A remark on finite difference schemes for the phase separation equation coupled with viscoelasticity in two-space dimension

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Abstract

This is a resume article of the result [1] in which the authors give a numerical analysis for the phase separation equation coupled with viscoelasticity in two-space dimension. The system was formulated by Pawłow and Zajączkowski [8], and we proposed a numerical scheme for the system and gave its mathematical analysis, such as the existence of a solution for the scheme and the error estimate in [1].

1 Introduction

We give the numerical analysis for the problem in two space-dimension:

$$\partial_t \boldsymbol{u} = \nabla \cdot \frac{\partial w}{\partial \varepsilon}(\varepsilon(\boldsymbol{u}), \chi) + \nu \partial_t \nabla \cdot (A\varepsilon(\boldsymbol{u})), \tag{1.1}$$

$$\partial_t \chi = \Delta p, \tag{1.2}$$

$$p = -\gamma \Delta \chi + \psi'(\chi) + \frac{\partial w}{\partial \chi}(\varepsilon(\boldsymbol{u}), \chi), \qquad (t, \boldsymbol{x}) \in (0, T] \times \Omega, \tag{1.3}$$

$$\boldsymbol{u} = \boldsymbol{0}, \quad \frac{d\chi}{d\boldsymbol{n}} = \frac{dp}{d\boldsymbol{n}} = 0, \qquad (t, \boldsymbol{x}) \in [0, T] \times \partial\Omega, \tag{1.4}$$

$$\boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0, \quad \partial_t \boldsymbol{u}(0, \cdot) = \boldsymbol{v}_0, \quad \chi(0, \cdot) = \chi_0, \qquad \boldsymbol{x} \in \Omega, \tag{1.5}$$

where ν and γ are positive constants, $\Omega \subset \mathbb{R}^2$ is a bounded domain with a boundary $\partial\Omega$, $\boldsymbol{n}$ is an outward unit normal vector on $\partial\Omega$, and $\boldsymbol{x} = (x, y)^T \in \mathbb{R}^2$. The system introduced by Pawlow-Zajaczkowski [8] describes a phase separation process in a binary deformable alloy quenched below a critical temperature. The unknowns $\boldsymbol{u} = (u_x, u_y)^T$, χ and p are the displacement vector, the order parameter (or phase ratio) and the chemical potential difference between the components, respectively. The unknown shear strain tensor $\varepsilon = \varepsilon(\boldsymbol{u})$ is defined by

$$\varepsilon := \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{pmatrix}, \quad \varepsilon_{xx} := \partial_x u_x, \quad \varepsilon_{yy} := \partial_y u_y, \quad \varepsilon_{xy} := \frac{\partial_x u_y + \partial_y u_x}{2}.$$

Let us denote the scalar product for tensors ε and $\tilde{\varepsilon}$ by $\varepsilon : \tilde{\varepsilon} = \sum_{i,j=x,y} \varepsilon_{ij}\tilde{\varepsilon}_{ij}$. We regards that $\chi = -1$ and $\chi = 1$ correspond to the phases a and b of binary a-b alloy, respectively. The elastic energy $w(\varepsilon, \chi)$ is assumed to be the form;

$$w(\varepsilon, \chi) := \frac{1}{2}(\varepsilon - \overline{\varepsilon}(\chi)) : A(\varepsilon - \overline{\varepsilon}(\chi)),$$

where the fourth order tensor $A = \{A_{ijkl}\}_{i,j,k,l=x,y}$ is an *elastic moduli tensor* defined by

$$A_{ijkl} := \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

with the Kronecker delta δ_{ij} and the Lamé contants λ and μ. From the definition, we can confirm that

$$\begin{gathered} A_{xxxx} = A_{yyyy} = \lambda + 2\mu, \qquad A_{xxyy} = A_{yyxx} = \lambda, \\ A_{xxxy} = A_{xxyx} = A_{xyxx} = A_{yxxx} = A_{yyyx} = A_{yyxy} = A_{yxyy} = A_{xyyy} = 0, \\ A_{xyxy} = A_{yxyx} = A_{xyyx} = A_{yxxy} = \mu. \end{gathered}$$

Throughout this article, we assume $\lambda > 0$ and $\lambda + \mu > 0$ which assure the strong ellipticity of A, that is, there exist positive numbers a_* and a^* such that for any ε

$$a_*\varepsilon : \varepsilon \leq \varepsilon : (A\varepsilon) \leq a^*\varepsilon : \varepsilon. \tag{1.6}$$

The function $\overline{\varepsilon}$ denotes the *stress-free strain* corresponding to the order parameter χ, defined by

$$\overline{\varepsilon}(\chi) = (1 - z(\chi))\overline{\varepsilon}_a + z(\chi)\overline{\varepsilon}_b$$

with constant tensors $\overline{\varepsilon}_a = \begin{pmatrix} \varepsilon_{a,xx} & \varepsilon_{a,xy} \\ \varepsilon_{a,xy} & \varepsilon_{a,yy} \end{pmatrix}$, $\overline{\varepsilon}_b = \begin{pmatrix} \varepsilon_{b,xx} & \varepsilon_{b,xy} \\ \varepsilon_{b,xy} & \varepsilon_{b,yy} \end{pmatrix}$ which represent eigenstrain tensors of phases a and b, and sufficiently smooth given function z satisfying

$$z(\chi) = \begin{cases} 0 & \text{for } \chi \leq -1, \\ 1 & \text{for } \chi \geq 1. \end{cases}$$

In addition, for simplicity, we assume that $z \in [0, 1]$ and $\overline{\varepsilon}$, $\overline{\varepsilon}'$ and $\overline{\varepsilon}''$ are uniformly bounded, throughout this article. We suppose that the chemical energy ψ is the general double-well form:

$$\psi(\chi) = \frac{1}{p+1}|\chi|^{p+1} - \frac{1}{2}\chi^2, \tag{1.7}$$

In the case of $p = 3$, the phase separation equation is called the Cahn-Hilliard equation. However, from the technical reason, here we assume $p \in [2, 3)$. Observe that

$$\frac{\partial w}{\partial \varepsilon}(\varepsilon, \chi) = \begin{pmatrix} \frac{\partial w}{\partial \varepsilon_{xx}} & \frac{\partial w}{\partial \varepsilon_{xy}} \\ \frac{\partial w}{\partial \varepsilon_{yx}} & \frac{\partial w}{\partial \varepsilon_{yy}} \end{pmatrix}, \qquad \frac{\partial w}{\partial \chi}(\varepsilon, \chi) = -\overline{\varepsilon}'(\chi) : A\left(\varepsilon - \overline{\varepsilon}(\chi)\right),$$

$$\psi'(\chi) = |\chi|^{p-1}\chi - \chi.$$

It is easily checked that

$$\frac{\partial w}{\partial \varepsilon_{xx}} = (\lambda + 2\mu)(\varepsilon_{xx} - \overline{\varepsilon}_{xx}(\chi)) + \lambda(\varepsilon_{yy} - \overline{\varepsilon}_{yy}(\chi)),$$

$$\frac{\partial w}{\partial \varepsilon_{xy}} = \frac{\partial w}{\partial \varepsilon_{yx}} = 2\mu(\varepsilon_{xy} - \overline{\varepsilon}_{xy}(\chi)),$$

$$\frac{\partial w}{\partial \varepsilon_{yy}} = (\lambda + 2\mu)(\varepsilon_{yy} - \overline{\varepsilon}_{yy}(\chi)) + \lambda(\varepsilon_{xx} - \overline{\varepsilon}_{xx}(\chi)),$$

which implies $\frac{\partial w}{\partial \varepsilon} = A(\varepsilon - \overline{\varepsilon}(\chi))$.

We introduce the background of the problem (1.1)–(1.5). Gurtin in [4] proposed the general model of the phase separation equation coupled with elasticity, with additional anisotropic, heterogeneous and kinetic effects, by using his thermodyanamical theory based on a microforce balance. The results for the system under the quasi-stationary approximation of elastic equation are found in a lot of references, e.g. Garcke [2, 3] (for more precise references we refer to [1]). On the other hand, Pawłow and Zajązkowski in a series of their articles [5, 6, 7, 8, 9] studied the non-stationary problem (1.1)–(1.5) in order to observe a formulation of the microstructure on a very fast time scale at the initial stages of the phase separation process. We remark that the some physical effects such as the anisotropic effect are omitted in the model for simplicity. The system is coupled of the (visco)-elastic equation and the Cahn-Hilliard equation (i.e. $p = 3$). In [10, 11] the numerical analyses for the problem (1.1)–(1.5) in the one-space dimension case are given, that is, the structure-preserving finite difference schemes are proposed and the existence of the solution and the error estimate are proved by applying the energy method for the structure-preserving finite difference methods (see e.g. [12, 13]).

2 A Finite Difference Scheme

Our aim of this talk is to propose a finite difference scheme for the system (1.1)–(1.5). Before introducing the scheme, we first reformulate the system (1.1)–(1.5) into the simplified version suitable for subsequent analysis, following the results by Pawłow and Zajązkowski [8]. Using the elastic operator Q defined by

$$Q\boldsymbol{u} := \mu\Delta\boldsymbol{u} + (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) = \begin{pmatrix} (\lambda + 2\mu)\partial_x^2 + \mu\partial_y^2 & (\lambda + \mu)\partial_x\partial_y \\ (\lambda + \mu)\partial_x\partial_y & (\lambda + 2\mu), \partial_y^2 + \mu\partial_x^2 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix},$$

we represent $Q\boldsymbol{u} = \nabla \cdot A\varepsilon$. Setting the first order differential operator

$$B = \begin{pmatrix} (\lambda+2\mu)\partial_x & \lambda\partial_x & 2\mu\partial_y \\ \lambda\partial_y & (\lambda+2\mu)\partial_y & 2\mu\partial_x \end{pmatrix},$$

the following identity

$$\nabla \cdot A\overline{\varepsilon}(\chi) = B \begin{pmatrix} \overline{\varepsilon}_{xx}(\chi) \\ \overline{\varepsilon}_{yy}(\chi) \\ \overline{\varepsilon}_{xy}(\chi) \end{pmatrix}$$

holds. By using the above notations, we simplify the equations (1.1)–(1.3) such as

$$\partial_t^2 \begin{pmatrix} u_x \\ u_y \end{pmatrix} - Q \begin{pmatrix} u_x \\ u_y \end{pmatrix} - \nu\partial_t Q \begin{pmatrix} u_x \\ u_y \end{pmatrix} = B \begin{pmatrix} \overline{\varepsilon}_{xx}(\chi) \\ \overline{\varepsilon}_{yy}(\chi) \\ \overline{\varepsilon}_{xy}(\chi) \end{pmatrix}, \tag{2.1}$$

$$\partial_t\chi + \gamma\Delta^2\chi + \Delta\chi = \Delta\left(|\chi|^{p-1}\chi - \overline{\varepsilon}'(\chi) : A\left(\varepsilon - \overline{\varepsilon}(\chi)\right)\right). \tag{2.2}$$

Let us first give a setting for the finite difference method. Throughout this paper, we assume that Ω is a two-dimensional rectangle as $\Omega := [0, L_x] \times [0, L_y] \ni (x, y)$. We denote the numbers of split with respect to t, x and y by N, K and M, respectively. Denote the split sizes by $\Delta t = T/N$, $\Delta x = L_x/K$ and $\Delta y := L_y/M$. We will use various kinds of difference operartors such as the first order difference operators $\delta_x^+, \delta_y^+, \delta_t^+, \delta_x^-, \delta_y^-, \delta_t^-, \delta_x^{\langle 1\rangle}, \delta_y^{\langle 1\rangle}, \delta_t^{\langle 1\rangle}$ and the second order difference operators $\delta_x^{\langle 2\rangle}, \delta_y^{\langle 2\rangle}, \delta_{xy}^{\langle 2\rangle}$. We first define the forward difference operators δ_x^+, δ_y^+ and δ_t^+ and the backward difference operators δ_x^-, δ_y^- and δ_t^- by

$$\begin{aligned}
\delta_x^+ f &:= \frac{f(n,k+1,m) - f(n,k,m)}{\Delta x}, & \delta_x^- f &:= \frac{f(n,k,m) - f(n,k-1,m)}{\Delta x}, \\
\delta_y^+ f &:= \frac{f(n,k,m+1) - f(n,k,m)}{\Delta y}, & \delta_y^- f &:= \frac{f(n,k,m) - f(n,k,m-1)}{\Delta y}, \\
\delta_t^+ f &:= \frac{f(n+1,k,m) - f(n,k,m)}{\Delta t}, & \delta_t^- f &:= \frac{f(n,k,m) - f(n-1,k,m)}{\Delta t}.
\end{aligned}$$

The first order central difference operators are defined by $\delta_\xi^{\langle 1\rangle} := \frac{\delta_\xi^+ + \delta_\xi^-}{2}$ for $\xi = x, y, t$, and the second order difference operators are defined by

$$\delta_\xi^{\langle 2\rangle} := \delta_\xi^+\delta_\xi^- = \delta_\xi^-\delta_\xi^+ \quad (\xi = x, y, t), \qquad \delta_{xy}^{\langle 2\rangle} := \frac{\delta_x^+\delta_y^- + \delta_x^-\delta_y^+}{2}.$$

Let us define the discrete domains Ω_d and $\overline{\Omega}_d$ by

$$\begin{aligned}
\Omega_d &:= \{(k,m) \mid k = 1, 2, \ldots, K-1,\ m = 1, 2, \ldots, M-1\}, \\
\overline{\Omega}_d &:= \{(k,m) \mid k = 0, 1, 2, \ldots, K,\ m = 0, 1, 2, \ldots, M\},
\end{aligned}$$

and the discrete boundaries $\partial\Omega_{d,x}$, $\partial\Omega_{d,y}$ and $\partial\Omega_d$ by

$$\partial\Omega_{d,x} := \{(k,m) \mid k = 0, K,\ m = 0,1,\dots,M\},$$
$$\partial\Omega_{d,y} := \{(k,m) \mid k = 0,1,\dots,K,\ m = 0, M\},$$

and $\partial\Omega_d := \partial\Omega_{d,x} \cup \partial\Omega_{d,y}$. Remark that $\partial\Omega_{d,x} \cap \partial\Omega_{d,y} = \{(0,0),\ (M,0),\ (0,K),\ (K,M)\}$. Obviously it holds that $\overline{\Omega}_d := \Omega_d \cup \partial\Omega_d$. We also define the exterior boundary point sets $\partial\Omega^e_{d,x}$ and $\partial\Omega^e_{d,y}$ by

$$\partial\Omega^e_{d,x} := \{(k,m) \mid k = -1, K+1,\ m = 0,1,\dots,M\},$$
$$\partial\Omega^e_{d,y} := \{(k,m) \mid k = 0,1,\dots,K,\ m = -1, M+1\},$$

Using the abbreviation $f(n)$ for $f(n,k,m)$, our scheme corresponding to (2.1) and (2.2) is as follows: for $n = 0,1,\dots,N-1$

$$\delta_t^{\langle 2\rangle}\begin{pmatrix} U_x(n) \\ U_y(n) \end{pmatrix} - Q_d\begin{pmatrix} \frac{U_x(n+1)+U_x(n)}{2} \\ \frac{U_y(n+1)+U_y(n)}{2} \end{pmatrix} - \nu\delta_t^{\langle 1\rangle} Q_d \begin{pmatrix} U_x(n) \\ U_y(n) \end{pmatrix} = B_d \begin{pmatrix} \overline{\varepsilon}_{xx}(\mathcal{X}(n+1)) \\ \overline{\varepsilon}_{yy}(\mathcal{X}(n+1)) \\ \overline{\varepsilon}_{xy}(\mathcal{X}(n+1)) \end{pmatrix}, \qquad (k,m) \in \Omega_d, \quad (2.3)$$

$$\delta_t^+ \mathcal{X}(n) = \Delta_d P(n), \quad (2.4)$$

$$P(n) = -\gamma\Delta_d\left(\frac{\mathcal{X}(n+1)+\mathcal{X}(n)}{2}\right) + \psi'\left(\frac{\mathcal{X}(n+1)+\mathcal{X}(n)}{2}\right) + \frac{1}{4}\sum_{\mathcal{E}\in\Lambda(n)} \frac{\partial w}{\partial \chi}\left(\frac{\mathcal{X}(n+1)+\mathcal{X}(n)}{2}, \mathcal{E}\right), \qquad (k,m) \in \overline{\Omega}_d, \quad (2.5)$$

where Q_d, B_d, Δ_d and $\Lambda(n)$ are defined by

$$Q_d = \begin{pmatrix} (\lambda+2\mu)\delta_x^{\langle 2\rangle} + \mu\delta_y^{\langle 2\rangle} & (\lambda+\mu)\delta_{xy}^{\langle 2\rangle} \\ (\lambda+\mu)\delta_{xy}^{\langle 2\rangle} & (\lambda+2\mu)\delta_y^{\langle 2\rangle} + \mu\delta_x^{\langle 2\rangle} \end{pmatrix},$$

$$B_d = \begin{pmatrix} (\lambda+2\mu)\delta_x^{\langle 1\rangle} & \lambda\delta_x^{\langle 1\rangle} & 2\mu\delta_y^{\langle 1\rangle} \\ \lambda\delta_y^{\langle 1\rangle} & (\lambda+2\mu)\delta_y^{\langle 1\rangle} & 2\mu\delta_x^{\langle 1\rangle} \end{pmatrix},$$

$$\Delta_d = \delta_x^{\langle 2\rangle} + \delta_y^{\langle 2\rangle}, \qquad \Lambda(n) := \{\mathcal{E}^+(n), \mathcal{E}^-(n), \mathcal{E}^+(n-1), \mathcal{E}^-(n-1)\},$$

$$\mathcal{E}^\pm(n) = \mathcal{E}^\pm(\boldsymbol{U}(n)) = \begin{pmatrix} \delta_x^\pm U_x(n) & \frac{1}{2}\left(\delta_x^\pm U_y(n) + \delta_y^\pm U_x(n)\right) \\ \frac{1}{2}\left(\delta_x^\pm U_y(n) + \delta_y^\pm U_x(n)\right) & \delta_y^\pm U_y(n) \end{pmatrix}.$$

Observe that for $\nabla_d^\pm := \begin{pmatrix} \delta_x^\pm \\ \delta_y^\pm \end{pmatrix}$ the following relation holds

$$Q_d\begin{pmatrix} U_x \\ U_y \end{pmatrix} = \frac{1}{2}(\nabla_d^+ \cdot A\mathcal{E}_- + \nabla_d^- \cdot A\mathcal{E}_+),$$

The discrete boundary conditions are as follows:

$$\boldsymbol{U}(n,k,m) = \mathbf{0} \qquad (k,m) \in \partial\Omega_d, \quad n = 0,1,\ldots,N, \tag{2.6}$$

$$\begin{cases} s_x^{\langle 1\rangle}\boldsymbol{U}(n,k,m) = \mathbf{0} & (k,m) \in \partial\Omega_{d,x}, \\ s_y^{\langle 1\rangle}\boldsymbol{U}(n,k,m) = \mathbf{0}, & (k,m) \in \partial\Omega_{d,y}, \end{cases} \quad n = 0,1,\ldots,N, \tag{2.7}$$

$$\begin{cases} \delta_x^{\langle 1\rangle}\mathcal{X}(n,k,m) = \delta_x^{\langle 1\rangle}P(n,k,m) = 0 & (k,m) \in \partial\Omega_{d,x}, \\ \delta_y^{\langle 1\rangle}\mathcal{X}(n,k,m) = \delta_y^{\langle 1\rangle}P(n,k,m) = 0 & (k,m) \in \partial\Omega_{d,y}, \end{cases} \quad n = 0,1,\ldots,N, \tag{2.8}$$

where $\boldsymbol{U} = (U_x, U_y)^T$, $s_x^{\langle 1\rangle}f(k,m) := \frac{f(k+1,m)+f(k-1,m)}{2}$, $s_y^{\langle 1\rangle}f(k,m) := \frac{f(k,m+1)+f(k,m-1)}{2}$, and the initial conditions are

$$\begin{aligned} &\boldsymbol{U}(0,k,m) = \boldsymbol{u}_0(k\Delta x, m\Delta y), \\ &\boldsymbol{U}(-1,k,m) = (\boldsymbol{u}_0 - \Delta t\boldsymbol{v}_0)(k\Delta x, m\Delta y) \quad \text{on } \Omega_d, \\ &\mathcal{X}(0,k,m) = \chi_0(k\Delta x, m\Delta y) \quad \text{on } \overline{\Omega_d}. \end{aligned} \tag{2.9}$$

There are $4\{(K+1)(M+1) + 2(K+1) + 2(M+1)\}$ unknowns in the system (2.3)–(2.5). Indeed, there are $(K+1)(M+1)$ lattice points on $\overline{\Omega}_d$, in addition, $2(K+1) + 2(M+1)$ lattice points on $\Omega^e_{d,x}$ and $\Omega^e_{d,y}$ required in (2.4) and (2.5), and there are 4 unknowns U_x, U_y, $\mathcal{X}$ and P. There are the same number of equations (2.3)–(2.8) as the unknowns, because there are $2(K-1)(M-1)$ equations in (2.3), $2(K+1)(M+1)$ equations in (2.4) and (2.5), $4K + 4M$ equations in (2.6), $4(K+1) + 4(M+1)$ equations in (2.7), and $4(K+1) + 4(M+1)$. t In the actual numerical calculation, we first obtain $\mathcal{X}(1)$ by solving the nonlinear equations (2.4) and (2.5) with (2.8) at $n = 0$ for given $\mathcal{X}(0)$, $\boldsymbol{U}(0)$ and $\boldsymbol{U}(-1)$, where we need to fill their exterior values by (2.6)–(2.8) beforehand. Next, we pursue $\boldsymbol{U}(1)$ in Ω_d by solving the linear equation (2.3) for given $\boldsymbol{U}(0)$ and $\boldsymbol{U}(-1)$ and $\mathcal{X}(1)$ obtained in the previous step. Lastly, with the help of the boundary conditions (2.6) and (2.7) again, we fill the exterior and boundary values $\boldsymbol{U}(1)$ in $\partial\Omega \cup \partial\Omega^e_{d,x} \cup \partial\Omega^e_{d,y}$. In the same fashion, repeating the procedures from top to bottom step by step:

- Solve the nonlinear equation (2.4) at n by the iterative method introduced later, and get the value $\mathcal{X}(n+1)$ in $\overline{\Omega}_d$ from known values $\mathcal{X}(n)$, $\boldsymbol{U}(n)$ and $\boldsymbol{U}(n-1)$.

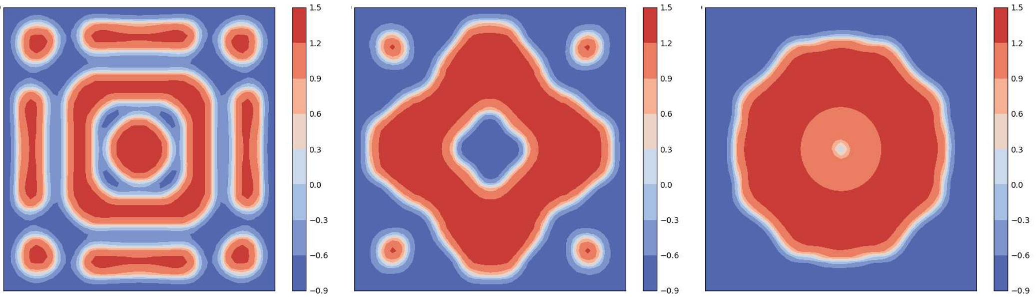
- Solve the linear equation (2.3) at n, and get the value $\boldsymbol{U}(n+1)$ in Ω_d from known values $\boldsymbol{U}(n)$ and $\boldsymbol{U}(n-1)$ and $\mathcal{X}(n+1)$.

- Fill the exterior and boundary values $U(n+1)$ on $\partial\Omega_d \cup \partial\Omega^e_{d,x} \cup \partial\Omega^e_{d,y}$ by (2.6) and (2.7).

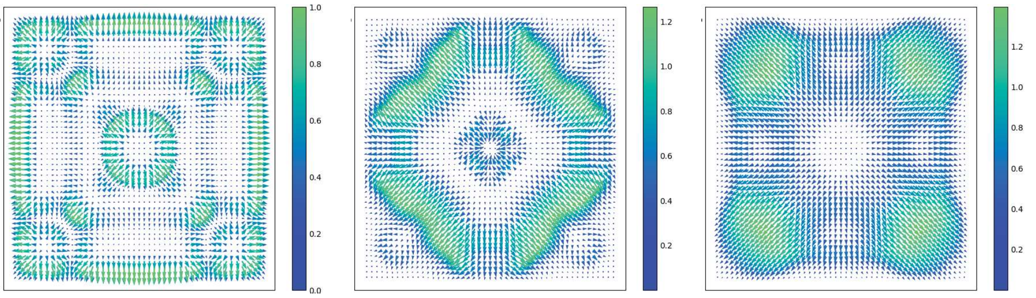
3 Numerical Simulation

Although we will give several numerical results in the actual talk, in this abstract we restrict ourselves to give only a figure of numerical simulations for the scheme.

Set the initial value as $\boldsymbol{u}_0 = \boldsymbol{0}$, $\boldsymbol{v}_0 = \boldsymbol{0}$ and $\chi_0 = \frac{1}{2}\sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi y}{2}\right)$, the domain as $\Omega = [0,1]\times[0,1]$, and the eigen-strains as $\overline{\varepsilon}_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\overline{\varepsilon}_b = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.



χ at $T = 0.25$ $\qquad$ χ at $T = 0.5$ $\qquad$ χ at $T = 2$



$\boldsymbol{u}$ at $T = 0$ $\qquad$ $\boldsymbol{u}$ at $T = 0.5$ $\qquad$ $\boldsymbol{u}$ at $T = 2$

4 Mathematical Results

Applying the energy method introduced in [12] and [13], we can show the existence of unique solution for the numerical scheme (2.3)–(2.9)

Theorem 4.1 (Existence of Unique Solution)**.** *For any given initial data satisfying* $\chi_0, \mathcal{E}^{\pm}(0), \mathcal{E}^{\pm}(-1), \boldsymbol{v}_0 \in L^2_d$*, there exists some constant* C_{ex} *determined by the initial data such that if* $\Delta t < C_{ex}$ *there exists a unique solution* $(\boldsymbol{U}(n), \mathcal{X}(n))$ *satisfying the difference equations* (2.3) *and* (2.4) *under the boundary conditions* (2.5) *and* (2.6) *and the initial condition* (2.7)*.*

In a similar manner, we can also prove the error estimate between the exact solution for (1.1)–(1.5) and the solution for (2.3)–(2.9) in the following sense.

Theorem 4.2 (Error Estimate)**.** *Denote the errors* $U_x(n,k,m) - u_x(n\Delta t, k\Delta x, m\Delta y)$, $U_y(n,k,m) - u_y(n\Delta t, k\Delta x, m\Delta y)$ *and* $\mathcal{X}(n,k,m) - \chi(n\Delta t, k\Delta x, m\Delta y)$ *by* $e_{ux}(n,k,m)$, $e_{uy}(n,k,m)$ *and* $e_\chi(n,k,m)$*, respectively. Assume that* (1.1)*–*(1.5) *has sufficiently smooth solution* (u, χ) *and*

$$\|\mathcal{X}(n)\|_{L^\infty_d}, \quad \|\varepsilon(n)\|_{L^\infty_d} \le C_{bdd}.$$

Then for some constant $C_{err} = C_{err}(L, T, C_{bdd})$*, if* $\Delta t < C_{err}$*, it holds that*

$$\|\delta_t^+ \boldsymbol{e}_u(n)\|_{L_d^2} + \|\boldsymbol{e}_\varepsilon(n)\|_{L_d^2} + \|e_\chi(n)\|_{L_d^2} \leq C(\Delta t + \Delta x^2 + \Delta y^2), \qquad n = 1, 2, \ldots, N,$$

where $\boldsymbol{e}_u(n) := (e_{ux}, e_{uy})^T$ *and* $\|\boldsymbol{e}_\varepsilon\|_{L_d^2}^2 := \frac{\|e_\varepsilon^+\|_{L_d^2}^2 + \|e_\varepsilon^-\|_{L_d^2}^2}{2}$ *with*

$$e_\varepsilon^\pm := \mathcal{E}^\pm(\boldsymbol{e}_u(n)) = \begin{pmatrix} \delta_x^\pm e_{ux}(n) & \frac{1}{2}\left(\delta_x^\pm e_{uy}(n) + \delta_y^\pm e_{ux}(n)\right) \\ \frac{1}{2}\left(\delta_x^\pm e_{uy}(n) + \delta_y^\pm e_{ux}(n)\right) & \delta_y^\pm e_{uy}(n) \end{pmatrix}.$$

For more detail, we refer to the original article [1].

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