in (\mathrm{PP})_{u_0,\,f+D_0 h}$. We then set $\Lambda(h) := v$. Here, a straightforward computation shows that the mapping $w \mapsto \frac{D_0}{2}\,\|w\|_H^2 + D_0$ is convex and Fréchet differentiable in H, and its Fréchet derivative is given by $D_0\, I_H$, where $I_H\colon H \to H$ denotes the identity mapping on H. Thus, for each $t \in [0,T]$, we have $D(\partial\varphi^t) = D(\partial\psi^t)$ and

$$\partial\psi^t(w) = \partial\varphi^t(w) + \{D_0\, w\}$$

for all $w \in D(\partial\varphi^t) = D(\partial\psi^t)$ (see, e.g., [5, 16]). In order to complete the proof, it suffices to show that there exists a unique function $v_* \in L^2(0,T;H)$ such that $\Lambda(v_*) = v_*$. Let $\beta \in (0,\infty)$ be a constant which will be determined later, and set $\mathfrak{X}_\beta := L^2(0,T;H)$ equipped with a norm given by

$$\|w\|_{\mathfrak{X}_\beta} := \operatorname*{ess\,sup}_{t\in(0,T)} \left| \mathrm{e}^{-\beta t} \int_0^t \|w(s)\|_H^2 \,\mathrm{d}s \right| = \left\| \mathrm{e}^{-\beta \bullet} \left[1 * \|w\|_H^2 \right](\bullet) \right\|_{L^\infty(0,T)}$$

for $w \in \mathfrak{X}_\beta$. Then $(\mathfrak{X}_\beta, \|\bullet\|_{\mathfrak{X}_\beta})$ is a Banach space. Let $\kappa_0 \in (0,1)$ be fixed. Then we can choose $\beta \in (0,\infty)$ sufficiently large such that

$$\|\Lambda(w_1) - \Lambda(w_2)\|_{\mathfrak{X}_\beta} \le \kappa_0\, \|w_1 - w_2\|_{\mathfrak{X}_\beta}$$

for all $w_1, w_2 \in \mathfrak{X}_\beta = L^2(0,T;H)$. In particular, Λ is a contraction mapping on the Banach space $(\mathfrak{X}_{\beta_0}, \|\bullet\|_{\mathfrak{X}_{\beta_0}})$. By virtue of Banach's fixed point theorem, there exists a unique function $v_* \in \mathfrak{X}_{\beta_0} = L^2(0,T;H)$ satisfying $\Lambda(v_*) = v_*$. This completes the outline of proof.

8. Application

In this section, we apply the abstract results obtained so far to the Cauchy–Dirichlet problem for certain nonlinear parabolic equations on moving domains.

We denote by $d \in \mathbb{N}$, by $x = (x_1,\dots,x_d)$ a generic point of $\mathbb{R}^d$, by $\gamma = (\gamma_1,\dots,\gamma_d)$ a multi-index and by D_x^γ the differentiation,

$$\frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1}\cdots\partial x_d^{\gamma_d}},$$

where $|\gamma| = \sum_{j=1}^d \gamma_j$. Let $U \subset \mathbb{R}^d$ be a bounded domain with smooth boundary ∂U, and for each $t \in [0,T]$, let $\Omega_t \subset U$ be a bounded domain of $\mathbb{R}^d$ with smooth boundary $\partial\Omega_t$. We impose the following assumption:

- (B) It holds that $\overline{\Omega_t} \subset U$ for all $t \in [0,T]$. Furthermore, there exists a diffeomorphism $\Theta_t = (\theta_1^t,\dots,\theta_d^t)$ of class C^1 from $\overline{U}$ onto itself with $\Theta_t(\Omega_0) = \Omega_t$ for every $t \in [0,T]$ such that Θ_0 is the identity on $\overline{U}$ and $D_x^\gamma \theta_i^t$ is continuously differentiable in t on $[0,T] \times \overline{U}$ for every multi-index γ with $|\gamma| \le 1$ and $i = 1,\dots,d$.

Let $Q \subset [0,T] \times U$ and $\partial Q \subset [0,T] \times \overline{U}$ be defined by

$$Q := \bigcup_{t\in[0,T]} (\{t\} \times \Omega_t), \qquad \partial Q := \bigcup_{t\in[0,T]} (\{t\} \times \partial\Omega_t).$$

We now consider the following Cauchy–Dirichlet problem:

$$\left.\begin{aligned} \partial_t^\alpha (u - u_0) - \Delta_p u = f \quad &\text{in } Q, \\ u = 0 \quad &\text{on } \partial Q, \end{aligned}\right\} \tag{CDP}$$

where $\alpha \in (0,1)$, $p \in [2,\infty)$, and $u_0 = u_0(x)$ and $f = f(t,x)$ are prescribed. Moreover, $\partial_t^\alpha (u - u_0) := \partial_t \left[k_{1-\alpha} * (u - u_0) \right]$ denotes the α-th order Riemann–Liouville derivative of $u - u_0$, and Δ_p is the so-called *p-Laplacian* given as $\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$. For each $t \in [0,T]$, let $K(t) \subset W^{1,p}(U)$ be defined by

$$K(t) := \{w \in W^{1,p}(U) \colon w|_{\Omega_t} \in W_0^{1,p}(\Omega_t),\ w(x) = 0 \text{ for a.e. } x \in U \setminus \Omega_t\}.$$

We are concerned with the definition of L^2-solutions to (CDP).

DEFINITION 8.1 (L^2-solutions to (CDP)). *Let $u_0 \in L^2(U)$, and let $f \in L^2(0,T;L^2(U))$. A function $u \in L^2\big(0,T;L^2(U)\big)$ is called an* L^2-solution *to* (CDP) *if the following conditions are all satisfied*:

(i) *It holds that $k_{1-\alpha} * (u-u_0) \in W^{1,2}\big(0,T;L^2(U)\big)$, $[k_{1-\alpha} * (u-u_0)](0) = 0$ and $u(t,\bullet) \in K(t)$ for a.e. $t \in (0,T)$.*

(ii) *There exists a null set $N \subset (0,T)$ such that for all $t \in (0,T)\setminus N$, the following properties are satisfied: $\tilde{u}(t,\bullet) := u(t,\bullet)|_{\Omega_t} \in W_0^{1,p}(\Omega_t)$ and $\Delta_p \tilde{u}(t,\bullet) \in L^2(\Omega_t)$, and*

$$\partial_t\big[k_{1-\alpha} * (u-u_0)\big](t,\bullet) - \Delta_p \tilde{u}(t,\bullet) = f(t,\bullet) \quad \text{in } L^2(\Omega_t).$$

Let $\{\Omega_t\}_{t\in[0,T]}$ be a family of bounded domains such that (B) holds, and set $H := L^2(U)$. For each $t \in [0,T]$, we define $\varphi_p^t \colon L^2(U) \to [0,\infty]$ by

$$\varphi_p^t(w) := \begin{cases} \dfrac{1}{p}\displaystyle\int_{\Omega_t} |\nabla w|^p \,\mathrm{d}x & \text{if } w \in K(t), \\ \infty & \text{otherwise} \end{cases}$$

for $w \in L^2(U)$. Then for each $t \in [0,T]$, φ_p^t is proper, lower-semicontinuous, and convex in $L^2(U)$, and moreover, $\partial\varphi_p^t(w)$ coincides with $-\Delta_p w$ equipped with the homogeneous Dirichlet boundary condition in the distributional sense for $w \in D(\partial\varphi_p^t)$, where

$$D(\partial\varphi_p^t) = \{w \in L^2(U) \cap K(t) \colon w|_{\Omega_t} \in W_0^{1,p}(\Omega_t) \text{ and } \Delta_p(w|_{\Omega_t}) \in L^2(\Omega_t)\}.$$

Hence the Cauchy–Dirichlet problem (CDP) is reduced to the following abstract Cauchy problem:

$$\partial_t\big[k_{1-\alpha} * (u-u_0)\big](t) + \partial\varphi_p^t\big(u(t)\big) = f(t) \quad \text{in } L^2(U) \quad \text{for } t \in (0,T).$$

Moreover, the assumption (B) guarantees that both (A1) and (A2) are satisfied. Indeed, from the assumption (B), we can define a continuous mapping $\Psi_p \colon [0,T] \times [0,T] \times L^2(U) \to L^2(U)$ by $\Psi_p(t,s,w(\bullet)) := w\big(\Theta(s,\Theta^{-1}(t,\bullet))\big)$ for $(t,s,w) \in [0,T] \times [0,T] \times L^2(U)$. Then, there exists a constant $C \in [0,\infty)$ such that

$$\|\Psi_p(t,s,w) - w\|_{L^2(U)} \le C\,|t-s|\big(1 + |\varphi_p^s(w)|\big)^{1/2},$$
$$\varphi_p^t\big(\Psi_p(t,s,w)\big) \le \varphi_p^s(w) + C\,|t-s|\big(1 + |\varphi_p^s(w)|\big)$$

for all $(t,s,w) \in [0,T] \times [0,T] \times L^2(U)$ (see [11]). Hence $(A\varphi^t)_{NL}$ holds. From Remark 2.4, these estimates enable us to verify that conditions (A1) and (A2) are fulfilled. Thus we can apply Theorems 2.3 and 2.5 to conclude the existence of L^2-solutions to (CDP).

THEOREM 8.2 (Existence of L^2-solutions to (CDP)). *Let $\{\Omega_t\}_{t\in[0,T]}$ be a family of bounded domains of $\mathbb{R}^d$ such that* (B) *holds. Then, for every $u_0 \in L^2(U)$ satisfying $u_0|_{\Omega_0} \in W_0^{1,p}(\Omega_0)$ and $f \in L^2\big(0,T;L^2(U)\big)$, the Cauchy–Dirichlet problem* (CDP) *admits a unique L^2-solution. In addition, if $f \in W^{1,2}(0,T;L^2(U))$, then the unique solution u satisfies the following properties*:

$$k_\alpha * \big\|\partial_t\big[k_{1-\alpha} * (u-u_0)\big]\big\|_{L^2(U)}^2 \in L^\infty(0,T),$$
$$\operatorname*{ess\,sup}_{t\in(0,T)} \|u(t,\bullet)\|_{W_0^{1,p}(\Omega_t)} < \infty.$$

Acknowledgements

This work forms part of the author's Ph.D. thesis. The author would like to express his sincere gratitude to Professor Goro Akagi for many stimulating discussions and valuable comments on earlier versions of this work. This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located at Kyoto University. The author is also grateful to the Erwin Schrödinger International Institute for Mathematics and Physics for its hospitality during the thematic program "Free Boundary Problems", where part of this work was presented.

Acknowledgements

This work is a part of the author's Ph.D. thesis. The author would like to express sincere gratitude to his supervisor, Professor Goro Akagi for numerous discussions, which provided valuable insights. Furthermore, the author is deeply grateful for the careful reading of earlier drafts and for many valuable comments and suggestions. This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. The author is also grateful for the hospitality of the Erwin Schrödinger International Institute for Mathematics and Physics, where the author had an occasion to present this work during the thematic program "Free Boundary Problems".

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