UNIQUENESS BY NOISE FOR STOCHASTIC EVOLUTION EQUATIONS

LUCA SCARPA

Abstract. We give an overview of some recent results on uniqueness by noise for stochastic evolution equations in Hilbert spaces. This is a resumé of selected achievements of the papers [1, 6] and presents the contents of the talk by the author at the RIMS symposium "Evolution Equations and Related Topics – Abstract Structures and Versatilities" (6th– 8th October 2025, Kyoto, Japan). This note is mainly focused on the motivation and the intuitive ideas behind the results, without going into the technical details of the proofs, for which we refer to the above-mentioned contributions.

1. Introduction

Uniqueness of solutions is a delicate issue arising in the study of evolution equations and has important consequences both from a mathematical and application perspective. Non-uniqueness is known even in the most simple scenarios of ordinary differential equations, such as the classical Peano example

$$\frac{\mathrm{d}}{\mathrm{d}t}X = |X|^{\frac{1}{2}}, \qquad X(0) = 0.$$

It is well-known that the equation has infinitely-many solutions, all in the form

$$X(t) = \frac{1}{4}(t - t_0)^2 \mathbb{1}_{(t_0,+\infty)}(t),$$

for some fixed $t_0 \in [0, +\infty]$, with $t_0 = +\infty$ corresponding to the constant solution $X \equiv 0$. Nevertheless, if one considers the same equation with an additional stochastic forcing, i.e.

$$\mathrm{d}X = |X|^{\frac{1}{2}}\,\mathrm{d}t + \mathrm{d}W, \qquad X(0) = 0,$$

where W is a real Brownian motion, then pathwise uniqueness holds [24, 22]. Such behaviour is usually referred to as uniqueness-by-noise phenomenon, and denotes the possibility to restore uniqueness by adding a stochastic forcing term to an ill-posed deterministic equation.

Let us try to convey an informal idea of why one can expect to recover uniqueness by adding noise. The main reason for the lack of uniqueness in the Peano example is the fact that the initial datum 0 is a critical point for the evolution, in the sense that the drift fails to be Lipschitz in any neighbourhood of 0. Generally speaking, by adding a stochastic force to the equation, one is essentially injecting energy into the system: as a result, the solution to the corresponding stochastic equation cannot stay at the critical point 0 for too long. More precisely, it can be shown that it instantly detaches from 0 and enters the Lipschitz regime. From these heuristic considerations, it is clear that in order to achieve uniqueness

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the roughness of the noise should be strong enough, and that in general the rougher the noise is, the easier it is to obtain uniqueness.

Moving to partial differential equations, non-uniqueness is more delicate, as it arises in numerous contexts and can be related to different pathological reasons: it is the case, e.g., of reaction-diffusion equations with Hölder-continuous forcings, fluid-dynamical models, and doubly nonlinear evolution equations. More generally, typical PDEs with no unique solution are the ones described by evolution equations in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}X + AX = B(X),$$

where A is linear maximal monotone operator on a Hilbert space and B is a nonlinear operator on H, possibly unbounded, which is only Hölder-continuous with respect to some metrics.

In this note, we focus on the stochastic counterparts of such dynamics, i.e. on evolution equations in the form

$$\mathrm{d}X + AX\,\mathrm{d}t = B(X)\,\mathrm{d}t + A^{-\delta}\,\mathrm{d}W\,, \qquad X(0) = x\,, \tag{1}$$

where A is a positive self-adjoint linear operator on a separable Hilbert space H with domain $D(A)$, W is a cylindrical Wiener process, and $x \in H$ is given. The nonlinear operator B is defined only on a subspace of H and may take values in dual spaces, i.e. $B : D(A^\alpha) \to D(A^{-\beta})$ for some $\alpha \in [0, 1)$ and $\beta \in \left[0, \frac{1}{2} - \delta\right]$. Here, the case $\beta = \frac{1}{2} - \delta$ will be referred to as critical. Moreover, B is only required to be locally θ-Hölder-continuous for some $\theta \in [0, 1)$, so that the corresponding deterministic equation in general lacks uniqueness. The admissible range of δ is $\left(-\frac{1}{2} + \alpha, \frac{1}{2}\right]$, meaning that the noise may be either coloured ($\delta > 0$), cylindrical ($\delta = 0$), or rougher-than-cylindrical ($\delta < 0$).

The study of regularisation effects by addition of noise in infinite dimensions started with the pioneering contribution [14]. More recently, uniqueness in distribution has been dealt with in [4, 5, 8, 23] with Hölder drift in the case $\alpha = \beta = 0$, in [16, 17, 18] locally Hölder drift in the setting $\alpha = \delta = 0$ and $\beta = \frac{1}{2}$, and in [6] with no Hölder assumption on the drift the sub-critical case $\beta < \frac{1}{2}$. Pathwise uniqueness was considered in [2, 3, 7, 9, 10] for Hölder drift in the setting $\alpha = \beta = 0$, in [11, 12] for measurable drift, and in [1] for $\alpha, \beta > 0$.

Let us give a further intuitive explanation of the regularisation effect provided by the noise in the infinite-dimensional setting. Still arguing formally, given a sufficiently smooth function u defined on the space H of initial data and a sufficiently regular solution X of (1), the Itô formula yields, for every $t \geq 0$,

$$\mathbb{E}[u(X(t))] + \mathbb{E}\int_0^t \langle AX(s) - B(X(s)), Du(X(s))\rangle\,\mathrm{d}s = u(x) + \frac{1}{2}\mathbb{E}\int_0^t \mathrm{Tr}[A^{-2\delta}D^2u(X(s))]\,\mathrm{d}s.$$

The second-order term on the right-hand side represents the energy proliferation conveyed by the noise, and confirms the intuition that the stochastic forcing injects energy into the system. This can be challenging in proving existence of solutions to (1) via classical energy methods, since the energy proliferation should be controlled. Nonetheless, the second-order term has some advantages too. Indeed, the Itô formula provides information on how u is transformed throughout the evolution along the trajectories of X, or, in other words, on the action of the transition semigrouop on u. In the purely deterministic case the transition semigrouop simply acts by transporting u along trajectories, with infinitesimal generator given by the transport operator

$$(Lu)(x) = \langle Ax - B(x), Du(x)\rangle, \quad x \in H,$$

whereas in the stochastic case the generator of the transition semigroup acts instead as the second order operator

$$(Lu)(x) = -\frac{1}{2}\operatorname{Tr}[A^{-2\delta}D^2u(x)] + \langle Ax - B(x), Du(x)\rangle, \quad x \in H.$$

Consequently, while for the deterministic evolution no regularisation effects take place along the flow, in the stochastic case the transition semigrouop may provide regularisation along trajectories. The addition of noise in the stochastic evolution equation can be then interpreted as an elliptic regularisation term in the so-called Kolmogorov operator L. This confirms again the main intuition behind the results, i.e. that the lower δ is, the rougher the noise is, and the easier it is to achieve uniqueness.

The main theorems highlighted in this note are selected results taken from [1, 6] and concern uniqueness in distribution and pathwise uniqueness for (1). The former is achieved under wide generality, possibly including the case $\theta = 0$ in the sub-critical case $\beta + \delta < \frac{1}{2}$, while the latter requires a specific tuning of the parameters in the form $(1-\theta)(\delta-\alpha) < \frac{\theta}{2}$. The constraint is extremely natural, since it becomes less restrictive as θ approaches 1, i.e. when the operator B approaches the Lipschitz regime, for which uniqueness is known to hold also without the noise.

Important novel contributions to the uniqueness of selected SPDEs are obtained for differential perturbation of the heat equation, the Burgers and the Navier-Stokes equations, perturbed Cahn-Hilliard equations, and reaction-diffusion equations. We refer to [1, 6] for a detailed presentation of the main applications and examples.

The note is organised as follows. In Section 2 we specify and comment the main assumptions, state some selected uniqueness results obtained in [1, 6], and give an intuitive idea of the strategy of the proof, without going into the technical details. In Section 3 we discuss some further extensions, work in progress, future developments, and open problems.

2. Some selected achievements

Let H be a separable Hilbert space, with scalar product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. Let also A be a linear self-adjoint operator on H with effective domain $D(A)$ densely embedded in H. We assume that there exists a complete orthonormal system $(e_k)_{k\in\mathbb{N}}$ of H, made of eigenvectors of A, with respective eigenvalues given by the increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$, such that $\lambda_k > 0$ for all $k \in \mathbb{N}$ and $\lambda_k \nearrow +\infty$ as $k \to \infty$.

This implies that the operator $-A$ generates a strongly continuous analytic semigroup $(e^{-tA})_{t\geq 0}$ of contractions on H. Furthermore, for every $s \in (0,1)$ we define the fractional powers A^s according to the classical spectral theory, as well as the corresponding fractional domains $D(A^s)$. The inverse operator A^{-s} is defined accordingly, and the negative domain $D(A^{-s})$ is set as the dual space of $D(A^s)$. We use the same symbol $\langle\cdot,\cdot\rangle$ to denote the duality pairing between $D(A^{-s})$ and $D(A^s)$, for any $s \in (0,1)$ for which it makes sense.

For every filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, a H-cylindrical Wiener process is formally defined as

$$W = \sum_{k\in\mathbb{N}} W_k e_k,$$

where $(W_k)_{k\in\mathbb{N}}$ is a sequence of real independent Brownian motions. We recall that such definition is formal, since the series above only converges in any space $\tilde{H}$ such that the inclusion $H \subset \tilde{H}$ is Hilbert-Schmidt. We also recall that for every $T > 0$ and for every progressively measurable process $G \in L^2(\Omega; L^2(0,T; \mathscr{L}^2(H,H)))$, where $\mathscr{L}^2(H,H)$ denotes the space of Hilbert-Schmidt operators on H, the stochastic integral $\int_0^{\cdot} G(s)\,\mathrm{d}W(s)$ is a well-defined adapted process in $L^2(\Omega; C^0([0,T]; H))$.

The parameters $\alpha, \beta, \delta, \theta$ are fixed and satisfy

$$\alpha \in [0,1), \qquad \delta \in \left(-\frac{1}{2} + \alpha, \frac{1}{2}\right], \qquad \beta \in \left[0, \frac{1}{2} - \delta\right], \qquad \theta \in [0,1).$$

We assume the following hypotheses:

(I) there exists $\eta \in (0,1)$ such that $A^{-(1+2\delta)+2\alpha+\eta}$ is a trace-class operator on H;

(II) the operator $B : D(A^\alpha) \to D(A^{-\beta})$ is locally bounded and locally-θ-Hölder-continuous, with $\theta > 0$ if $\beta + \delta = \frac{1}{2}$.

Assumption (I) ensures that the so-called stochastic convolution

$$t \mapsto \int_0^t e^{-(t-s)A} A^{-\delta}\,\mathrm{d}W(s)$$

is a well defined adapted process with trajectories in $C^0([0,+\infty); D(A^\alpha))$: see e.g. [13, Ch. 4]. Assumption (II) allows for the case $\theta = 0$ (i.e. B continuous) only in the sub-critical case $\beta + \delta < \frac{1}{2}$. In the critical case $\beta + \delta = \frac{1}{2}$, B has to be at least locally Hölder (i.e. $\theta > 0$).

Let us specify now the definition of solution for (1).

Definition 2.1. *A weak mild solution to* (1) *is a family* $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}, W, X)$ *where* $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ *is a filtered probability space satisfying the usual conditions,* W *is a* H*-cylindrical Wiener process on* $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$*,* X *is a* H*-valued progressively measurable process with*

$$X \in C^0([0,+\infty); H) \cap C^0((0,+\infty); D(A^\alpha)) \quad \mathbb{P}\text{-a.s.} \qquad \text{if } x \in H,$$
$$X \in C^0([0,+\infty); D(A^\alpha)) \quad \mathbb{P}\text{-a.s.} \qquad \text{if } x \in D(A^\alpha),$$

such that for every $t > 0$ *it holds* $s \mapsto e^{-(t-s)A} B(X(s)) \in L^1(0,t;H)$*,* $\mathbb{P}$*-a.s., and*

$$X(t) = e^{-tA}x + \int_0^t e^{-(t-s)A} B(X(s))\,\mathrm{d}s + \int_0^t e^{-(t-s)A} A^{-\delta}\,\mathrm{d}W(s) \tag{2}$$

for every $t \geq 0$*,* $\mathbb{P}$*-almost surely.*

We note that the terms appearing in the mild formulation (2) are well-defined in our setting. Indeed, we have already pointed out that the stochastic convolution is a continuous process with values in $D(A^\alpha)$. Moreover, by the properties of semigroups the function $t \mapsto e^{-tA}x$ is either in $C^0([0,+\infty); H) \cap C^0((0,+\infty); D(A^\alpha))$ if $x \in H$ or in $C^0([0,+\infty); D(A^\alpha))$ if $x \in D(A^\alpha)$. Lastly, the deterministic convolution term is always continuous process in with values in H by the integrability requirement on in Definition 2.1, and if also $x \in D(A^\alpha)$ then it is also continuous in $D(A^\alpha)$ by deterministic maximal regularity since $\alpha + \beta < 1$.

For equation (1) we focus on two classical types of uniqueness: weak uniqueness and pathwise uniqueness.

Definition 2.2. *Weak uniqueness, or uniqueness in distribution, holds for* (1) *with initial data in $U \subset H$ if for every $x \in U$ and for every weak mild solutions X_1 and X_2 of* (1) *with $X_1(0) = X_2(0) = x$, it holds that X_1 and X_2 have the same probability distribution on $C^0([0,+\infty); H)$, i.e. for every measurable bounded $\Phi : C^0([0,+\infty); H) \to \mathbb{R}$ it holds that*

$$\mathbb{E}\left[\Phi(X_1)\right] = \mathbb{E}\left[\Phi(X_2)\right].$$

Definition 2.3. *Pathwise uniqueness, or strong uniqueness, holds for* (1) *with initial data in $U \subset H$ if for every $x \in U$ and for every weak mild solutions X_1 and X_2 of* (1) *with $X_1(0) = X_2(0) = x$, with the same H-cylindrical process W and defined on the same probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, it holds that*

$$\mathbb{P}\left(X_1(t) = X_2(t) \ \ \forall\, t \geq 0\right) = 1.$$

The main focus of this note is uniqueness: we summarise here some selected results obtained in the papers [1, 6]. For what concerns existence of weak mild solutions to (1), this is more classical and can be shown via standard techniques according to the specific form of the operator B. A general result on existence can be found e.g. in [6, Thm. A1].

Theorem 2.4. *Under (I)–(II), weak uniqueness holds for* (1) *with initial data in $D(A^\alpha)$, and also with initial data in H if B is bounded.*

For pathwise uniqueness, we need some further assumptions, namely:

(III) it holds that $\theta > 0$, and in the critical case $\beta + \delta = \frac{1}{2}$ there exists $z_0 \in D(A^\alpha)$ such that $\sup_{x\in D(A^\alpha)} \|B(x) - z_0\|_{D(A^{-\beta})} < K$, where K depends only on $\alpha, \beta, \theta, \delta$;

(IV) there exists $\varepsilon \in (0, 1)$ such that $A^{-(1+\theta)+2\beta+2(1-\theta)\delta+2\theta\alpha+2\varepsilon}$ is a trace-class operator on H.

Note that hypothesis (III) implies that B is bounded in the critical case $\beta+\delta = \frac{1}{2}$ and requires a smallness condition on the range of B: its precise value is specified in [1]. Hypothesis (IV) can be interpreted as a summability property on the eigenvalues of A, and can be shown to be satisfied in the typical case of the negative Laplacian: see [1].

Theorem 2.5 (Pathwise uniqueness)**.** *Under (I)–(II)–(III)–(IV), let $(1 - \theta)(\delta - \alpha) < \frac{\theta}{2}$. Then, pathwise uniqueness holds for* (1) *with initial data in $D(A^\alpha)$, and also with initial data in H if B is bounded and $\theta\alpha + (1 - \theta)\delta < \frac{\theta}{2}$.*

2.1. **Formal ideas of the proof.** Let us sketch here a formal idea of the proof of Theorems 2.4–2.5. We argue formally, as the rigorous proof is rather technical, and we refer the reader to [1, 6] for precise details.

The main idea is to consider the elliptic Kolmogorov equation associated to the stochastic equation (1), namely

$$\lambda u(x) - \frac{1}{2}\,\mathrm{Tr}[A^{-2\delta}D^2u(x)] + \langle Ax - B(x), Du(x)\rangle = f(x), \quad x \in H,$$

where $\lambda > 0$ is fixed and $f \in C_b^0(H)$ is a given forcing term. Note that such elliptic equation is infinite dimensional, in the sense that the space variable x lives in the infinite-dimensional space H. Let us suppose for a moment that one is able to obtain a sufficiently regular solution u to the Kolmogorov equation, e.g. such that $u \in C_b^2(H)$ and with $Du \in C_b^0(H; D(A^\beta))$. Then, if X is a sufficiently regular solution to (1), e.g. if X is analytically strong, an application of Itô formula for $u(X)$ yields, for every $t \geq 0$,

$$\mathbb{E}[u(X(t))] + \mathbb{E}\int_0^t \langle AX(s), Du(X(s))\rangle \,\mathrm{d}s \\ = u(x) + \mathbb{E}\int_0^t \langle B(X(s)), Du(X(s))\rangle \,\mathrm{d}s + \frac{1}{2}\int_0^t \mathrm{Tr}[A^{-2\delta} D^2 u(X(s))]\,\mathrm{d}s.$$

Since u solves the Kolmogorov equation, some of the terms in Itô formula cancel out and one obtains

$$\mathbb{E}[u(X(t))] - \lambda \int_0^t \mathbb{E}[u(X(s))]\,\mathrm{d}s + \int_0^t \mathbb{E}[f(X(s))]\,\mathrm{d}s = 0 \quad \forall\, t \geq 0,$$

from which it follows the identity

$$u(x) = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}[f(X(s))]\,\mathrm{d}s \quad \forall\, f \in C_b^0(H).$$

Due to the arbitrariness of the forcing term f, this implies uniqueness of the probability distribution of X. Indeed, if Y is another solution of (1) starting from x, by the same argument it follows that

$$\int_0^{+\infty} e^{-\lambda t}\mathbb{E}[f(X(s))]\,\mathrm{d}s = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}[f(Y(s))]\,\mathrm{d}s \quad \forall\, f \in C_b^0(H). \tag{3}$$

By the properties of the Laplace transform this readily implies that X and Y have the same one-dimensional marginal distributions, hence also the same distribution on $C^0([0, +\infty); H)$ by continuity of the trajectories.

This argument of course is only formal, firstly because in general the solution u to the Kolmogorov equation may not be C^2, and secondly because the solution X is not necessarily analytically strong. In order to make the argument rigorous, one needs to introduce suitable approximations, one on the Kolmogorov equation

$$\lambda u_n(x) - \frac{1}{2}\mathrm{Tr}[A^{-2\delta} D^2 u_n(x)] + \langle A_n x - B_n(x), Du_n(x)\rangle = f(x), \quad x \in H_n,$$

where H_n is finite dimensional and B_n approximates B, and one on the stochastic evolution equation

$$\mathrm{d}X_n + A_n X_n \,\mathrm{d}t = B_n(X)\,\mathrm{d}t + A_n^{-2\delta}\,\mathrm{d}W, \qquad X_n(0) = x_n.$$

By suitably choosing the approximated operators A_n and B_n, it is clear that one can recover enough regularity on u_n when n is fixed and perform the computations above. In particular, Itô formula for $u_n(X_n)$ yields

$$\mathbb{E}[u_n(X_n(t))] + \mathbb{E}\int_0^t \langle AX_n(s), Du_n(X_n(s))\rangle \,\mathrm{d}s$$

$$= u_n(x_n) + \mathbb{E}\int_0^t \langle B(X(s)), Du_n(X_n(s))\rangle \,\mathrm{d}s + \frac{1}{2}\int_0^t \mathrm{Tr}[A^{-2\delta}D^2u_n(X_n(s))]\,\mathrm{d}s,$$

and by the approximated Kolmogorov equation we get

$$\mathbb{E}[u_n(X_n(t))] - \lambda\int_0^t \mathbb{E}[u_n(X_n(s))]\,\mathrm{d}s + \int_0^t \mathbb{E}[f(X_n(s))]\,\mathrm{d}s$$
$$= \mathbb{E}\int_0^t \langle B_n(X(s)) - B_n(X_n(s)), Du_n(X_n(s))\rangle\,\mathrm{d}s.$$

Since $X_n \to X$ in natural topologies, one can pass to the limit as $n \to \infty$ in the equality above by dominated convergence theorem provided to have a uniform bound on the first-order terms, namely $(u_n)_n$ in $C^1_b(H_n)$ and $(Du_n)_n$ in $C^0_b(H_n; D(A^\beta))$. Passing to the limit as $n \to \infty$ yields (3), and one can conclude in a rigorous way. Let us point out that the C^2-regularity on u_n is needed only to apply Itô formula, but not to pass to the limit in n, since the second-order operator is canceled out thanks to the fact that u_n solves the Kolmogorov equation.

Let us stress that the possibility of solving the Kolmogorov equation entirely relies on the presence of the noise, which provides a second-order elliptic regularisation on the Kolmogorov equation and allows to rely on Schauder-type estimates. The analogous argument would not be doable in the deterministic case. The requirement on the parameters $\beta + \delta \in [0, \frac{1}{2}]$ are needed in order to solve the Kolmogorov equation, since it ensures that the transition semigroup associated to (1) has the right regularising properties. In this interpretation, the case $\beta + \delta = \frac{1}{2}$ is critical in the sense that the corresponding elliptic estimates on the Kolmogorov equation turn out to be sharp.

The proof of pathwise uniqueness is much more delicate. The idea is similar, i.e. to exploit the Kolmogorov equation, but it is performed in a more subtle way. By using the same finite dimensional approximation as before, for every $k \in \{1, \dots, n\}$ one can consider the Kolmogorov equation

$$c\lambda_k u_{n,k}(x) - \frac{1}{2}\,\mathrm{Tr}\left[A_n^{-2\delta}D^2u_{n,k}(x)\right] + \langle A_n x - B_n(x), Du_{n,k}(x)\rangle = \langle B(x), e_k\rangle,$$

where $c > 0$ is a fixed constant. Roughly speaking, this means that instead of having a general forcing term f, one uses B as forcing term. By setting $\mathfrak{u} := \sum_{k=1}^n u_{n,k}e_k : H_n \to H_n$, the Kolmogorov equation can be written in compact form as an infinite-dimensional vector-valued equation

$$cA_n\mathfrak{u}_n(x) + \mathfrak{L}\mathfrak{u}_n(x) = D\mathfrak{u}_n(x)[B_n(x)] + B_n(x), \quad x \in H_n,$$

where $\mathfrak{L}$ is the natural vector-valued extension of L. By using the Itô formula on $\mathfrak{u}_n(X_n)$ and by exploiting the Kolmogorov equation, similarly as before one obtains

$$\int_0^t B_n(X_n(s))\,\mathrm{d}s = \mathfrak{u}_n(x_n) - \mathfrak{u}_n(X_n(t)) + \int_0^t D\mathfrak{u}_n(X_n(s))[B_n(X(s)) - B_n(X_n(s))]\,\mathrm{d}s$$
$$+ c\int_0^t A_n\mathfrak{u}_n(X_n(s))\,\mathrm{d}s + \int_0^t D\mathfrak{u}_n(X_n(s))[A^{-\delta}\,\mathrm{d}W(s)].$$

Looking back at the stochastic evolution equation (1), this allows to substitute the term involving B (i.e. the one responsible for non-uniqueness) with other terms involving $\mathfrak{u}_n$. It is clear now that if $(\mathfrak{u}_n)_n$ is uniformly bounded in $C^2_b(H_n)$, then all the additional terms turn out to be uniformly Lipschitz with respect to X_n. Hence, uniqueness can be achieved also pathwise via standard arguments on the difference $X - Y$. The substitution technique is usually referred-to as Itô-Tanaka trick, and consists in replacing the nonlinear term B by means of addends which depends on the (more regular) solution to the Kolmogorov equation itself.

Let us point out that in order to obtain pathwise uniqueness, more information is needed on the solution to the Kolmogorov equation, compared to the case of weak uniqueness, since here one needs to control also the second-order terms $(D^2\mathfrak{u}_n)_n$. This calls for ad-hoc Schauder estimates on the Kolmogorov equation, which require B to be at least θ-Hölder continuous for some positive θ.

3. Some selected extensions and work-in-progress

We discuss here some extensions of the uniqueness results contained in Theorems 2.4–2.5, as well as possible developments and work-in-progress.

- Continuous dependence on the initial data. In the paper [1], more precise results are obtained for stochastic evolution equations in the form (1). In particular, continuous dependence of the solution with respect to the initial datum is also proved: the main idea of the proof is analogous to the one of pathwise uniqueness, and relies on the Itô-Tanaka trick.
- Uniqueness by noise for doubly nonlinear equations. Deterministic doubly nonlinear evolution equations are known not to have a unique solution. These are equations in the form
$$\mathfrak{A}(\partial_t X) + \mathfrak{B}(X) \ni f, \qquad X(0) = x,$$
where $\mathfrak{A}, \mathfrak{B}$ are maximal monotone nonlinear operators on a Hilbert space, and explicit counterexamples for uniqueness are available. The stochastic counterpart of such equations has been recently introduced in [19, 20, 21] and read
$$X = X^d + \int_0^\cdot G\,\mathrm{d}W(s), \qquad \mathfrak{A}(\partial_t X^d) + \mathfrak{B}(X) \ni f, \qquad X(0) = x,$$
where G is a given Hilbert-Schmidt linear operator. The main issue in the direction of studying uniqueness by noise is that the formulation of such equations in so-called normal form (1) requires $\alpha = 1$, which is critical and not covered by the results of [1, 6]. Very recently, a partial result has been obtained in [15], ensuring weak stability by noise for families of approximations of doubly nonlinear equations.
- The unbounded case in the critical regime. An interesting topic under investigation is the study of the critical regime $\beta + \delta = \frac{1}{2}$ without the smallness condition on the drift B in assumption (III): for weak uniqueness this has been done in [1, 6], while for pathwise uniqueness the study is more delicate and is currently carried out.
- The non-Hölder case in the critical regime. A standing open problem in the context of regularisation by noise is another issue appearing in the critical regime $\beta + \delta = \frac{1}{2}$ and is the question whether uniqueness continues to hold if $\theta = 0$ only, i.e. if it possible to relax assumption (II) is the critical case. Here the problem is open both for weak and pathwise uniqueness: while for the former the situation is delicate, for the latter

one may try to obtain partial results, such as pathwise uniqueness for almost every initial data with respect to a Gaussian invariant measure, in the spirit of [11].

- Regularisation via multiplicative noise. The results presented in this note cover the case where the noise is independent of the solution X. However, in explicit models one is required to consider stochastic forcing of multiplicative type, i.e. where the noise intensity depends on X itself. Regularisation properties here are extremely more delicate, as the second-order elliptic regularisation on the Kolmogorov operator depends now on a mobility function, which in principle may also be degenerate.

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(Luca Scarpa) Department of Mathematics, Politecnico di Milano, Via E. Bonardi 9, 20133 Milano, Italy.
Email address: luca.scarpa@polimi.it
URL: https://sites.google.com/view/lucascarpa