On the Hamilton-Jacobi flow starting from the Takagi function

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1. Introduction

In this paper, we are concerned with the time evolution of the Hamilton-Jacobi flow starting from the Takagi function. This paper is based on Fujita, Hamamuki and the author [**5, 6**], and Fujita and the author [**7**].

We first introduce the Hamilton-Jacobi flow. Let $BUC(\mathbb{R})$ denote the set of all bounded uniformly continuous functions on $\mathbb{R}$. For $f \in BUC(\mathbb{R})$, the Hamilton-Jacobi flow $\{H_t f\}_{t>0}$ starting from initial datum f is defined by

$$H_t f(x) := \inf_{y\in\mathbb{R}} P_f(t,x;y), \quad (t,x) \in (0,\infty)\times\mathbb{R}, \tag{1.1a}$$

where $\{P_f(t,x;y)\}_{y\in\mathbb{R}}$ is the family of parabolas as follows.

$$P_f(t,x;y) := f(y) + \frac{(x-y)^2}{2t}, \quad (t,x,y) \in (0,\infty)\times\mathbb{R}\times\mathbb{R}. \tag{1.1b}$$

The function $H_t f$ is also called the *inf-convolution* or *Moreau envelope* of f.

It is well known that $u(t,x) := H_t f(x)$ is the unique viscosity solution to the following initial value problem of the Hamilton-Jacobi equation in the class of bounded and uniformly continuous functions on $[0,T)\times\mathbb{R}$ for each $T \in (0,\infty)$ (see e.g., Evans [**4**, Chapters 3 and 10]):

$$\frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\left(\frac{\partial u}{\partial x}(t,x)\right)^2 = 0, \qquad (t,x) \in (0,T)\times\mathbb{R},$$

$$u(0,x) = f(x), \quad x \in \mathbb{R}.$$

For a general initial datum $f \in BUC(\mathbb{R})$, (1.1) implies that $u(t,x) := H_t f(x)$ converges locally uniformly in $x \in \mathbb{R}$ to $m_f := \inf_{x\in\mathbb{R}} f(x)$ as $t \to \infty$. Moreover, it is known that $H_t f(x)$ is Lipschitz continuous for every $t > 0$, and hence $H_t f(x)$ is differentiable almost everywhere for $t > 0$. However, our interest lies in the finer behavior of the solution $u(t,x) = H_t f(x)$. In particular, when the initial datum is represented as a superposition of a fundamental wave and infinitely many higher-frequency waves, we would like to understand how H_t acts on different frequency scales. We focus on the Hamilton-Jacobi flow $\{H_t \tau\}_{t>0}$ starting from *the Takagi function* τ, which is a natural prototype for such an initial datum.

The Takagi function was first introduced by T. Takagi [**12**] in 1903 as a simple example of a continuous nowhere differentiable function on [0, 1], and has since appeared in various areas of mathematics. We refer to Allaart and Kawamura [**2**], Lagarias [**10**], and Jarnicki and Pflug [**8**] for general overviews of the Takagi function.

The Takagi function is defined by

$$\tau(x) := \sum_{j=0}^{\infty} \frac{1}{2^j} d(2^j x), \quad x \in \mathbb{R},$$

where $d(x) = \min\{|x-z| \mid z \in \mathbb{Z}\}$ is the distance from x to the nearest integer. d is known as a triangle wave. Figure 1.1 illustrates the graph of τ on [0, 1].

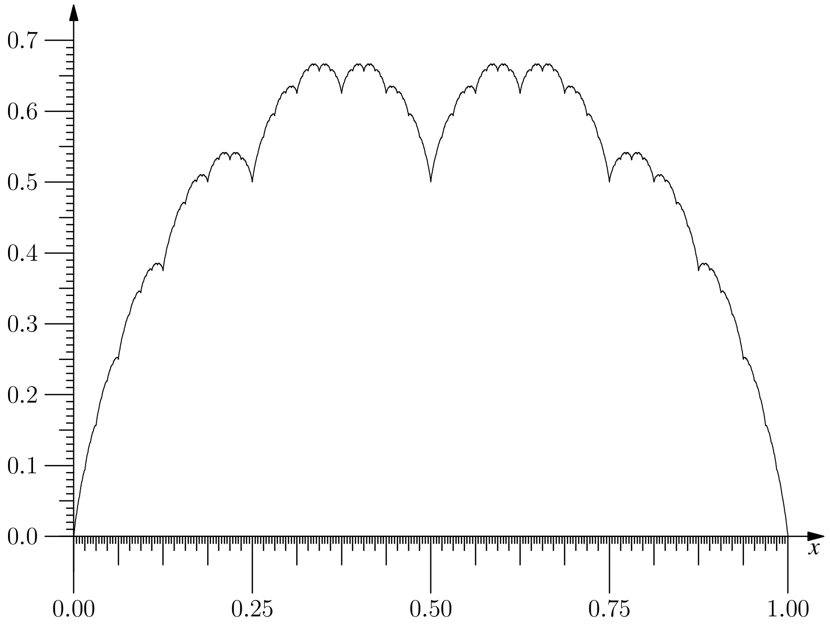


FIGURE 1.1. Graph of the Takagi function

Let $C_p(\mathbb{R})$ denote the set of all continuous, non-negative, and 1-periodic functions f on $\mathbb{R}$ satisfying $f(0) = 0$ and $f \not\equiv 0$. Since $d \in C_p(\mathbb{R})$ is the triangle wave with period 1, the function τ can be regarded as a superposition of the fundamental wave d together with infinitely many waves of increasing frequency and decreasing amplitude.

In order to state our results precisely, we shall introduce the *deficient digit function* $D_n(\cdot)$, which plays an important role in our analysis.

Definition 1.1 (Lagarias [**10**]). For $x \in [0, 1)$, let $x = \sum_{j=1}^{\infty} \frac{b_j}{2^j} = (0.b_1b_2b_3\dots)_{(2)}$ be the binary expansion of x with $b_j \in \{0, 1\}$. When $x \in (0, 1)$ has the form $x = \frac{k}{2^n}$ for some $k, n \in \mathbb{N}$, x has two different binary expansions. In this case, we use the expansion ending in zeros. For each $n \in \mathbb{N}$, we define the *deficient digit function* $D_n(x)$ by

$$D_n(x) := n - 2\sum_{j=1}^{n} b_j.$$

We call a dyadic rational $\frac{k}{2^n}$ with $D_n\big(\frac{k}{2^n}\big) = 0$ a *balanced* dyadic rational.

Remark 1.2. From Definition 1.1, we see that $D_n(x) \in \mathbb{Z}$ and $-n \le D_n(x) \le n$.

For $n \in \mathbb{N}$ and $t > 0$, we define constants $\{\theta_n^k(t)\}_{k=-1}^{2^n}$ by

$$\theta_n^k(t) := \frac{2k+1}{2^{n+1}} + tD_n\Big(\frac{k}{2^n}\Big), \quad k \in \{0, 1, \dots, 2^n - 1\} \tag{1.2}$$

and $\theta_n^{-1}(t) = 0$, $\theta_n^{2^n} = 1$. For $n \in \mathbb{N}, k \in \{0, 1, \dots, 2^n - 1\}$ and $t > 0$, $\theta_n^k(t)$ is the unique x-coordinate of the point of intersection between $P_\tau\big(t, \cdot; \frac{k}{2^n}\big)$ and $P_\tau\big(t, \cdot; \frac{k+1}{2^n}\big)$. By Lemma 3.4 below, for any $n \in \mathbb{N}$ and $t \in \big(0, \frac{1}{2^{n+1}}\big)$, we see that

$$0 = \theta_n^{-1}(t) < \theta_n^0(t) < \theta_n^1(t) < \cdots < \theta_n^{2^n-1}(t) < \theta_n^{2^n}(t) = 1. \tag{1.3}$$

The next theorem is the main result of this paper, which provides an explicit representation formula for $H_t\tau(x)$ for dyadic time intervals $[\frac{1}{2^{n+2}}, \frac{1}{2^{n+1}})$ for each $n \in \mathbb{N}$.

Theorem 1.3 (Fujita *et al.* [**6**]). *Let $n \in \mathbb{N}$ and $x \in [0,1]$. When $\frac{1}{2^{n+2}} \le t < \frac{1}{2^{n+1}}$, we have*

$$u(t,x) = P_\tau\Big(t,x;\frac{k}{2^n}\Big), \quad x \in \big[\theta_n^{k-1}(t), \theta_n^k(t)\big],\ k \in \{0,1,2,\dots,2^n\}. \tag{1.4}$$

In particular, when $t \ge \frac{1}{4}$,

$$u(t,x) = \begin{cases} P_\tau(t,x;0) = \dfrac{x^2}{2t} & \Big(0 \le x \le \dfrac{1}{2}\Big), \\ P_\tau(t,x;1) = \dfrac{(x-1)^2}{2t} & \Big(\dfrac{1}{2} \le x \le 1\Big). \end{cases}$$

Theorem 1.3 yields various properties of $H_t\tau$.

Remark 1.4. Theorem 1.3 ensures that $u(t,\cdot)$ is a piecewise quadratic function consisting of parabolas P_τ, and that the point of intersection between adjacent parabolas are given by $\{\theta_n^k(t)\}$.

This paper is organized as follows. In Section 2, we summarize fundamental properties of the Takagi function and the Hamilton-Jacobi flow $H_t f$, where $f \in C_p(\mathbb{R})$. In Section 3, we prove Theorem 1.3 and introduce some remarkable properties of $H_t\tau$. In Section 4, we provide a formal result concerning the inviscid Burgers equation associated with $H_t\tau$.

2. Preliminaries

2.1. **Fundamental properties of the Takagi function.** The Takagi function $\tau(x)$ can be obtained as a limit of piecewise linear approximations. Let $n \in \mathbb{N}$. The *partial Takagi function of level n* is given by:

$$\tau_n(x) := \sum_{j=0}^{n-1} \frac{1}{2^j} d(2^j x). \tag{2.1}$$

Lemma 2.1 (Lagarias [**10**]). *The functions $\tau_n(x)$ approximate the Takagi function monotonically from below:*

$$\tau_1(x) \le \tau_2(x) \le \cdots$$

The values $\{\tau_n(x)\}_{n\ge 1}$ converge uniformly to $\tau(x)$ with

$$|\tau(x) - \tau_n(x)| \le \frac{2}{3}\cdot\frac{1}{2^n}.$$

For any $n \in \mathbb{N}$, $\tau_n \in C_p(\mathbb{R})$, $\tau_n(0) = 0$ and $\tau_n(x) = \tau_n(1-x)$ for any $x \in \mathbb{R}$. τ_n is piecewise linear on each dyadic interval $\big[\frac{k}{2^n}, \frac{k+1}{2^n}\big]$ ($k \in \mathbb{Z}$) and τ_n has integer slope between $-n$ and n on each such interval.

Lemma 2.2. *The Takagi function τ has the following properties.*

(T1) $\tau \in C_p(\mathbb{R})$ *and τ is nowhere differentiable;*
(T2) $\tau(0) = 0$ *and $\tau(x) = \tau(1-x)$ for any $x \in \mathbb{R}$, i.e., τ on $[0,1]$ is symmetric with respect to $\frac{1}{2}$;*
(T3) $\tau(x) \ge 2x(1-x)$ *for any $x \in [0,1]$.*
(T4) (perfect approximation) *For a dyadic rational $x = \frac{k}{2^n}$, we have $\tau(x) = \tau_m(x)$ for any $m \ge n \in \mathbb{N}$.*

(T3) follows from the fact that $\tau(x) \ge \tau_2(x) = 2x(1-x)$ for any $x \in [0,1]$.

From the definition of the Takagi function, τ has a kind of self-similar structure. In particular, τ has the following self-affine property.

Lemma 2.3 (Lagarias [**10**]). *For an arbitrary dyadic rational $\frac{k}{2^n}$, there holds*

$$\tau\Big(\frac{k+y}{2^n}\Big) = \tau\Big(\frac{k}{2^n}\Big) + \frac{1}{2^n}\left(\tau(y) + D_n\Big(\frac{k}{2^n}\Big)y\right), \quad 0 \le y \le 1. \tag{2.2}$$

Remark 2.4. In particular, if $\frac{k}{2^n}$ is a balanced dyadic rational, then we have

$$\tau\Big(\frac{k+y}{2^n}\Big) = \tau\Big(\frac{k}{2^n}\Big) + \frac{1}{2^n}\tau(y), \quad 0 \le y \le 1.$$

Furthermore, taking $y = 1$ in (2.2), we have

$$D_n\Big(\frac{k}{2^n}\Big) = 2^n\left(\tau\Big(\frac{k+1}{2^n}\Big) - \tau\Big(\frac{k}{2^n}\Big)\right). \tag{2.3}$$

2.2. **Fundamental properties of the Hamilton-Jacobi flow.** We summarize the fundamental properties of $u(t,x) = H_t f(x)$ for $f \in C_p(\mathbb{R})$.

Proposition 2.5. *Let $f \in C_p(\mathbb{R})$. Then we have*

- (H1) (Periodicity and smoothing) *$H_t f \in C_p(\mathbb{R})$ for all $t \ge 0$ and it is differentiable almost everywhere with respect to x for any $t > 0$;*
- (H2) (Monotonicity with respect to time) *For $0 < s < t$, $H_t f(x) \le H_s f(x) \le f(x)$ for each $x \in \mathbb{R}$;*
- (H3) (Comparison principle) *Let $f, g \in C_p(\mathbb{R})$. If $f(x) \le g(x)$ for every $x \in \mathbb{R}$, then $H_t f(x) \le H_t g(x)$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$;*
- (H4) (Semigroup property) *For any $t, s \ge 0$, $H_{t+s} f(x) = H_t(H_s f)(x)$ and $\lim_{t\to 0^+} \|H_t f - f\|_{L^\infty} = 0$;*
- (H5) (1.1a) *is reduced to*

$$H_t f(x) = \min_{y\in[0,1)}\left(f(y) + \frac{(x-y)^2}{2t}\right), \quad (t,x) \in (0,\infty) \times \mathbb{R};$$

- (H6) (Large time behavior) $\|H_t f\|_{L^\infty} = O\big(t^{-1}\big)$ *as $t \to \infty$.*
  Note that $m_f = \min_{y\in[0,1)} f(y) = f(0) = 0$, because $f \in C_p(\mathbb{R})$.

3. Hamilton-Jacobi flow $H_t\tau$

In this section, we prove Theorem 1.3 and discuss some properties of $H_t\tau$. From Lemma 2.2 and Proposition 2.5, $u(t,x) := H_t\tau(x)$ is given by

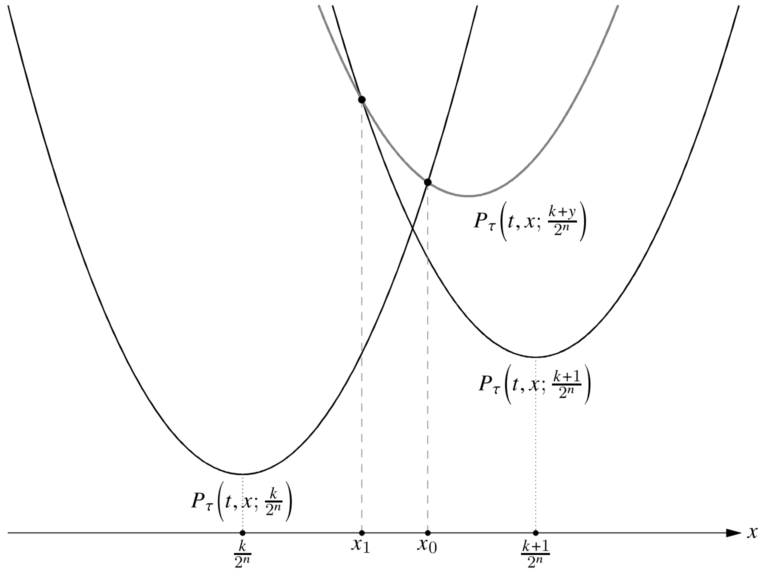
$$u(t,x) = H_t\tau(x) = \min_{y\in[0,1)}\left(\tau(y) + \frac{(x-y)^2}{2t}\right), \quad (t,x) \in (0,\infty) \times [0,1). \tag{3.1}$$

3.1. **Proof of Theorem 1.3.** This subsection is devoted to the proof of Theorem 1.3. We first introduce the following lemma.

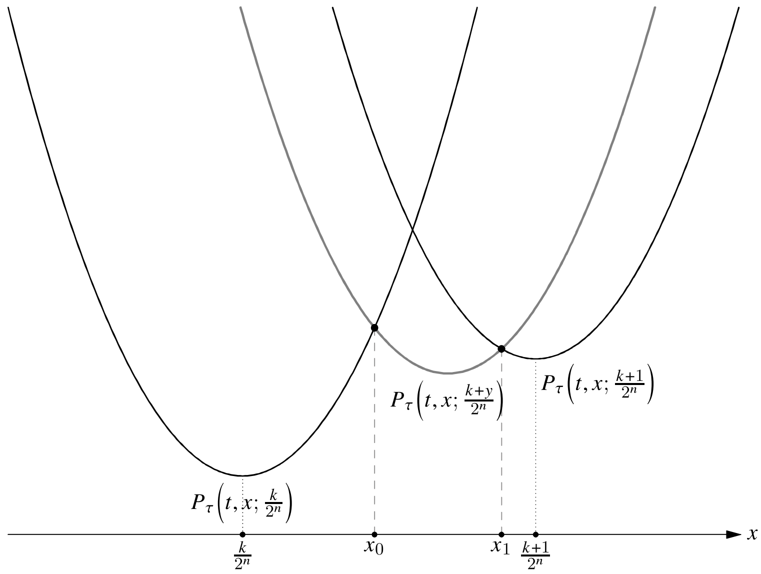
Lemma 3.1 (Fujita *et al.* [**6**]). *For any $x \in \mathbb{R}, n \in \mathbb{N}, t \ge \frac{1}{2^{n+2}}, k \in \{0, 1, \ldots, 2^n - 1\}$ and $y \in (0,1)$, there holds*

$$P_\tau\Big(t, x; \frac{k+y}{2^n}\Big) \ge \min\left\{P_\tau\Big(t, x; \frac{k}{2^n}\Big), P_\tau\Big(t, x; \frac{k+1}{2^n}\Big)\right\} \tag{3.2}$$

This lemma ensures that $P_\tau(t,x;z)$ attains its minimum at the endpoints for $z \in \big[\frac{k}{2^n}, \frac{k+1}{2^n}\big]$ for $t \ge \frac{1}{2^{n+2}}$ (see Figure 3.1).



(A) The case $x_1 \leq x_0$



(B) The case $x_1 \geq x_0$

FIGURE 3.1. Comparison of three parabolas

Proof. Fix $n \in \mathbb{N}, t \geq \frac{1}{2^{n+2}}, k \in \{0, 1, \dots, 2^n - 1\}$ and $y \in (0, 1)$ arbitrarily.

Let x_0 be the x-coordinate of the point of intersection between $P_\tau\big(t, x; \frac{k+y}{2^n}\big)$ and $P_\tau\big(t, x; \frac{k}{2^n}\big)$, and let x_1 be the x-coordinate of the point of intersection between $P_\tau\big(t, x; \frac{k+y}{2^n}\big)$ and $P_\tau\big(t, x; \frac{k+1}{2^n}\big)$.

From elementary computations, we have

$$x_0 = \frac{y}{2^{n+1}} + \frac{\tau(y)}{y} t + D_n\Big(\frac{k}{2^n}\Big) t, \tag{3.3}$$

$P_\tau\big(t, x; \frac{k+y}{2^n}\big) \lesseqgtr P_\tau\big(t, x; \frac{k}{2^n}\big)$ if and only if $x \lesseqgtr x_0$,

$$x_1 = \frac{1+y}{2^{n+1}} + \frac{\tau(y)}{1-y} t + D_n\Big(\frac{k}{2^n}\Big) t \tag{3.4}$$

and $P_\tau\big(t,x;\frac{k+y}{2^n}\big) \lesseqgtr P_\tau\big(t,x;\frac{k+1}{2^n}\big)$ if and only if $x \gtreqless x_1$, where the inequality signs are taken in the opposite order.

It remains to show that $x_0 \geq x_1$, provided that $t \geq \frac{1}{2^{n+2}}$. Note that $\tau(y) \geq 2y(1-y)$ for $y \in [0,1]$ (see (T3)). Hence by (3.3) and (3.4), we have

$$\begin{aligned} x_0 - x_1 &= -\frac{1}{2^{n+1}} + \frac{\tau(y)}{y(1-y)}t \\ &\geq -\frac{1}{2^{n+1}} + 2t \geq -\frac{1}{2^{n+1}} + 2\cdot\frac{1}{2^{n+2}} = 0 \end{aligned}$$

for $t \geq \frac{1}{2^{n+2}}$. This completes the proof. □

From Lemma 3.1 and (3.1), we have the following proposition.

Proposition 3.2. *For any $x \in [0,1]$, $n \in \mathbb{N}$ and $t \geq \frac{1}{2^{n+2}}$, we have*

$$H_t\tau(x) = \min_{k\in\{0,1,2,\dots,2^n\}} P_\tau\Big(t,x;\frac{k}{2^n}\Big).$$

This proposition implies that the number of points to be investigated decreases over time. In other words, the shape of $u(t,\cdot)$ becomes simpler as time progresses.

We recall the constants $\theta_n^k(t)$ of (1.2) for $n \in \mathbb{N}, k \in \{0,1,\dots,2^n-1\}$, and $t > 0$.

Lemma 3.3. *For any $n \in \mathbb{N}, k \in \{0,1,\dots,2^n-1\}$, and $t > 0$, $\theta_n^k(t)$ is the unique x-coordinate of the point of intersection between $P_\tau\big(t,\cdot;\frac{k}{2^n}\big)$ and $P_\tau\big(t,\cdot;\frac{k+1}{2^n}\big)$.*

Proof. Let $n \in \mathbb{N}, k \in \{0,1,\dots,2^n-1\}$ and $t > 0$. Let x satisfy the equality $P_\tau\big(t,\cdot;\frac{k}{2^n}\big) = P_\tau\big(t,\cdot;\frac{k+1}{2^n}\big)$. Then, from (1.1b) and (2.3), solving the quadratic equation:

$$\frac{1}{2t}\Big(x-\frac{k}{2^n}\Big)^2 + \tau\Big(\frac{k}{2^n}\Big) = \frac{1}{2t}\Big(x-\frac{k+1}{2^n}\Big)^2 + \tau\Big(\frac{k+1}{2^n}\Big),$$

we obtain

$$x = \frac{2k+1}{2^{n+1}} + tD_n\Big(\frac{k}{2^n}\Big)$$

Thus, we have $\theta_n^k(t)$ of (1.2). □

Lemma 3.4. *For any $n \in \mathbb{N}$ and $t \in \big(0,\frac{1}{2^{n+1}}\big)$, we have* (1.3)*:*

$$0 = \theta_n^{-1}(t) < \theta_n^0(t) < \theta_n^1(t) < \cdots < \theta_n^{2^n-1}(t) < \theta_n^{2^n}(t) = 1. \tag{1.3}$$

Proof. From Definition 1.1, we have $D_n(0) = n, D_n\big(\frac{2^n-1}{2^n}\big) = D_n\big(1-\frac{1}{2^n}\big) = -n$. Hence we have by (1.2),

$$\begin{aligned} \theta_n^0(t) &= \frac{1}{2^{n+1}} + nt > 0 = \theta_n^{-1}(t), \\ \theta_n^{2^n-1}(t) &= 1 - \frac{1}{2^{n+1}} - nt < 1 = \theta_n^{2^n}(t). \end{aligned}$$

For $k \in \{1,2,\dots,2^n-1\}$,

$$\begin{aligned} \theta_n^k(t) - \theta_n^{k-1}(t) &= \frac{2k+1}{2^{n+1}} + tD_n\Big(\frac{k}{2^n}\Big) - \left(\frac{2k-1}{2^{n+1}} + tD_n\Big(\frac{k-1}{2^n}\Big)\right) \\ &= \frac{1}{2^n} + t\left(D_n\Big(\frac{k}{2^n}\Big) - D_n\Big(\frac{k-1}{2^n}\Big)\right) \geq \frac{1}{2^n} - 2t > 0. \end{aligned}$$

This completes the proof. □

Summarizing the above results, we have Theorem 1.3.

3.2. **Properties of $H_t\tau$.** In this subsection, we briefly introduce some properties of $H_t\tau$.

As an application of Theorem 1.3, we see that $u(t,x) = H_t\tau(x)$ inherits the self-affine property of the Takagi function τ in the following sense.

Theorem 3.5 (Fujita *et al.* [**6**]). *Let $n \in \mathbb{N}$. If $k \in \{1, 2, 3, \ldots, 2^n - 1\}$ satisfies $D_n(\frac{k}{2^n}) = 0$, then we have*

$$u\Big(\frac{t}{2^n}, \frac{k+x}{2^n}\Big) = \tau\Big(\frac{k}{2^n}\Big) + \frac{1}{2^n}u(t,x), \quad (t,x) \in \Big[0, \frac{1}{4}\Big] \times \Big[0,1\Big]. \tag{3.5}$$

Theorem 3.5 yields that, for any $t \in [0, \frac{1}{4}]$, the graph of $u(\frac{t}{2^n}, \cdot)$ in the dyadic interval $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ is a miniature version of the graph of $u(t,\cdot)$ in the interval $[0,1]$. The miniature is obtained by scaling 2^{-n} and shifting it vertically by $\tau(\frac{k}{2^n})$. Furthermore, the upper bound $t = \frac{1}{4}$ in (3.5) is optimal (see [**6**, Theorem 1.4]).

From the explicit representation formula (1.4), we can visualize $u(t,x) := H_t\tau(x)$ for $t > 0$ and $x \in [0,1]$. Figures 3.2 and 3.3 show the graphs of $u(t,x) = H_t\tau(x)$ on $[0,1]$ at some dyadic times $t = 2^{-n}$ ($n = 2, 3, \ldots, 8$).

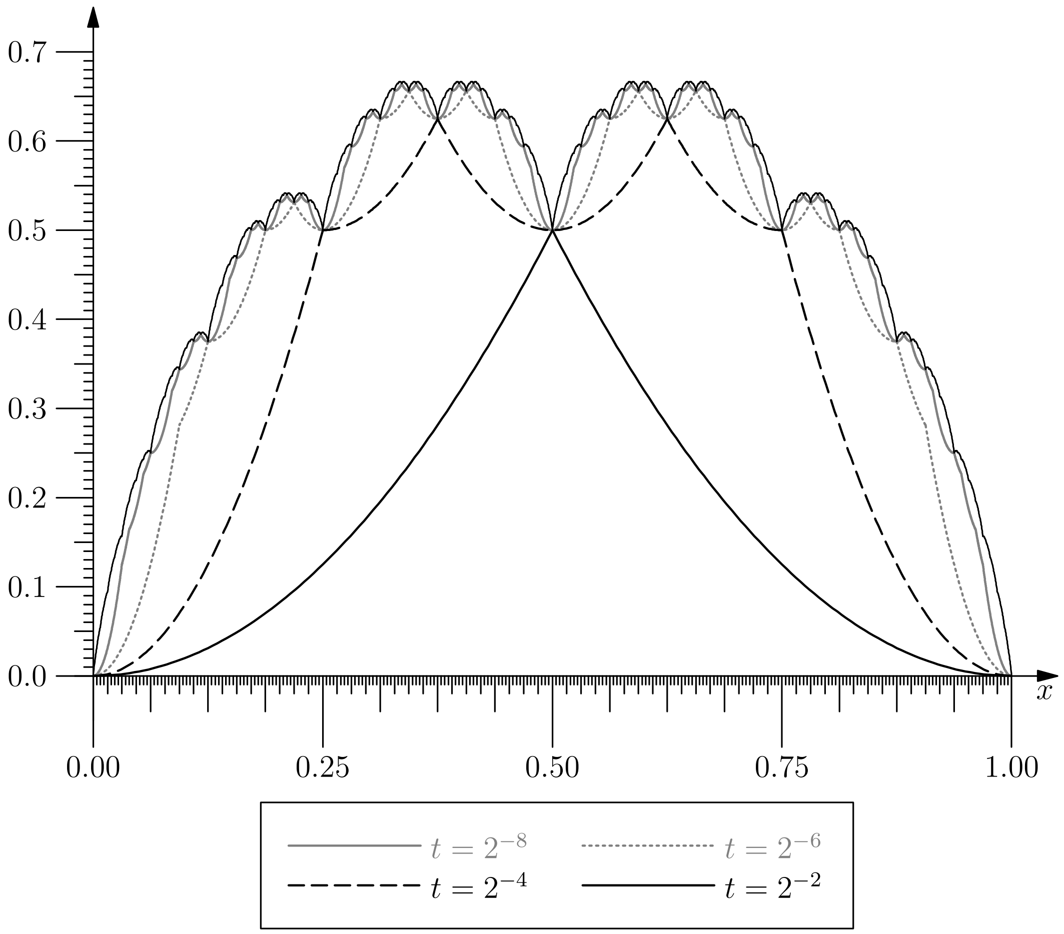


FIGURE 3.2. Graph of $u(t,\cdot)$ for $t = 2^{-2}, 2^{-4}, 2^{-6}, 2^{-8}$

Figure 3.2 illustrates the graph of $u(t,\cdot)$ on $[0,1]$ at the dyadic times $t = 2^{-2}, 2^{-4}, 2^{-6}$ and 2^{-8}. From Theorem 3.5, the equality

$$u(t,x) = 2^n u\Big(\frac{t}{2^n}, \frac{k+x}{2^n}\Big) - 2^n\tau\Big(\frac{k}{2^n}\Big), \quad (t,x) \in \Big[0, \frac{1}{4}\Big] \times [0,1) \tag{3.6}$$

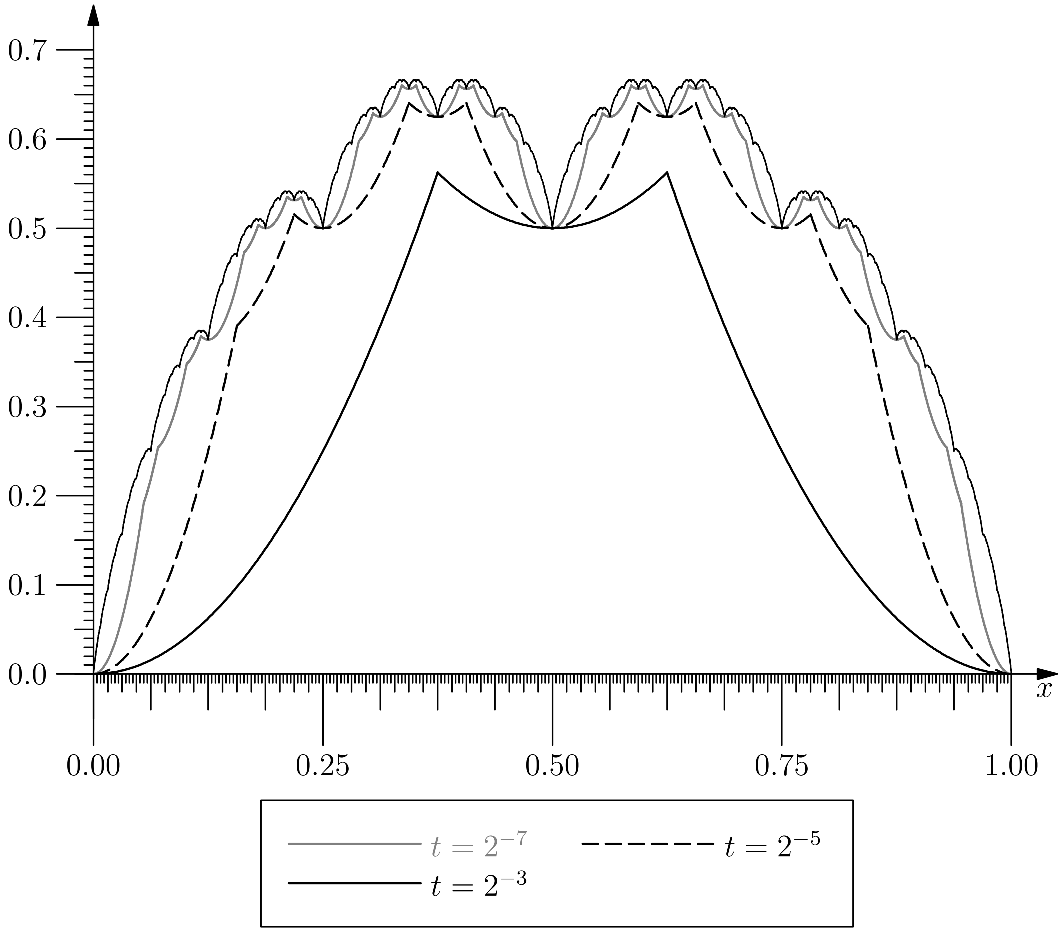


FIGURE 3.3. Graph of $u(t,\cdot)$ for $t = 2^{-3}, 2^{-5}, 2^{-7}$

holds for $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2^n - 1\}$ with $D_n\left(\frac{k}{2^n}\right) = 0$. (3.6) implies that the graph of $u(t,\cdot)$ on $[0,1]$ is obtained by the graph of $u\left(\frac{t}{2^n},\cdot\right)$ on the dyadic interval $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ with suitable scaling and translation. In fact, one can see that the same shape of the graph of $u\left(\frac{1}{2^n},\cdot\right)$ on $[0,1]$ appears in the graph of $u\left(\frac{1}{2^{n+2}},\cdot\right)$ on the dyadic interval $\left[\frac{k}{2^2}, \frac{k+1}{2^2}\right]$ for $n = 2, 4, 6$. Similar phenomena can be observed in Figure 3.3 for dyadic times $t = 2^{-3}, 2^{-5}, 2^{-7}$.

The next result is concerning a perfect approximation of the flow $H_t\tau$ by flows starting from partial Takagi functions $\{\tau_n\}_{n\in\mathbb{N}}$. From Lemma 2.1 and Proposition 2.5, for any $n \in \mathbb{N}$, we have $H_t\tau_n(x) \le H_t\tau(x)$ for any $t \ge 0$ and $x \in [0,1)$. By (T4), for dyadic rationals $x = \frac{k}{2^n}$, $\tau(x) = \tau_m(x)$ holds for any $m \ge n \in \mathbb{N}$. Combining these two facts and Theorem 1.3, we obtain

Theorem 3.6. *Let $n \in \mathbb{N}$. Then*

$$H_t\tau(x) = H_t\tau_n(x), \quad x \in \mathbb{R}, t \ge \frac{1}{2^n}.$$

Here τ_n is the partial Takagi function defined by (2.1).

This theorem implies that the effects of the high-frequency components successively disappear from the time evolution of $H_t\tau$.

Remark 3.7. Esteve and Zuazua [**3**] studied an inverse problem for the Hamilton-Jacobi equation. They showed that for given Lipschitz initial data f and $T > 0$, one can characterize a function g satisfying $H_T f = H_T g$. Although the Takagi function is *not* Lipschitz, the above theorem provides a concrete example associated with the result of [**3**].

Finally, we discuss the propagation of singularities. Let $\Sigma(u)$ denote the set of singular points.

$$\Sigma(u) := \{(t, x) \in (0, \infty) \times [0, 1) \mid u \text{ is not differentiable at } (t, x)\}$$

By Theorem 1.3, we can completely understand the number, locations, directions, and velocities of the singular points on the interval $[0, 1)$ for each fixed time $t > 0$. In particular, when $t = \frac{1}{2^m}$ ($m \in \mathbb{N}$ and $m \geq 2$), the singularities start to move or merge with each other.

Figure 3.4 illustrates the evolution of singular points in the (t, x)-plane. The black solid lines of Figure 3.4 indicate $\Sigma(u)$. The singular points move along the generalized characteristics (see Albano, Cannarsa and Sinestrani [**1**]). For details, see Fujita *et al.* [**5**].

4. Inviscid Burgers equation associated with $H_t\tau$

In this section, we briefly comment on an application of Theorem 1.3 to the inviscid Burgers equation. The inviscid Burgers equation is known as a typical case of scalar conservation laws without dissipation (see e.g., Evans [**4**] and Smoller [**11**]).

From Proposition 2.5 (H1), $u(t, x) = H_t\tau(x)$ is differentiable almost everywhere with respect to x for any $t > 0$. Hence, at least *formally*, its spatial derivative $v := \partial_x u$ provides a solution to the corresponding initial value problem for the inviscid Burgers equation:

$$\begin{aligned} \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0, \qquad (t, x) \in (0, T) \times \mathbb{R}, \\ v(0, x) = \tau'(x), \quad x \in \mathbb{R}. \end{aligned} \tag{4.1}$$

Since $u(0, x) = \tau(x)$ is not differentiable in the classical sense, the initial data for v should be interpreted in the sense of distributions.

Here we discuss (4.1) only at a *formal* level and focus on a consequence suggested by the above correspondence. Since the inviscid Burgers equation arises from fluid dynamics, we are naturally interested in its energy behavior, in particular through the L^2 norm of v. The following result *formally* describes the singular behavior of the energy near the initial time.

Theorem 4.1 (Formal statement)**.** *Let $v(t, x)$ be a formal solution to* (4.1)*. Then, $\|v(t)\|^2_{L^2(\mathbb{T})}$ exhibits a logarithmic singularity near the initial time in the sense that*

$$\lim_{t \to 0+} \frac{\|v(t)\|^2_{L^2(\mathbb{T})}}{\log_2(1/t)} = 1.$$

Remark 4.2. Such a singularity arises from the non-differentiability of the Takagi function. See e.g., Kôno [**9**] and Lagarias [**10**].

The above statement can be made rigorous. The proof relies on Theorem 1.3 together with a careful analysis of the associated flow, but the detailed argument requires further analysis. It will be presented in our forthcoming paper [**7**].

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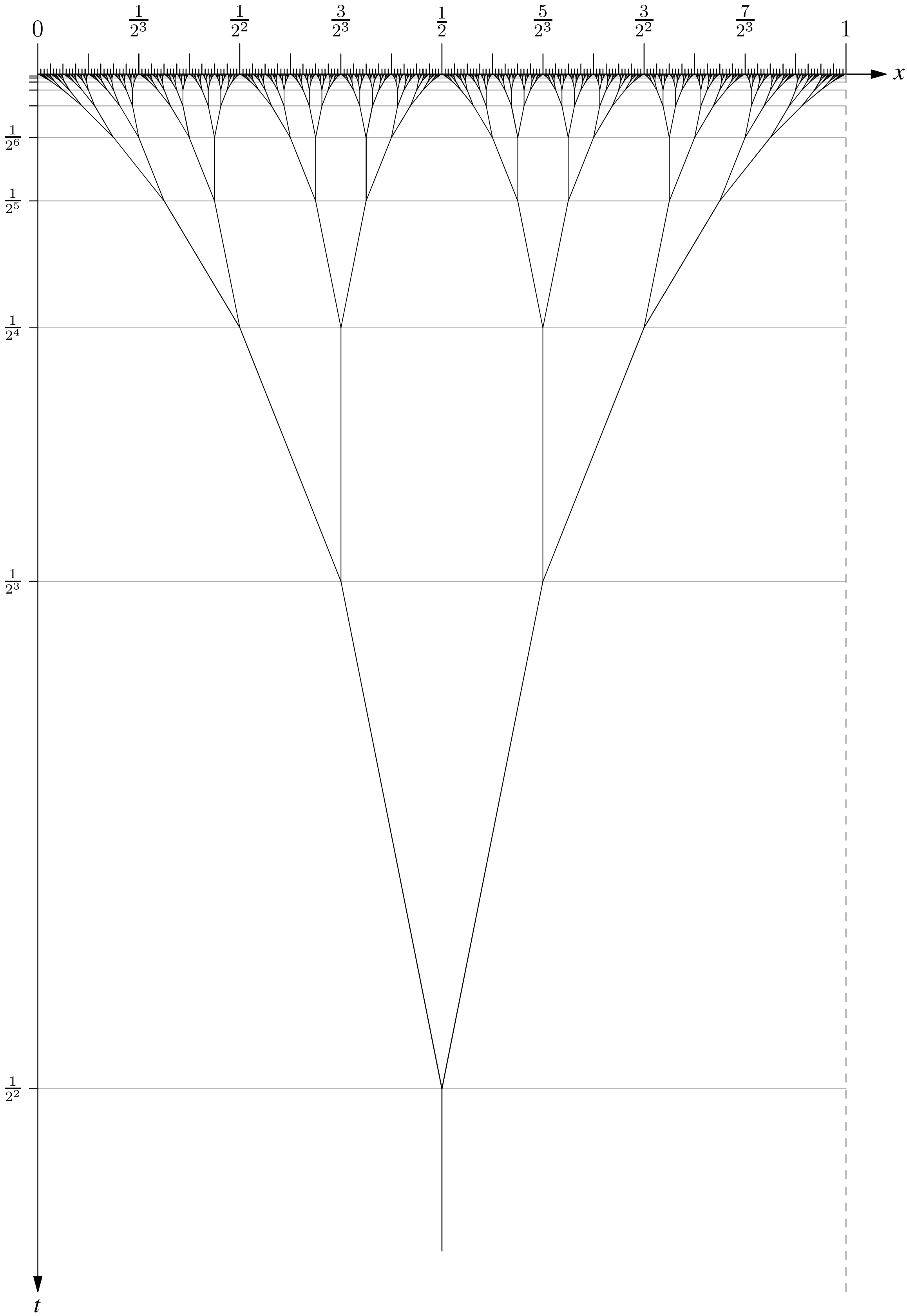

FIGURE 3.4. Evolution of singular points in the (t, x)-plane

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