Bifurcation analysis for the Chafee-Infante problem on a graph

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1 Introduction

This article is based on the joint work with Toru Kan (Osaka Metropolitan University). In [9] the following boundary value problem has been introduced

$$\begin{cases} u_{xx} + \lambda f(u) = 0, \quad x \in (-1,1) \setminus \{0\}, \\ u_x(-1) = u_x(1) = 0, \\ u(-0) + a u_x(-0) = u(+0) - a u_x(+0), \\ u_x(-0) = u_x(+0), \end{cases} \tag{1.1}$$

where $f \in C^2(\mathbf{R})$, $\lambda > 0$ is a bifurcation parameter and $a > 0$ is a fixed constant. The function f is supposed to be a bistable nonlinearity. For simplicity, we only discuss a typical case $f(u) = u - u^3$ throughout this article.

In a special case $a = 0$, (1.1) is equivalent to a classical Chafee-Infante problem

$$\begin{cases} w_{xx} + \mu f(w) = 0, \quad x \in (-1,1), \\ w_x(-1) = w_x(1) = 0, \end{cases} \tag{1.2}$$

where $\mu > 0$. It admits two stable constant solutions $w = \pm 1$ and one unstable constant solution $w = 0$ for every $\mu > 0$. To this problem (1.2), the existence of nonconstant solutions and their stability are main issues. For a general nonlinearity f, any nonconstant solution of (1.2) is known to be unstable; the homogeneous Neumann boundary condition plays a crucial role (see [1]). On the other hand, the solution structure of (1.2) which includes unstable nonconstant solutions, is interesting from the viewpoint of bifurcation. It is shown in [2] that there exists a sequence

$$0 = \mu_0 < \mu_1 < \mu_2 < \cdots < \cdots < \mu_n < \cdots$$

with $\mu_n \to \infty$ as $n \to \infty$ such that if $\mu \in (\mu_m, \mu_{m+1}]$, then (1.2) has *exactly* $2m$ nonconstant solutions $\pm w_n = \pm w_n(x;\mu)$ $(n = 1, \cdots, m)$, where $w_n : [-1,1] \times (\mu_n, \infty) \to \mathbf{R}$ for $n \in \mathbf{N}$. The two functions w_n and $-w_n$ are called *the n-mode solutions*; we can take $w_n(-1) > 0$ without loss of generality. In a framework of bifurcation theory $(\mu_n, 0) \in \mathbf{R}_+ \times C^2[-1,1]$ is a bifurcation point; it lies on a branch for a trivial solution $\{(\mu, w) | \ \mu > 0, w = 0\}$ and the branches for the n-mode solutions bifurcate from $(\mu_n, 0)$. In addition, we have the following:

- If n is odd, then $w_n(x;\mu) \sim (-1)^{\frac{n+1}{2}} \sqrt{\mu - \mu_n} \sin n\pi x \quad$ as $\mu \to \mu_n$.

- If n is even, then $w_n(x;\mu) \sim (-1)^{\frac{n}{2}}\sqrt{\mu-\mu_n}\cos n\pi x \quad$ as $\mu \to \mu_n$.

The problem (1.1) is derived as a limit problem of a higher-dimensional case of (1.2); it is expected that (1.1) possesses the stable nonconstant solutions (see [9] and also [8]). This problem also admits two stable stationary solutions $u = \pm 1$ and one unstable stationary solution $u = 0$ for every $\lambda > 0$. We note that solutions u to (1.1) could be discontinuous at $x = 0$; any continuous solution u must be an even function. A precise bifurcation structure of (1.1) is discussed in [9]. It is shown that there exists a sequence

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \cdots < \lambda_n < \cdots$$

with $\lambda_n \to \infty$ as $n \to \infty$ such that if $\lambda \in (\lambda_m, \lambda_{m+1}]$, then (1.1) has *at least* $2m$ nonconstant solutions $\pm u_n(x;\lambda)$ $(n = 1, \cdots, m)$, where $u_n : [-1,1] \times (\lambda_n, \infty) \to \mathbf{R}$ for $n \in \mathbf{N}$. The two functions u_n and $-u_n$ are called *the n-mode solutions*; we can take $u_n(-1) > 0$ without loss of generality. More precisely, it is proved:

- If n is odd, then u_n is odd and discontinuous at $x = 0$; $\lambda_n \in (\mu_{n-1}, \mu_n)$ and $u_n(x;\lambda) \neq w_n(x;\lambda)$.
- If n is even, then u_n is even and continuous at $x = 0$; $\lambda_n = \mu_n$ and $u_n(x;\lambda) = w_n(x;\lambda)$.

In the same way as the Chafee-Infante problem (1.2), $(\lambda_n, 0)$ is a bifurcation point (in an appropriate Banach space) and the branches $\{(\lambda, u) | \lambda > \lambda_n,\ u = \pm u_n(\cdot;\lambda)\}$ bifurcate from $(\lambda_n, 0)$. Furthermore, it is shown that the (local) secondary bifurcation occurs for the branches $\{(\lambda, w) | \lambda > \lambda_{2k-1},\ u = \pm u_{2k-1}(\cdot;\lambda)\}$ $(k = 1, 2, \ldots)$ exactly once, while it never occurs for the branches $\{(\lambda, w) | \lambda > \lambda_{2k},\ u = \pm u_{2k}(\cdot;\lambda)\}$ $(k = 1, 2, \ldots)$. This implies, in particular, that 1-mode solutions $\pm u_1(x;\lambda)$ are stable when λ is sufficiently large.

In this article we are interested in the bifurcation structure of generalized problems of (1.1). Here, we consider the three-component model of the Chafee-Infante type

$$\begin{cases} u^{(j)}_{xx} + \lambda f(u^{(j)}) = 0, \quad x \in I_j, \quad (j = 1, 2, 3), \\ u^{(1)}_x(0) = 0, \quad u^{(3)}_x(0) = 0, \\ u^{(1)}(1) + a u^{(1)}_x(1) = u^{(2)}(-1) - a u^{(2)}_x(-1), \\ u^{(1)}_x(1) = u^{(2)}_x(-1), \\ u^{(2)}(1) + a u^{(2)}_x(1) = u^{(3)}(-1) - a u^{(3)}_x(-1), \\ u^{(2)}_x(1) = u^{(3)}_x(-1), \end{cases} \tag{1.3}$$

where $I_1 = (0, 1)$, $I_2 = (-1, 1)$ and $I_3 = (-1, 0)$, respectively. The problem (1.3) is an artificially extended problem based on (1.1); the three intervals I_1, I_2 and I_3 are chosen suitably from a symmetric point of view.

To the problem (1.3) we will apply a bifurcation analysis to show the existence of branches of the bifurcating solutions from constant solutions. Both problems (1.1) and (1.3) are regarded as the Chafee-Infante problems on the simple metric graphs (with the nodes of the degrees 1 and 2). Recently, reaction-diffusion systems on metric graphs have been investigated by many researchers. See [10], [7] for the stability of solutions, [3], [6] for the variational approaches, and [4], [5] for the propagation on unbounded graphs, and so on. In many papers, the Kirchhoff boundary conditions are imposed at the nodes. In our model the matching conditions at the nodes are different from the Kirchhoff conditions. Also, we are interested in (1.3) from the viewpoint of bifurcation and stability.

The organization of this article is as follows. In Section 2 we will give the main theorem. In Section 3 we will introduce a shooting method and reduce our problem into the finite dimensional equations. In Section 4 we will give an outline of the proof of the main theorem. Finally, the nondegeneracy of solutions will be discussed in Section 5.

2 Main result

We first introduce a framework with a certain symmetry in order to discuss solutions to (1.3). Denote

$$X_0 := C^2(\bar{I}_1) \times C^2(\bar{I}_2) \times C^2(\bar{I}_3).$$

and

$$\mathbf{u} := (u^{(1)}(x), u^{(2)}(x), u^{(3)}(x)) \in X_0.$$

The function space X_0 becomes a Banach space with a norm introduced by a standard C^2-topology. We say that $\mathbf{u}$ is *odd* if

$$\begin{cases} u^{(1)}(x) = -u^{(3)}(-x) & \text{for } x \in (0,1), \\ u^{(2)}(x) = -u^{(2)}(-x) & \text{for } x \in (0,1), \end{cases}$$

and that $\mathbf{u}$ is *even* if

$$\begin{cases} u^{(1)}(x) = u^{(3)}(-x) & \text{for } x \in (0,1), \\ u^{(2)}(x) = u^{(2)}(-x) & \text{for } x \in (0,1). \end{cases}$$

We focus on the solutions of (1.3) in the set $X \subset X_0$ defined by

$$X := \{\mathbf{u} \in X_0 \mid -1 < u^{(j)}(x) < 1 \text{ for } x \in I_j \ (j = 1,2,3)\}.$$

Let us denote the set of all pairs $(\lambda, \mathbf{u}) \in \mathbf{R}_+ \times X$ of (1.3) as follows:

$$\mathcal{S} := \textstyle\bigcup_{\lambda \in (0,\infty)} \{\lambda\} \times \mathcal{S}_\lambda, \quad \mathcal{S}_\lambda := \{\mathbf{u} \in X \mid u_1,\ u_2 \text{ and } u_3 \text{ satisfy(1.3)}\}.$$

Note that $\mathcal{S}$ contains a half-line which corresponds to a constant solution $\mathbf{u} = (0,0,0)$. Moreover, set

$$\mathcal{S}_\lambda^o := \{\mathbf{u} \in \mathcal{S}_\lambda \mid \mathbf{u} \text{ is odd}\}, \quad \mathcal{S}_\lambda^e := \{\mathbf{u} \in \mathcal{S}_\lambda \mid \mathbf{u} \text{ is even}\}.$$

We focus on the structure of $\mathcal{S}_\lambda^o$. Our main result is given by the following theorem.

Theorem 1. *There is $\tilde{\lambda}_1 > 0$ such that for $\lambda > \tilde{\lambda}_1$ there exist $\pm\mathbf{u}_1 = \pm\mathbf{u}_1(x;\lambda) \in \mathcal{S}_\lambda^o$. In addition, $\mathbf{u}_1(x;\lambda) \to (0,0,0)$ as $\lambda \to \tilde{\lambda}_1$ in the topology of X_0.*

In Theorem 1 we can take $u_1^{(1)}(0) > 0$ without loss of generality. Note that $u_1^{(j)}$ is decreasing in $\bar{I}_j$ $(j = 1,2,3)$; the solutions $\pm\mathbf{u}_1$ is regarded as 1-mode solution of (1.3). Moreover, $(\tilde{\lambda}_1, (0,0,0))$ is a bifurcation point in $\mathbf{R}_+ \times X$ and the branches $\{(\lambda, \mathbf{u})|\lambda >$

$\tilde{\lambda}_1,\ \mathbf{u} = \pm\mathbf{u}_1(\cdot;\lambda)\}$ bifurcate from $(\tilde{\lambda}_1, 0)$. The existence of the other *odd* solutions corresponding to the higher modes ($n = 3, 5, \dots$) remains an open problem.

Let us consider the following problem

$$\begin{cases} v_{xx} + \lambda f(v) = 0, \quad x \in (-1,1) \setminus \{0\}, \\ v_x(-1) = 0,\ v(1) = 0, \\ v(-0) + a v_x(-0) = v(+0) - a v_x(+0), \\ v_x(-0) = v_x(+0). \end{cases} \tag{2.1}$$

The difference between (1.1) and (2.1) is the boundary condition at $x = 1$. There is the one to one correspondence between $\mathcal{S}^o_\lambda$ and the corresponding solution set of (2.1). In particular, we have the following relation between the two solutions of both problems: for any solution v of (2.1), $\mathbf{u} = (u^{(1)}(x), u^{(2)}(x), u^{(3)}(x)) \in X_0$ with

$$\begin{cases} u^{(1)}(x) := v(x-1), \quad x \in I_1^* := [-1, 0), \\ u^{(2)}(x) := v(x+1), \quad x \in I_{2,-}^* := (-1, 0), \\ u^{(2)}(x) := -v(-x+1), \quad x \in I_{2,+}^* := [0, 1), \\ u^{(3)}(x) := -v(-x-1), \quad x \in I_3^* := (-1, 0], \end{cases} \tag{2.2}$$

is an *odd* solution of (1.3). Conversely, for any *odd* solution $\mathbf{u}$ to (1.3),

$$v(x) := \begin{cases} u^{(1)}(x+1) & x \in [-1, 0), \\ u^{(2)}(x-1) & x \in (0, 1]. \end{cases} \tag{2.3}$$

solves (2.1).

To this end, we give a remark on the structure of $\mathcal{S}^e_\lambda$. There is a one to one correspondence between $\mathcal{S}^e_\lambda$ and the correspoinding solution set of (1.1). In particular, we have the following relation between the two solutions of the both problems: for any solution u of (1.1), $\mathbf{u} = (u^{(1)}(x), u^{(2)}(x), u^{(3)}(x)) \in X_0$ with

$$\begin{cases} u^{(1)}(x) := u(x-1), \quad x \in I_1^* := [-1, 0), \\ u^{(2)}(x) := u(x+1), \quad x \in I_{2,-}^* := (-1, 0), \\ u^{(2)}(x) := u(-x+1), \quad x \in I_{2,+}^* := [0, 1), \\ u^{(3)}(x) := u(-x-1), \quad x \in I_3^* := (-1, 0], \end{cases} \tag{2.4}$$

is an *even* solution of (1.3). Conversely, for any *even* solution $\mathbf{u}$ to (1.3),

$$u(x) := \begin{cases} u^{(1)}(x+1) & x \in [-1, 0), \\ u^{(2)}(x-1) & x \in (0, 1]. \end{cases} \tag{2.5}$$

solves (1.1). Hence the analysis for even solutions of (1.3), except the stability and the secondary bifurcation, is completely reduced into the analysis for solutions of (1.1).

3 The shooting method

In this section we introduce the shooting argument developed in [9] in order to show Theorem 1. Let $U(\,\cdot\,;\gamma,\delta)$ be the solution of the initial value problem

$$\begin{cases} U'' + f(U) = 0, \\ U(0) = \gamma, \quad U'(0) = \delta. \end{cases} \tag{3.1}$$

For a convenience, we set

$$F(u) := 2\int_0^u f(s)ds, \quad \beta_0 := \sqrt{F(\pm 1)}.$$

A bounded solution of (3.1) satisfies

$$(U'(y))^2 + F(U(y)) = \delta^2 + F(\gamma) \ \le \beta_0^2 \quad \text{for all } y \in \mathbf{R}.$$

We note that $U(y;0,\pm\beta_0)$ are the heteroclinic solutions satisfying $U(+\infty) = \pm 1$ and $U(-\infty) = \mp 1$, respectively.

Define

$$\Sigma_0 := \{(\gamma,\delta) \mid \gamma \in (-1,1),\ \delta^2 + F(\gamma) \ < \beta_0^2\}.$$

In a standard shooting method we construct solutions of (1.3) in terms of U with suitable γ and δ. Considering that $I_1 = (0,1)$, $I_2 = (-1,1)$ and $I_3 = (-1,0)$ respectively, we look for the solutions to (1.3) of the form

$$\mathbf{u}(\gamma_1,\gamma_2,\gamma_3,\delta;\lambda) := \big(U(\sqrt{\lambda}x;\gamma_1,0), U(\sqrt{\lambda}x;\gamma_2,\delta), U(-\sqrt{\lambda}x;\gamma_3,0)\big) \tag{3.2}$$

with $(\gamma_1,\gamma_2,\gamma_3,\delta) \in \Sigma$, where

$$\Sigma := \{(\gamma_1,\gamma_2,\gamma_3,\delta) \in (-1,1)^3 \times (-\beta_0,\beta_0) \mid (\gamma_2,\delta) \in \Sigma_0\}.$$

To discuss the matching conditions of $\mathbf{u}(\gamma_1,\gamma_2,\gamma_3,\delta;\lambda)$ at the boundary of I_1 and I_2, and at the boundary of I_2 and I_3 respectively, we define the functions $P^{\pm}, Q^{\pm} : \Sigma \to \mathbf{R}$ for each $\lambda > 0$;

$$P^{\pm}(\gamma,\delta;\lambda) := U(\pm\sqrt{\lambda};\gamma,\delta) \pm a\sqrt{\lambda}U'(\pm\sqrt{\lambda};\gamma,\delta)$$

and

$$Q^{\pm}(\gamma,\delta;\lambda) := U'(\pm\sqrt{\lambda};\gamma,\delta).$$

Moreover, we define $\mathcal{R} : \Sigma \to \mathbf{R}^4$

$$\mathcal{R}(\gamma_1,\gamma_2,\gamma_3,\delta;\lambda) := \begin{pmatrix} P^+(\gamma_1,0;\lambda) - P^-(\gamma_2,\delta;\lambda) \\ Q^+(\gamma_1,0;\lambda) - Q^-(\gamma_2,\delta;\lambda) \\ P^+(\gamma_2,\delta;\lambda) - P^-(\gamma_3,0;\lambda) \\ Q^+(\gamma_2,\delta;\lambda) - Q^-(\gamma_3,0;\lambda) \end{pmatrix}. \tag{3.3}$$

The following proposition is fundamental (see Lemma 2.3 in [9]).

Proposition 3.1. *Fix* $\lambda > 0$. *Let* $\mathbf{u} = \mathbf{u}(\gamma_1^*, \gamma_2^*, \gamma_3^*, \delta^*; \lambda)$. *Then,* $\mathbf{u} \in \mathcal{S}_\lambda$ *if and only if*

$$\mathcal{R}(\gamma_1^*, \gamma_2^*, \gamma_3^*, \delta^*; \lambda) = \mathbf{0}. \tag{3.4}$$

Moreover, $\mathbf{u}$ *is nondegenerate if and only if*

$$\det\left[\mathcal{R}_{(\gamma_1,\gamma_2,\gamma_3,\delta)}(\gamma_1^*, \gamma_2^*, \gamma_3^*, \delta^*; \lambda)\right] \neq 0, \tag{3.5}$$

where $\mathcal{R}_{(\gamma_1,\gamma_2,\gamma_3,\delta)}$ *denotes the Jacobi matrix of* $\mathcal{R}$.

When we consider the *odd* solutions of (1.3), we obtain the following proposition by combining Proposition 3.1 with a symmetry of U (in y, γ and δ).

Proposition 3.2. *Fix* $\lambda > 0$. *Let* $\mathbf{u} = \mathbf{u}(\gamma_1^*, \gamma_2^*, \gamma_3^*, \delta^*; \lambda)$. *Then,* $\mathbf{u} \in \mathcal{S}_\lambda^o$ *if and only if* $\gamma_1^* = -\gamma_3^*$, $\gamma_2^* = 0$ *and*

$$\begin{pmatrix} P^+(\gamma_1^*, 0; \lambda) + P^+(0, \delta^*; \lambda) \\ Q^+(\gamma_1^*, 0; \lambda) - Q^+(0, \delta^*; \lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.6}$$

Moreover, $\mathbf{u}$ *is nondegenerate if and only if*

$$P_\gamma^+(\gamma_1^*, 0; \lambda)Q_\gamma^+(0, \delta^*; \lambda) + P_\gamma^+(0, \delta^*; \lambda)Q_\gamma^+(\gamma_1^*, 0 : \lambda) \neq 0 \tag{3.7}$$

and

$$P_\gamma^+(\gamma_1^*, 0, \lambda)Q_\delta^+(0, \delta^*; \lambda) + P_\delta^+(0, \delta^*; \lambda)Q_\gamma^+(\gamma_1^*, 0; \lambda) \neq 0. \tag{3.8}$$

Now we give a new representation of $\mathbf{u}(\gamma_1, \gamma_2, \gamma, 3, \delta; \lambda)$ by introducing a function $G : [-\beta_0, \beta_0] \to [-1, 1]$; G is the inverse of the mapping $u \mapsto \operatorname{sgn}(u)\sqrt{F(u)}$. It follows from $v^2 = F(G(v)) = F(u)$ that

$$G'(v) = \frac{\operatorname{sgn}(u)\sqrt{F(u)}}{f(u)} \quad \text{for } v \in (-\beta_0, \beta_0).$$

it is easy to see that G is odd and G' is even. Moreover,

$$G''(v) = \frac{1}{f(u)}\left(1 - \frac{f'(u)F(u)}{f(u)^2}\right) \quad \text{for } v \in (-\beta_0, \beta_0) \setminus \{0\}.$$

and we also see that $G''(v) > 0$ for $v \in (0, \beta_0)$.

Proposition 3.3. *Suppose* $(\gamma, \delta) \in \Sigma \setminus \{(0, 0)\}$. *Then, there exists* $(\beta, \theta_0) \in (-\beta_0, \beta_0) \times \mathbf{R} \setminus \{(0, 0)\}$ *such that*

$$\begin{cases} U(y; \gamma, \delta) = G(\beta \cos\theta(y; \beta, \theta_0)), \\ U_y(y; \gamma, \delta) = -\beta \sin\theta(y; \beta, \theta_0) \end{cases}$$

where $\theta(y) := \theta(y; \beta, \theta_0)$ *is defined by*

$$\int_{\theta_0}^{\theta} G'(\beta \cos\tau) d\tau = y.$$

Proof. We see that G is bijective and that $(G^{-1}(u))^2 = F(u)$ for $u \in (-1, 1)$. For any $(\gamma, \delta) \in \Sigma \setminus \{(0, 0)\}$, we choose $(\beta, \theta_0) \in (-\beta_0, \beta_0) \times \mathbf{R} \setminus \{(0, 0)\}$ as a solution of

$$\begin{cases} G^{-1}(\gamma) = \beta \cos \theta_0, \\ \delta = -\beta \sin \theta_0. \end{cases} \tag{3.9}$$

Then we will show that the function $\tilde{U}(y) = G(\beta \cos \theta(y))$ solves (3.1). Since $\theta(y)$ is the unique solution of

$$\frac{d\theta}{dy} = \frac{1}{G'(\beta \cos \theta(y))}, \quad \theta(0) = \theta_0,$$

we obtain

$$\begin{aligned} \tilde{U}'(y) &= G'(\beta \cos \theta(y)) \cdot (-\beta \sin \theta(y)) \cdot \frac{d\theta}{dy} \\ &= -\beta \sin \theta(y). \end{aligned}$$

Also, since

$$\begin{aligned} \beta \cos \theta(y) = G^{-1}(\tilde{U}(y)) & \\ &= \operatorname{sgn}(\tilde{U}(y)) \sqrt{F(\tilde{U}(y))} \\ &= f(\tilde{U}(y)) \cdot G'(\beta \cos \theta(y)), \end{aligned}$$

we are led to

$$\begin{aligned} \tilde{U}''(y) &= -\beta \cos \theta(y) \cdot \frac{1}{G'(\beta \cos \theta(y))} \\ &= -f(U(y)). \end{aligned}$$

Moreover, $\tilde{U}(0) = \gamma$ and $\tilde{U}'(0) = \delta$ come from (3.11). Thus it completes a proof. □

In Proposition 3.3 we can choose β and θ_0 as follows:

- $\beta = G^{-1}(\gamma)$ and $\theta_0 = 0$ if $\delta = 0$,
- $\beta = -\delta > 0$ and $\theta_0 = \dfrac{\pi}{2}$ and if $\gamma = 0$ with $\delta < 0$.

Set $\mathbf{u}^o(\alpha, \delta; \lambda) := \mathbf{u}(G^{-1}(\alpha), 0, -G^{-1}(\alpha), \delta; \lambda)$ for $\alpha \in (0, \beta_0)$ and $\delta \in (-\beta_0, 0)$; it follows from Proposition 3.3 that

$$\mathbf{u}^o(\alpha, \delta; \lambda) = \Big(G\big(\alpha \cos \theta_1(\sqrt{\lambda}x)\big), G\big(-\delta \cos \theta_2(\sqrt{\lambda}x)\big), -G\big(\alpha \cos \theta_1(\sqrt{\lambda}x)\big)\Big), \tag{3.10}$$

where $\theta_1(y)$ and $\theta_2(y)$ are defined by

$$\int_0^{\theta_1} G'(\alpha \cos \tau) d\tau = y \quad \text{and} \quad \int_{\frac{\pi}{2}}^{\theta_2} G'(\delta \cos \tau) d\tau = y,$$

respectively (note that G' is even). Since G' is positive, the functions θ_1 and θ_2 are increasing in y.

By Proposition 3.2 we are led to the following lemma.

Lemma 3.1. *Let* $\lambda > 0$*,* $\alpha \in (0, \beta_0)$ *and* $\delta \in (-\beta_0, 0)$*. Then,* $\mathbf{u}^o(\alpha, \delta; \lambda)$ *is an odd solution of* (1.3) *if and only if* $(\alpha, \delta, \lambda)$ *satisfies*

$$\begin{cases} \dfrac{G(\alpha\cos\phi_1)}{\displaystyle\int_0^{\phi_1} G'(\alpha\cos\tau)d\tau \cdot \alpha\sin\phi_1} + \dfrac{G(\delta\cos\phi_2)}{\displaystyle\int_{\frac{\pi}{2}}^{\phi_2} G'(\delta\cos\tau)d\tau \cdot \delta\sin\phi_2} = 2a, \\ \alpha\sin\phi_1 + \delta\sin\phi_2 = 0, \\ \displaystyle\int_0^{\phi_1} G'(\alpha\cos\tau)d\tau = \sqrt{\lambda}, \\ \displaystyle\int_{\frac{\pi}{2}}^{\phi_2} G'(\delta\cos\tau)d\tau = \sqrt{\lambda}. \end{cases} \tag{3.11}$$

with some $\phi_1 > 0$ *and* $\phi_2 > \dfrac{\pi}{2}$*.*

Remark 3.1. Suppose $(\alpha, \delta, \lambda)$ be a solution of (3.11) with $\phi_1 \in \left(0, \dfrac{\pi}{2}\right)$ and $\phi_2 \in \left(\dfrac{\pi}{2}, \pi\right)$. Then it follows from Proposition 3.3 and (3.10) that $(u^{(j)})'(x) < 0$ for $x \in I_j$ $(j = 1, 2, 3)$.

Suppose $(\alpha, \delta, \lambda)$ be a solution of (3.11) with $\phi_1 \in \left(0, \dfrac{\pi}{2}\right)$ and $\phi_2 \in \left(\dfrac{\pi}{2}, \pi\right)$. Then, we see that

$$G(\alpha\cos\phi_1) > G(\delta\cos\phi_2). \tag{3.12}$$

It follows from a monotonicity of G that $\alpha\cos\phi_1 > \delta\cos\phi_2 > 0$. Moreover, it follows from $G''(s) > 0$ for $s \in (0, \beta_0)$ that

$$G(\alpha\cos\phi_1) - G'(\alpha\cos\phi_1)\alpha\cos\phi_1 < G(\delta\cos\phi_2) - G'(\delta\cos\phi_2)\delta\cos\phi_2 < 0. \tag{3.13}$$

4 Sketch of Proof

In this section we show a sketch of proof for Theorem 1.

In order to prove Theorem 1, we prove that the system (3.11) as in Lemma 3.1 has a solution for each $\lambda > 0$. More precisely, we regard $\alpha \in (0, \beta_0)$ as a new parameter instead of λ, and construct one parameter families $\delta(\alpha)$, $\phi_1(\alpha)$, $\phi_2(\alpha)$ and $\lambda(\alpha)$ satisfying some suitable properties.

The following Lemmas 4.1 and 4.2 are proved by a standard continuity argument.

Lemma 4.1. *Fix* $\alpha \in (0, \beta_0)$*,* $\delta \in (-\beta_0, 0)$ *arbitrarily. Consider the equations for* $0 < \phi_1 < \dfrac{\pi}{2}$ *and* $\dfrac{\pi}{2} < \phi_2 < \pi$*:*

$$\begin{cases} \alpha\sin\phi_1 + \delta\sin\phi_2 = 0, \\ \displaystyle\int_0^{\phi_1} G'(\alpha\cos\tau)d\tau - \int_{\pi/2}^{\phi_2} G'(\delta\cos\tau)d\tau = 0. \end{cases} \tag{4.1}$$

Then, the following (i) and (ii) hold.

(i) *If* $\delta = -\alpha$*, then there exist a unique* $\theta_*(\alpha) \in \left(0, \frac{\pi}{2}\right)$ *such that* $(\phi_1, \phi_2) = (\theta_*(\alpha), \pi - \theta_*(\alpha))$ *satisfy* (4.1).

(ii) *If* $\delta > -\alpha$*, then there exist a solution* $(\phi_1, \phi_2) = (\phi_1(\alpha, \delta), \phi_2(\alpha, \delta))$ *of* (4.1).

Remark 4.1. A pair $(\alpha, -\alpha, \theta_*(\alpha), \pi - \theta_*(\alpha))$ as in Lemma 4.1 does not satisfy (3.11) for all $\alpha \in (0, \beta_0)$.

By the implicit function, $\phi_1(\alpha, \delta)$ and $\phi_2(\alpha, \delta)$ are C^1-function of (α, δ) and

$$\lim_{\delta \to -\alpha} \phi_1(\alpha, \delta) = \theta_*(\alpha) \text{and} \lim_{\delta \to -\alpha} \phi_1(\alpha, \delta) = \pi - \theta_*(\alpha).$$

By using a continuity argument again with the inqualities (3.12) and (3.13), we obtain the following two lemmas.

Lemma 4.2. *Fix* $\alpha \in (0, \beta_0)$ *arbitrarily. Consider the equation for* $\delta \in (-\alpha, 0)$

$$\frac{G(\alpha \cos \phi_1(\alpha, \delta)) - G(\delta \cos \phi_2(\alpha, \delta))}{\int_0^{\phi_1(\alpha,\delta)} G'(\alpha \cos \tau) d\tau \cdot \alpha \sin \phi_1(\alpha, \delta)} - 2a = 0, \tag{4.2}$$

where $(\phi_1(\alpha, \delta), \phi_2(\alpha, \delta))$ *is defined in (ii) of Lemma 4.1. Then, there exists a solution* $\delta = \delta(\alpha) \in (-\alpha, 0)$ *to* (4.2).

Lemma 4.3. *Fix* $\alpha \in (0, \beta_0)$ *arbitrarily. Suppose that* $(\phi_1(\alpha, \delta), \phi_2(\alpha, \delta))$ *is defined in (ii) of Lemma 4.1 and that* $\delta(\alpha)$ *is defined in Lemma 4.2. Set* $\lambda(\alpha)$ *by*

$$\lambda(\alpha) := \left(\int_0^{\phi_1(\alpha,\delta(\alpha))} G'(\alpha \cos \tau) d\tau \right)^2.$$

Then, the following hold:

(i) $(\alpha, \delta(\alpha), \lambda(\alpha))$ *is a solution of* (3.11) *with* $\phi_1 = \phi_1(\alpha, \delta(\alpha))$ *and* $\phi_2 = \phi_2(\alpha, \delta(\alpha))$,

(ii) $\frac{d\lambda}{d\alpha}(\alpha) > 0$ *for* $\alpha \in (0, \beta_0)$,

(iii) $\lambda(\alpha) \to \tilde{\lambda}_1$ *as* $\alpha \to 0$ *and* $\lambda(\alpha) \to \infty$ *as* $\alpha \to \beta_0$.

By Lemmas 3.1 and 4.2, it is concluded that $\pm \mathbf{u}^o(\alpha, \delta(\alpha); \lambda(\alpha))$ is a desired solution of (1.3) by (3.10), and the inequality $|G(v)| \leq M|v|$ with some $M > 0$; in the topology of X_0,

$$\mathbf{u}^o(\alpha, \delta(\alpha); \lambda(\alpha)) \to (0, 0, 0) \quad \text{as } \alpha \to 0.$$

See also, Remark 3.1.

5 Remark on nondegeneracy of solutions

In a bifurcation analysis it is also important to consider the nondegeneracy of the solutions; in general, a degenerate solution implies the secondary bifurcation. When we investigate the nondegeneracy of the solutions $\mathbf{u}(\gamma_1, \gamma_2, \gamma_3, \delta; \lambda)$ defined by (3.2) and the *odd* solutions $\mathbf{u}(\gamma_1, 0, -\gamma_1, \delta; \lambda)$, the conditions (3.5) and (3.7), (3.8) as in Propositions 3.1 and 3.2 respectively, play crucial role. For the odd solutions, the condition (3.8) is obtained while the condition (3.7) is not obtained yet.

By using this fact we have a partial result on the nondegeneracy of 1-mode solutions of (1.3).

Proposition 5.1. *Let* $\tilde{\lambda}_1 > 0$ *and* $\pm\mathbf{u}_1^o = \pm\mathbf{u}_1^o(x; \lambda) \in \mathcal{S}_\lambda^o$ *be as in Theorem 1, respectively. Suppose* $\lambda > \tilde{\lambda}_1$. *Then the solution* $\pm\mathbf{u}_1^o$ *is nondegenerate in the space*

$$X_0^o := \{\mathbf{u} \in X_0 \mid \mathbf{u} \text{ is } odd\}.$$

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