LONG TIME BEHAVIOR FOR DISSIPATIVE NONLINEAR SCHRÖDINGER EQUATIONS IN ANALYTIC SPACES

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Abstract. This paper studies the Cauchy problem for the nonlinear Schrödinger equation with a power-type nonlinearity $\lambda|u|^{p-1}u$. It is known that $p = 1+2/n$ is the critical exponent for the L^2-decay of solutions under the dissipative condition $\operatorname{Im}\lambda < 0$. In this work, we demonstrate that for small initial data in the subcritical case $p > 1 + 2/n$, solutions maintain a positive L^2-norm lower bound for all positive times. Moreover, in the critical case $p = 1 + 2/n$, we clarify that the L^2-decay rate of solutions depends on the smoothness of the initial data.

1. Introduction

We first consider the Cauchy problem for the nonlinear Schrödinger equation with a p-th power-type nonlinearity

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^{p-1}u, & t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ u(0) = u_0, & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $p > 1$, $\lambda \in \mathbb{C}$, $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ is the unknown function and $u_0 : \mathbb{R}^n \to \mathbb{C}$ is the initial data. We denote by ∂_t the partial derivative with respect to t, and by Δ the n-dimensional Laplacian.

When we study the Cauchy problem (1.1), it is natural to work in the energy spaces $L^2(\mathbb{R}^n)$ or $H^1(\mathbb{R}^n)$, since the problem (1.1) has the conservation laws:

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad t \in \mathbb{R}, \tag{1.2}$$

and

$$E[u(t)] \equiv \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{\lambda}{p+1}\|u(t)\|_{L^{p+1}}^{p+1} = E[u_0], \quad t \in \mathbb{R}. \tag{1.3}$$

Here, $L^2(\mathbb{R}^n)$ denotes the Banach space of square-integrable functions with respect to the Lebesgue measure in $x \in \mathbb{R}^n$, and $H^1(\mathbb{R}^n)$ consists of functions $f \in L^2(\mathbb{R}^n)$ such that the spatial derivatives of f of order one exist and belong to $L^2(\mathbb{R}^n)$.

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For $\lambda \in \mathbb{C} \setminus \{0\}$, the local well-posedness of (1.1) was established by Ginibre–Velo [10, 11], Kato [21], and Yajima [47] (see also [2], [7], [8]): If $1 < p < 1 + 4/(n-2)$, then for any $u_0 \in H^1(\mathbb{R}^n)$, there exist $T = T(\|u_0\|_{H^1}) > 0$ and a unique solution $u \in C([0,T); H^1(\mathbb{R}^n))$ to (1.1).

In the case where $\lambda \in \mathbb{C}$ and $\operatorname{Im}\lambda \leq 0$, Tsutsumi [45] showed that the Cauchy problem (1.1) is locally well-posed in $L^2(\mathbb{R}^n)$ and, using the mass conservation law (1.2), the local solution is extended globally in time when $1 < p < 1 + 4/n$. Cazenave–Weissler [8] generalized the result [45] to the Sobolev space $H^s(\mathbb{R}^n)$ for $s \geq 0$.

These results are obtained in the framework of the theory of evolution equations using the semigroup theory developed by Hille–Yosida, and a common idea is to introduce the notion of mild solutions: u is called a mild solution to (1.1) if

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)|u(s)|^{p-1}u(s)\,ds$$

in $C([0,T); L^2(\mathbb{R}^n))$, where $U(t) = e^{i\frac{t}{2}\Delta}$ is the free Schrödinger evolution operator.

2. L^2 lower bound of solutions to the dissipative nonlinear Schrödinger equation

In this section, we consider the global dynamics of solutions to (1.1) under the dissipative condition $\operatorname{Im}\lambda < 0$. We first review some known facts on the asymptotic behavior of solutions to (1.1) in the non-dissipative case. In this case, the exponent $p = 1 + 2/n$ is known to be the borderline for the scattering theory. Namely, when $p > 1 + 2/n$, the solution to (1.1) behaves like a free solution for large time, while there is no scattering state in the case $p \leq 1 + 2/n$ (see for details [1], [11], [17], [37], [46]). The limiting case $p = 1 + 2/n$ exhibits the so-called *nonlinear long-range scattering*, discovered by Ozawa [36] (cf. [9], [14], [35]).

For the case $\operatorname{Im}\lambda < 0$, there are several works on the large time behavior of dissipative solutions (see [3], [4], [5] and [16]), and the exponent $p = 1 + 2/n$ plays a critical role not for scattering, but for the large time behavior. Cazenave–Naumkin [6] and Hoshino [19] showed that there exists a scattering state $u_+ \in L^2(\mathbb{R}^n)$ for the dissipative solution $u(t)$ to (1.1) when $p > 1 + 2/n$, i.e.,

$$\|u(t) - U(t)u_+\|_{L^2} \to 0 \quad \text{as } t \to \infty. \tag{2.1}$$

However, the mass (L^2 norm) of solutions to (1.1) with $\operatorname{Im}\lambda < 0$ monotonically decreases because of the nonlinear dissipation, namely the dissipative solution satisfies

$$\|u(t)\|_{L^2}^2 + |\operatorname{Im}\lambda| \int_0^t \|u(\tau)\|_{L^{p+1}}^{p+1}\,d\tau = \|u_0\|_{L^2}^2, \quad t \geq 0. \tag{2.2}$$

Thus the solution has both a dispersive and a dissipative nature. Because of the dissipative structure (2.2), it is not *a priori* clear that the scattering state $u_+ \in L^2(\mathbb{R}^n)$ in (2.1) is nontrivial. We therefore show that the mass of dissipative solutions has a positive lower bound as $t \to \infty$, which ensures not only the existence of a nonzero scattering state u_+ but also shows that $p = 1+2/n$ is the critical exponent governing the large time behavior of dissipative solutions to (1.1) in the L^2 topology.

To state the main theorems, we define several functional spaces. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing functions. For any $f \in \mathcal{S}(\mathbb{R}^n)$, we define the Fourier transform and the inverse Fourier transform by

$$\begin{aligned} \mathcal{F}[f](\xi) = \widehat{f}(\xi) &\equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \\ \mathcal{F}^{-1}[f](x) &\equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\, d\xi. \end{aligned} \tag{2.3}$$

For $s > 0$ and $r > 0$, we define

$$H^s_r(\mathbb{R}^n) \equiv \Big\{ f \in H^s(\mathbb{R}^n);\ \|f\|_{H^s_r} \equiv \|\langle x\rangle^r \langle\nabla\rangle^s f\|_{L^2} < \infty \Big\},$$

where $\langle x\rangle \equiv (1+|x|^2)^{1/2}$ and $\langle\nabla\rangle^s f \equiv \mathcal{F}^{-1}[\langle\xi\rangle^s \widehat{f}\,]$. We also write $H^0_r(\mathbb{R}^n) = L^2_r(\mathbb{R}^n)$. Here, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are defined by (2.3).

We now state that any small solution to (1.1) with the dissipative nonlinearity has a positive L^2 lower bound when $p > 1 + 2/n$.

Theorem 2.1 ([42])**.** *Let $1 \le n \le 3$, $1+2/n < p < p(n)$, where $p(n) = \infty$ if $n = 1, 2$ and $p(n) = 1 + 4/(n-2)$ if $n = 3$, and let $n/2 < s < p$, and assume* $\operatorname{Im}\lambda < 0$*. Then there exists a small $\delta_0 > 0$ such that for any nontrivial $u_0 \in H^s(\mathbb{R}^n) \cap L^2_s(\mathbb{R}^n)$ with $\|u_0\|_{H^s} + \||x|^s u_0\|_{L^2} < \delta_0$, the Cauchy problem* (1.1) *has a global solution $u \in C([0,\infty); H^s(\mathbb{R}^n))$ such that $J(t)^s u \in C([0,\infty); L^2(\mathbb{R}^n))$, where $J(t) = x + it\nabla$, and it satisfies*

$$\sup_{t\ge 0} \Big(\|u(t)\|_{H^s} + \|J(t)^s u(t)\|_{L^2} \Big) \le C\big(\|u_0\|_{H^s} + \||x|^s u_0\|_{L^2}\big). \tag{2.4}$$

Moreover, the solution $u(t)$ satisfies the following lower bound:

$$\|u(t)\|_{L^2} \ge C, \quad t \ge 0, \tag{2.5}$$

where $C > 0$ is independent of t.

In the previous work [41], we needed to control the nonlinear dissipation by the size of the initial data in order to obtain an L^2 lower bound of solutions to (1.1). Indeed the solution is estimated by

$$\|u(t)\|^2_{L^2} \ge \|u_0\|^2_{L^2} - C\big(\|u_0\|_{H^1} + \||x|^s u_0\|_{L^2}\big)^{p+1}, \tag{2.6}$$

and we restricted the initial condition to the form $u_0 = \delta v_0$ for small $\delta > 0$ in order to obtain positivity of the right-hand side of (2.6). In Theorem 2.1 we remove this restriction on the initial data, and we show that the solution has a uniform lower bound for general small initial data (cf. [24], [28] and [32]).

We next define an energy of solutions to (1.1) with a complex coefficient λ as follows.

$$E[u(t)] \equiv \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(t,x)|^2\,dx + \frac{\operatorname{Re}\lambda}{p+1}\int_{\mathbb{R}^n} |u(t,x)|^{p+1}\,dx, \quad t \ge 0. \tag{2.7}$$

Large dissipative solutions to (1.1) also have a positive L^2 lower bound by adding a special oscillation to the initial datum.

Theorem 2.2 ([41, 42])**.** *Let $n \ge 1$, $1+2/n < p < 1+4/(n-2)$, $\operatorname{Re}\lambda > 0$ and $\operatorname{Im}\lambda < 0$. For any nontrivial $v_0 \in H^1(\mathbb{R}^n) \cap L^2_1(\mathbb{R}^n)$ and*

$$b_0 > \begin{cases} \dfrac{4|\operatorname{Re}\lambda|(p+1)E[v_0]}{n|\operatorname{Im}\lambda|(p-1-2/n)\|v_0\|_{L^2}^2}, & 1+2/n < p < 1+4/n, \\ \dfrac{2|\operatorname{Re}\lambda|(p+1)E[v_0]}{|\operatorname{Im}\lambda|\|v_0\|_{L^2}^2}, & 1+4/n \le p < 1+4/(n-2), \end{cases} \tag{2.8}$$

the Cauchy problem (1.1) *with $u_0 = e^{i\frac{|x|^2}{4}b_0} v_0$ has a unique global solution*

$$u \in C([0,\infty); H^1(\mathbb{R}^n)).$$

Moreover, the solution satisfies, for any $t \ge 0$,

$$\|u(t)\|_{L^2} \ge \sqrt{B_0},$$

where $B_0 > 0$ is given by

$$B_0 = \begin{cases} \|v_0\|_{L^2}^2 - \dfrac{4|\operatorname{Re}\lambda|(p+1)E[v_0]}{n|\operatorname{Im}\lambda|(p-1-2/n)b_0}, & 1+2/n < p < 1+4/n, \\ \|v_0\|_{L^2}^2 - \dfrac{2|\operatorname{Re}\lambda|(p+1)E[v_0]}{|\operatorname{Im}\lambda|b_0}, & 1+4/n \le p < 1+4/(n-2). \end{cases} \tag{2.9}$$

From Theorems 2.1 and 2.2, the dissipative solution to (1.1) has an L^2 lower bound for $p > 1+2/n$, while the solution to (1.1) with $p \le 1+2/n$ decays in L^2:

$$\|u(t)\|_{L^2} \to 0, \qquad t \to \infty,$$

which is obtained by Kita–Shimomura [25, 26] (cf. Shimomura [39] and Sunagawa [38], see also [29] and [33]). Thus Theorems 2.1 and 2.2 imply that $p = 1+2/n$ is a *critical exponent for the L^2-decay* of dissipative solutions to (1.1).

We prepare several lemmas to prove the main theorems.

Ozawa [36], Hayashi-Naumkin [14], Katayama–Li–Sunagawa [22], Kim [23] and Kita–Shimomura [25, 26] proved the following decay estimates.

Lemma 2.3 ([36], [14], [14], [22], [23], [25], [26])**.** *Let $n \ge 1$, $s > n/2$, $p > 1+2/n$, and $\tilde\gamma > 0$ be sufficiently small. Assume that $u \in \mathcal{S}(\mathbb{R}^n)$. Then there exists $C > 0$ such that for any $t \ge 1$,*

$$\|f\|_{L^\infty} \le t^{-\frac{n}{2}}\|\mathcal{F}[U^{-1}f]\|_{L^\infty} + C\,t^{-\frac{n}{2}-\tilde\gamma}\|J^s(t)f\|_{L^2}, \tag{2.10}$$

and

$$\left\| \mathcal{F}[U^{-1}(|u|^{p-1}u)] - t^{-\frac{n(p-1)}{2}} |\mathcal{F}[U^{-1}u]|^{p-1} (\mathcal{F}[U^{-1}u]) \right\|_{L^\infty \cap L^2} \le C\, t^{-\frac{n(p-1)}{2} - \tilde{\gamma}} \big(\|u\|_{L^2} + \|J^s(t)u\|_{L^2} \big)^p, \tag{2.11}$$

where $J(t) = x + it\nabla$ *is the generator of the Galilei transformation and*

$$\| \cdot \|_{L^\infty \cap L^2} \equiv \| \cdot \|_{L^\infty} + \| \cdot \|_{L^2}.$$

Lemma 2.4. *Let* $n \ge 1$, $s > n/2$. *There exists* $C > 0$ *such that for any* $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\|f\|_{L^2}^2 \le C \| |\nabla|^s f \|_{L^2} \| |x|^s f \|_{L^2}. \tag{2.12}$$

Proof of Lemma 2.4. It easily follows from the Hölder inequality and the fractional Hardy inequality that

$$\|f\|_{L^2}^2 \le \| |x|^{-s} f \|_{L^2} \| |x|^s f \|_{L^2} \le C \| |\nabla|^s f \|_{L^2} \| |x|^s f \|_{L^2}$$

for some $C > 0$ depending only on s. ❑

We now give the proof of Theorem 2.1.

Proof of Theorem 2.1. The upper bound for small solutions has already been obtained in the previous works [22], [34] and [40]. Let u be the solution to (1.1) in $C([0,\infty); L^2(\mathbb{R}^n))$ and set $U(t) \equiv e^{i\frac{t}{2}\Delta}$. Then $\mathcal{F}[U^{-1}u](t,\xi)$ satisfies

$$i\partial_t \mathcal{F}[U^{-1}u](t,\xi) = \frac{\lambda}{t^{\frac{n}{2}(p-1)}} |\mathcal{F}[U^{-1}u](t,\xi)|^{p-1} \mathcal{F}[U^{-1}u](t,\xi) + R(t,\xi), \tag{2.13}$$

where

$$R(t,\xi) = \lambda \mathcal{F}[U^{-1}(|u|^{p-1}u)] - \lambda t^{-\frac{n}{2}(p-1)} |\mathcal{F}[U^{-1}u](t,\xi)|^{p-1} \mathcal{F}[U^{-1}u](t,\xi).$$

Multiplying (4.3) by $\overline{\mathcal{F}[U^{-1}u](t,\xi)}$ and taking the imaginary part, we obtain

$$\frac{1}{2}\partial_t |\mathcal{F}[U^{-1}u](t,\xi)|^2 = -\frac{|\mathrm{Im}\,\lambda|}{t^{\frac{n}{2}(p-1)}} |\mathcal{F}[U^{-1}u](t,\xi)|^{p+1} + \mathrm{Im}\{R(t,\xi)\overline{\mathcal{F}[U^{-1}u](t,\xi)}\}. \tag{2.14}$$

By the decay estimate associated with the free Schrödinger evolution (2.11), $R(t,\xi)$ satisfies, for sufficiently small $\tilde{\gamma} > 0$,

$$\|R(t,\cdot)\|_{L^\infty} \le C M^p\, t^{-\frac{n}{2}(p-1) - \tilde{\gamma}}, \quad t \ge 1, \tag{2.15}$$

where $M = \|u_0\|_{H^s} + \| |x|^s u_0 \|_{L^2}$. Using Hölder's inequality and (2.15), we obtain

$$\frac{1}{2}\partial_t |\mathcal{F}[U^{-1}u](t,\xi)|^2 \le -\frac{|\mathrm{Im}\,\lambda|}{t^{\frac{n}{2}(p-1)}} |\mathcal{F}[U^{-1}u](t,\xi)|^{p+1} + C M^{p+1}\, t^{-\frac{n}{2}(p-1) - \tilde{\gamma}}. \tag{2.16}$$

Hence, by a similar argument to that in [28], we have $\|\mathcal{F}[U^{-1}u](t)\|_{L^\infty} \le CM$ and, by (2.10), for any $t \ge 0$,

$$\|u(t)\|_{L^\infty} \le CM(1+t)^{-\frac{n}{2}} + CM(1+t)^{-\frac{n}{2} - \tilde{\gamma}}, \tag{2.17}$$

where $C > 0$ is a constant. Combining (2.17) and the L^2-dissipative identity:

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 - 2|\mathrm{Im}\,\lambda| \int_0^t \|u(\tau)\|_{L^{p+1}}^{p+1}\, d\tau, \quad t \geq 0, \tag{2.18}$$

we deduce that

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_{L^2}^2 \geq & -2|\mathrm{Im}\,\lambda| \|u(t)\|_{L^\infty}^{p-1} \|u(t)\|_{L^2}^2 \\ \geq & -C\big(M(1+t)^{-\frac{n}{2}} + M(1+t)^{-\frac{n}{2}-\tilde{\gamma}}\big)^{p-1} \|u(t)\|_{L^2}^2, \end{aligned} \tag{2.19}$$

where, as above, $M \equiv \|u_0\|_{H^s} + \||x|^s u_0\|_{L^2}$. By solving (2.19), one obtains a positive L^2 lower bound for small solutions to (1.1), since $p > 1 + 2/n$. ❑

3. L^2-decay of dissipative solutions

In this section, we show a relation between the L^2-decay rate and the smoothness of dissipative solutions to (1.1) under the critical setting $p = 1 + 2/n$. Hayashi–Li–Naumkin [15] obtained an L^2-decay rate of dissipative solutions with $H^1(\mathbb{R}^n)$ regularity (see also [12, 13, 27]). For any $t \geq 0$ and $\varepsilon > 0$,

$$\|u(t)\|_{L^2} \leq \begin{cases} C_\varepsilon (1+t)^{-(\frac{1}{p-1} - \frac{n}{2})\frac{2}{n+2} + \varepsilon}, & p < 1 + 2/n, \\ C_\varepsilon \{\log(1+t)\}^{-\frac{n}{n+2}+\varepsilon}, & p = 1 + 2/n, \end{cases} \tag{3.1}$$

where $C_\varepsilon \to \infty$ as $\varepsilon \to 0$. Hoshino [20] showed the L^2-decay order of dissipative solutions in the Sobolev space $H^r(\mathbb{R}^n)$ for $0 < r < 1$. Ogawa–Sato [34] and Sato [40] showed L^2-decay rates for smooth solutions in higher order Sobolev spaces.

Proposition 3.1. *Let $n = 1, p = 3$ and $k \in \mathbb{N}$. Then, there exists a small constant $\varepsilon_0 > 0$ such that for any $u_0 \in H^{k+1}(\mathbb{R}) \cap H_1^k(\mathbb{R})$ with $\|u_0\|_{H^{k+1}} + \|u_0\|_{H_1^k} \leq \varepsilon_0$, the Cauchy problem* (1.1) *has a unique global solution $u(t,x)$ satisfying*

$$u \in C([0,\infty); H^{k+1}(\mathbb{R})), \quad J(t)\partial_x^k u \in C([0,\infty); L^2(\mathbb{R})).$$

Moreover, there exists $C > 0$ such that for any $t \geq 0$,

$$\|u(t)\|_{L^2} \leq C(\log t)^{-\frac{1}{2} + \frac{1}{2(2k+1)}}. \tag{3.2}$$

We can deduce Proposition 3.1 by combining the argument in [34] with the proof of Theorem 3.2.

We remark that the estimate (3.2) generalizes the result (3.1) by [15] and [20] for $k \geq 1$. Namely, higher regularity of the solution yields faster decay in the critical dissipative case. Since the regularity index can be taken arbitrarily large, one can expect that the decay rate of the solution to (1.1) should be $(\log t)^{-\frac{1}{2}+\varepsilon}$ for $0 < \varepsilon \ll 1$ as $k \to \infty$. Indeed, if dissipative solutions belong to the Gevrey class, which lies between the C^∞ class and the analytic class, defined by

$$G_v^s(\mathbb{R}) \equiv \left\{ f \in L^2(\mathbb{R}); \|f\|_{G_v^s} \equiv \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k f\|_{L^2} < \infty \right\}, \tag{3.3}$$

then the solution shows a slightly faster L^2-decay than dissipative solutions with only H^k regularity.

Theorem 3.2 ([34, 40, 42]). *Let $s \geq 1$ and consider the Cauchy problem* (1.1) *with the cubic power-type nonlinearity*

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda |u|^2 u, \qquad (t,x) \in \mathbb{R} \times \mathbb{R},$$

where $\lambda \in \mathbb{C}$ and $\operatorname{Im} \lambda < 0$. Then, there exists a small constant $\varepsilon_0 > 0$ such that for any $u_0 \in G_v^s(\mathbb{R})$ with

$$\|\partial_x u_0\|_{G_v^s} + \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|x \partial_x^k u_0\|_{L^2} \leq \varepsilon_0,$$

the Cauchy problem (1.1) *has a unique global solution u in $C([0,\infty); G_v^s(\mathbb{R}))$. Moreover, the solution satisfies*

$$\partial_x u \in C([0,\infty); G_v^s(\mathbb{R})), \quad \sum_{k=0}^{\infty} \frac{1}{(k!)^s} J(t)\partial_x^k u \in C([0,\infty); L^2(\mathbb{R}))$$

and there exists $C > 0$ such that

$$\|u(t)\|_{L^2} \leq C(\log\log t)^{\frac{s}{2}}(\log t)^{-\frac{1}{2}} \tag{3.4}$$

for any $t \geq e^e$.

It was shown in [34, 40] that dissipative solutions to (1.1) with the cubic nonlinearity $|u|^2 u$ exhibit L^2-decay with the same order as in (3.4) when $n = 1$ and $p = 3$. Moreover, it was revealed in [43, 44] that the decay rate (3.4) is optimal; namely, an associated L^2-lower bound with the same decay order as in (3.4) was established. In particular, the regularity assumption on the initial data was relaxed in [44], where only an exponential decay assumption in the frequency domain is imposed, whereas [43] assumed exponential decay of the initial data in both the physical and frequency domains. Such an exponential profile of the initial data is preserved in the solution for any positive time. This property enables us to obtain sharp upper and lower L^2-decay estimates, since the dissipative problem is dominated by a Bernoulli-type ordinary differential equation as $t \to \infty$. Furthermore, the positivity of the solutions allows us to solve this Bernoulli-type ordinary differential equation directly, and consequently, we obtain a new sharp L^2-lower estimate.

In [41], it was shown that

$$\liminf_{t\to\infty} \|u(t)\|_{L^2} > 0,$$

where u is the solution to (1.1) with a power-type nonlinearity $|u|^{p-1}u$ with $p > 1 + 2/n$. Thus, for supercritical powers $p > 1 + 2/n$ the L^2-mass does not decay to zero, while in the critical cubic case $p = 1 + 2/n$ one still observes L^2-decay of solutions.

The decay rate in (3.4) is not expected to be optimal in general. Indeed, Kita–Sato [28] showed the existence of solutions to (1.1) such that $\|u(t)\|_{L^2} = O((\log t)^{-1/2})$ as $t \to \infty$

when $n = 1$ and $p = 3$, and it was proved that this rate is optimal (cf. Li–Nishii–Sagawa–Sunagawa [31]).

We remark that constructing global solutions in $G^s_v(\mathbb{R})$ for $s \in (0, 1)$ is difficult because of the estimate

$$\frac{k!}{k_1!k_2!k_3!} \le \Big(\frac{k!}{k_1!k_2!k_3!}\Big)^s,$$

which holds only for $s \ge 1$ and any $k_1, k_2, k_3, k \in \mathbb{N} \cup \{0\}$ with $k_1 + k_2 + k_3 = k$.

In the analytic case $s = 1$, Ogawa–Sato [34] proved the L^2-decay of dissipative solutions to (1.1) with $n = 1$ and $p = 3$ by applying the operator $e^{iy\partial_x}$ introduced by Hoshino [18] to obtain the analytic smoothing effect of solutions to nonlinear Schrödinger systems.

To show the L^2-decay rate for the dissipative solution to (1.1) with the cubic nonlinearity in the Gevrey class, we prove a global *a priori* estimate of the solution in the Gevrey class defined by (3.3). Let u be the solution to (1.1) on $[0, T]$. For $0 < \gamma \le 1/4$, we define

$$\|u\|_{X_T} \equiv \sup_{t\in[0,T]} \Big\{(1+t)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(t)\|_{L^\infty} + (1+t)^{-\gamma}\Big(\|\partial_x u(t)\|_{G^s_v} + \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|J(t)\partial_x^k u(t)\|_{L^2}\Big)\Big\}.$$

For $s \ge 1$, set

$$\varepsilon_s \equiv \|\partial_x u_0\|_{G^s_v} + \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|x\partial_x^k u_0\|_{L^2}. \tag{3.5}$$

Lemma 3.3. *There exists a positive constant C, independent of $T > 0$, such that*

$$\|u\|_{X_T} \le C\varepsilon_s, \tag{3.6}$$

provided that ε_s is sufficiently small.

Proof of Lemma 3.3. Let $s \ge 1$. To prove the estimate (3.6), it suffices to show that there exists a constant $C > 0$ such that

$$\|u\|_{X_T} \le C\Big(\varepsilon_s + \|u\|^3_{X_T}\Big). \tag{3.7}$$

The estimate (3.7) yields the uniform Gevrey bound (3.6) if we choose ε_s sufficiently small. The existence of the global solution to the Cauchy problem (1.1) is then an immediate consequence of the *a priori* bound (3.6) and the standard local existence theorem proved by Ginibre–Velo [10], Yajima [47], Cazenave–Weissler [7].

In what follows, we denote by C various positive constants. We first find estimates for

$$\|\partial_x u(t)\|_{G^s_v}, \quad \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|J(t)\partial_x^k u(t)\|_{L^2}.$$

By the standard energy estimate for (1.1) and the unitarity of the free Schrödinger group, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|\partial_x^k u(t)\|_{\dot H^1} \le & \sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|\partial_x^k u_0\|_{\dot H^1} \\ & + C\int_0^t \sum_{k=0}^{\infty}\frac{1}{(k!)^s}\Big\|\partial_x^k\big(|u(\tau)|^2u(\tau)\big)\Big\|_{\dot H^1}\,d\tau.\end{aligned} \tag{3.8}$$

By Leibniz's rule, the cubic nonlinearity can be expanded as

$$\partial_x^k(|u|^2u) = \sum_{k_1+k_2+k_3=k}\frac{k!}{k_1!k_2!k_3!}\,\partial_x^{k_1}u\,\overline{\partial_x^{k_2}u}\,\partial_x^{k_3}u,$$

and, using $s\ge 1$, we have

$$\begin{aligned}&\int_0^t \sum_{k=0}^{\infty}\frac{1}{(k!)^s}\Big\|\sum_{k_1+k_2+k_3=k}\frac{k!}{k_1!k_2!k_3!}\big(\partial_x^{k_1}u(\tau)\,\overline{\partial_x^{k_2}u(\tau)}\,\partial_x^{k_3}u(\tau)\big)\Big\|_{\dot H^1}\,d\tau \\ &\le C\int_0^t \Big(\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|\partial_x^k u(\tau)\|_{\dot H^1}\Big)\Big(\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|\partial_x^k u(\tau)\|_{L^\infty}\Big)^2 d\tau \\ &\le C\|u\|_{X_T}^3\int_0^t (1+\tau)^{-1+\gamma}d\tau \\ &\le C\|u\|_{X_T}^3(1+t)^{\gamma}.\end{aligned}$$

Hence, from (3.8) we obtain

$$\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|\partial_x^k u(t)\|_{\dot H^1} \le \varepsilon_s + C(1+t)^{\gamma}\|u\|_{X_T}^3. \tag{3.9}$$

By the representation $J(t)f = e^{i\frac{x^2}{2t}}\,it\partial_x\big(e^{-i\frac{x^2}{2t}}f\big)$, a similar argument yields

$$\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|J(t)\partial_x^k u(t)\|_{L^2} \le \varepsilon_s + C(1+t)^{\gamma}\|u\|_{X_T}^3. \tag{3.10}$$

We next consider the L^∞-bound for the solution:

$$(1+t)^{\frac12}\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|\partial_x^k u(t)\|_{L^\infty} \le C\varepsilon_s + C\|u\|_{X_T}^3 \tag{3.11}$$

for any $t\ge 0$. For $t\in[0,1]$, the standard Sobolev embedding and (3.8) yield

$$(1+t)^{\frac12}\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|\partial_x^k u(t)\|_{L^\infty} \le C\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\big(\|\partial_x^k u(t)\|_{L^2} + \|\partial_x^k u(t)\|_{\dot H^1}\big). \tag{3.12}$$

By applying (2.12), (3.8), and (3.10), we have

$$
\begin{aligned}
\|\partial_x^k u(t)\|_{L^2} &= \|U(-t)[\partial_x^k u](t)\|_{L^2} \\
&\le \Big(\|J(t)\partial_x^k u(t)\|_{L^2}\|\partial_x^k u(t)\|_{\dot H^1}\Big)^{\frac12} \\
&\le C(k!)^s\big(\varepsilon_s + C\|u\|_{X_T}^3\big).
\end{aligned} \tag{3.13}
$$

From (3.12) and (3.13), the estimate (3.11) holds for any $t \in [0,1]$. Hence it remains to prove (3.11) for $t \ge 1$.

Let u be the solution to (1.1). Then for any $k \in \mathbb{N}\cup\{0\}$,

$$
\begin{aligned}
i\partial_t \mathcal{F}[U(-t)\partial_x^k u](t,\xi) &= \mathcal{F}\Big[U(-t)\Big\{\partial_x^k\Big(i\partial_t u + \frac12\partial_x^2 u\Big)\Big\}\Big] \\
&= \lambda\mathcal{F}\Big[U(-t)\partial_x^k\big(|u|^2u\big)\Big].
\end{aligned} \tag{3.14}
$$

Following the normal form argument in [34], one can write

$$
\lambda\mathcal{F}\Big[U(-t)\partial_x^k\big(|u|^2u\big)\Big] = \frac{\lambda}{t}|\mathcal{F}(U(-t)u)|^2\,\mathcal{F}(U(-t)\partial_x^k u) + R_k,
$$

where $R_k = R_k(t,\xi)$ is the remainder defined by the difference between the exact nonlinearity and its leading order term, and R_k satisfies

$$
\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|R_k(t)\|_{L^\infty\cap L^2} \le C(1+t)^{-1-\tilde\gamma+3\gamma}\,\|u\|_{X_T}^3 \tag{3.15}
$$

for any $t \ge 1$ and $\tilde\gamma > 3\gamma$.

Multiplying (3.14) by $\overline{\mathcal{F}[U(-t)\partial_x^k u](t,\xi)}$ and taking the imaginary part, using $\operatorname{Im}\lambda < 0$, we obtain

$$
\partial_t|\mathcal{F}(U(-t)\partial_x^k u)(t,\xi)|^2 \le 2|R_k(t,\xi)|\,\big|\mathcal{F}(U(-t)\partial_x^k u)(t,\xi)\big|. \tag{3.16}
$$

This is equivalent to

$$
\partial_t\big|\mathcal{F}(U(-t)\partial_x^k u)(t,\xi)\big| \le |R_k(t,\xi)|. \tag{3.17}
$$

Integrating (3.17) over $[1,t]$ and summing in k, we obtain

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\big|\mathcal{F}(U(-t)\partial_x^k u)(t,\xi)\big| &\le \sum_{k=0}^{\infty}\frac{1}{(k!)^s}\big|\mathcal{F}(U^{-1}\partial_x^k u)(1,\xi)\big| \\
&\quad + \int_1^t\sum_{k=0}^{\infty}\frac{1}{(k!)^s}|R_k(\tau,\xi)|\,d\tau.
\end{aligned} \tag{3.18}
$$

By (3.10) and the representation $J(t) = U(t)xU(-t)$, we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\big|\mathcal{F}(U^{-1}\partial_x^k u)(1,\xi)\big| &\le \sum_{k=0}^{\infty}\frac{1}{(k!)^s}\int_{\mathbb{R}}\big|U^{-1}[\partial_x^k u](1,x)\big|\,dx\\ &\le C\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\Big(\int_{\mathbb{R}}\langle x\rangle^2\big|U^{-1}[\partial_x^k u](1,x)\big|^2\,dx\Big)^{\frac{1}{2}}\\ &\le C\big(\varepsilon_s+\|u\|_{X_T}^3\big).\end{aligned}\tag{3.19}$$

Combining (3.18), (3.15) and (3.19), we obtain

$$\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|\mathcal{F}(U(-t)\partial_x^k u)(t)\|_{L^\infty} \le C\big(\varepsilon_s+\|u\|_{X_T}^3\big).$$

By (2.10) in Lemma 2.3,

$$(1+t)^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|\partial_x^k u(t)\|_{L^\infty} \le C\big(\varepsilon_s+\|u\|_{X_T}^3\big)\tag{3.20}$$

for any $t \ge 1$. Combining (3.9), (3.10), (3.11) and (3.20), we conclude (3.7), which completes the proof. ❑

We now prove the uniform bound of dissipative solutions to (1.1) in the Gevrey class by using the estimate (3.7).

Lemma 3.4. *Let u be the solution to* (1.1) *satisfying* (3.7). *Then there exists $C > 0$ such that for any $t \ge 0$,*

$$\|u(t)\|_{G_v^s} \le C\varepsilon_s,\tag{3.21}$$

where ε_s is sufficiently small, and is given by (3.5).

Proof of Lemma 3.4. From Plancherel's theorem and the unitarity of $U(t)$, we see that for any $f \in L^2(\mathbb{R})$, $\|\mathcal{F}(U(-t)f)(t)\|_{L^2} = \|f\|_{L^2}$. Hence, by integrating (3.16) over $\mathbb{R}_\xi \times [1,t]$ and applying Hölder's inequality to the remainder term, we obtain for any $k \in \mathbb{N}\cup\{0\}$ and $t \ge 1$,

$$\|\partial_x^k u(t)\|_{L^2}^2 \le \|\partial_x^k u(1)\|_{L^2}^2 + \int_1^t \|R_k(\tau)\|_{L^2}\|\partial_x^k u(\tau)\|_{L^2}\,d\tau,\tag{3.22}$$

and thus

$$\|\partial_x^k u(t)\|_{L^2} \le \|\partial_x^k u(1)\|_{L^2} + \Big(\int_1^t \|R_k(\tau)\|_{L^2}\|\partial_x^k u(\tau)\|_{L^2}\,d\tau\Big)^{\frac{1}{2}}.$$

Multiplying by $1/(k!)^s$ and summing, we obtain

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|\partial_x^k u(t)\|_{L^2}\\
(3.23)\qquad &\leq \sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|\partial_x^k u(1)\|_{L^2}\\
&\quad+\Big\{\int_1^t\Big(\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|R_k(\tau)\|_{L^2}\Big)\Big(\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|\partial_x^k u(\tau)\|_{L^2}\Big)\,d\tau\Big\}^{\frac{1}{2}}.
\end{aligned}
$$

By unitarity of $U(t)$, the representation $J(t)f = U(t)xU(-t)f$, the uncertainty principle (2.12), and the uniform bound (3.6), we see that for any $\tau \in [1, t]$,

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|\partial_x^k u(\tau)\|_{L^2} &= \sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|U(-\tau)(\partial_x^k u)(\tau)\|_{L^2}\\
(3.24)\qquad &\leq \sum_{k=0}^{\infty}\frac{1}{(k!)^s}\Big(\|J(\tau)\partial_x^k u(\tau)\|_{L^2}+\|\partial_x^k u(\tau)\|_{\dot H^1}\Big)\\
&\leq C(1+\tau)^{\gamma}\|u\|_{X_T}\leq C\varepsilon_s(1+\tau)^{\gamma}.
\end{aligned}
$$

By (**??**), (2.11), and (3.6), we also obtain for any $\tilde\gamma > 4\gamma$ and $\tau \in [1, t]$,

$$
\sum_{k=0}^{\infty}\frac{1}{(k!)^s}\|R_k(\tau)\|_{L^2}\leq C\varepsilon_s\,\tau^{-1-(\tilde\gamma-3\gamma)}. \tag{3.25}
$$

Combining (3.6), (3.23), (3.24), and (3.25), we obtain

$$
\|u(t)\|_{G_v^s}\leq C\varepsilon_s,\quad t\geq 1,
$$

where ε_s is sufficiently small. By (3.6), the estimate (3.21) holds for any $t \in [0, 1]$ as well.
❑

4. Proof of Theorem 3.2

Proof of Theorem 3.2. For $r > 0$, let

$$
\|f\|_{L^2([-r,r])}\equiv\Big(\int_{-r}^{r}|f(x)|^2\,dx\Big)^{\frac{1}{2}}.
$$

Then it follows that for any $r > 0$,

$$
\|f\|_{L^2([-r,r])}\leq (2r)^{\frac{1}{4}}\|f\|_{L^4}\quad\text{for any } f\in L^4(\mathbb{R}), \tag{4.1}
$$

and for any $a > 0$ there exists a constant $C > 0$ such that

$$
\|f\|_{L^2([-r,r])}^2\leq\frac{a^s}{2^{2-s}}|\mathrm{Im}\,\lambda|(\log\log t)^{-s}(\log t)\|f\|_{L^2([-r,r])}^4+C(\log\log t)^s(\log t)^{-1} \tag{4.2}
$$

for any $f \in L^2(\mathbb{R})$, by Young's inequality.

Let u be the global solution to (1.1) given by the *a priori* estimate (3.7). Then $\mathcal{F}[U(-t)u](t,\xi)$ satisfies

$$i\partial_t \mathcal{F}[U(-t)u](t,\xi) = \frac{\lambda}{t}|\mathcal{F}[U(-t)u]|^2\mathcal{F}[U(-t)u](t,\xi) + R(t,\xi), \tag{4.3}$$

where

$$R(t,\xi) = \lambda\mathcal{F}[U(-t)(|u|^2u)] - \frac{\lambda}{t}|\mathcal{F}[U(-t)u]|^2\mathcal{F}[U(-t)u].$$

Multiplying (4.3) by $\overline{\mathcal{F}[U(-t)u](t,\xi)}$ and taking the imaginary part, we obtain

$$\frac{1}{2}\partial_t|\mathcal{F}[U(-t)u]|^2 = -\frac{|\operatorname{Im}\lambda|}{t}|\mathcal{F}[U(-t)u]|^4 + \operatorname{Im}\{R\overline{\mathcal{F}[U(-t)u]}\}. \tag{4.4}$$

By (2.11) and (3.7), $R = R(t,\xi)$ satisfies, for any $\tilde{\gamma} > 3\gamma$,

$$\|R(t)\|_{L^2} \le C\,\varepsilon_s^3\, t^{-1-(\tilde{\gamma}-3\gamma)}$$

for any $t \ge 1$. This remainder is controlled by the initial size with cubic order (cf. [34, 40]). Using the L^2 *a priori* estimate $\|\mathcal{F}[U(-t)u](t)\|_{L^2} \le \|u_0\|_{L^2}$ and integrating (4.4) in ξ over $\mathbb{R}$, we obtain, for sufficiently small $\varepsilon_s > 0$,

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{F}[U(-t)u](t)\|_{L^2}^2 \le -\frac{|\operatorname{Im}\lambda|}{t}\|\mathcal{F}[U(-t)u](t)\|_{L^4}^4 + C\,\varepsilon_s\, t^{-1-(\tilde{\gamma}-3\gamma)}. \tag{4.5}$$

For $0 < a < 1$, let

$$r(t) \equiv \Big(\frac{1}{2a}\log\log t\Big)^s.$$

From (4.1), (4.2) and (4.5), we deduce, for sufficiently small $\varepsilon_s > 0$,

$$\begin{aligned}
&\frac{d}{dt}\{(\log\log t)^{-s}(\log t)^2\|\mathcal{F}[U(-t)u](t)\|_{L^2}^2\} \\
&\le 2(\log\log t)^{-s}(\log t)t^{-1}\Big(\|\mathcal{F}[U(-t)u](t)\|_{L^2([-r,r])}^2 + \|\mathcal{F}[U(-t)u](t)\|_{L^2(\mathbb{R}\setminus[-r,r])}^2\Big) \\
&\quad + (\log\log t)^{-s}(\log t)^2\Big(-\frac{|\operatorname{Im}\lambda|}{t}\|\mathcal{F}[U(-t)u](t)\|_{L^4}^4 + C\,t^{-1-(\tilde{\gamma}-3\gamma)}\Big) \\
&\le 2(\log\log t)^{-s}(\log t)t^{-1}\Big\{\Big(\frac{a^s}{2^{2-s}}|\operatorname{Im}\lambda|(\log\log t)^{-s}(\log t)\|\mathcal{F}[U(-t)u](t)\|_{L^2([-r,r])}^4 \\
&\qquad + C(\log\log t)^s(\log t)^{-1}\Big) + \|\mathcal{F}[U(-t)u](t)\|_{L^2(\mathbb{R}\setminus[-r,r])}^2\Big\} \\
&\quad - |\operatorname{Im}\lambda|(2r(t))^{-1}(\log\log t)^{-s}(\log t)^2t^{-1}\|\mathcal{F}[U(-t)u](t)\|_{L^2([-r,r])}^4 + Ct^{-1-\frac{\tilde{\gamma}-3\gamma}{2}} \\
&\le Ct^{-1} + C(\log\log t)^{-s}(\log t)t^{-1}\|\mathcal{F}[U(-t)u](t)\|_{L^2(\mathbb{R}\setminus[-r,r])}^2 + Ct^{-1-\frac{\tilde{\gamma}-3\gamma}{2}}.
\end{aligned} \tag{4.6}$$

To estimate the high-frequency part we use the following lemma.

Lemma 4.1. *Let $0 < a < 1$ and $s \ge 1$. Then the following inequality holds:*

$$\|e^{a|\xi|^{1/s}}\widehat{f}\|_{L^2} \le \Big(\frac{1}{1-a^{1-\frac{1}{s}}}\Big)^{1-\frac{1}{s}}\|f\|_{L^2}^{1-\frac{1}{s}}\|f\|_{G_v^s}^{\frac{1}{s}} \tag{4.7}$$

for any $f \in G_v^s(\mathbb{R})$.

Proof of Lemma 4.1. Let $0 < a < 1$ and $s \geq 1$. By Taylor expansion of $e^{a|\xi|^{1/s}}$ and Hölder's inequality, we obtain

$$\begin{aligned}\|e^{a|\xi|^{1/s}}\widehat{f}\|_{L^2} &= \Big\|\sum_{k=0}^{\infty}\frac{a^k}{k!}|\xi|^{\frac{k}{s}}\widehat{f}\Big\|_{L^2}\\ &\leq \Big\|\Big(\sum_{k=0}^{\infty}a^{\frac{k}{1-\frac{1}{s}}}|\widehat{f}|\Big)^{1-\frac{1}{s}}\Big(\sum_{k=0}^{\infty}\frac{1}{(k!)^s}|\xi|^k|\widehat{f}|\Big)^{\frac{1}{s}}\Big\|_{L^2}\\ &\leq \Big(\frac{1}{1-a^{1-\frac{1}{s}}}\Big)^{1-\frac{1}{s}}\|f\|_{L^2}^{1-\frac{1}{s}}\|f\|_{G_v^s}^{\frac{1}{s}}\end{aligned}$$

for any $f \in G_v^s(\mathbb{R})$. ❑

By (4.7) and (3.21), for $0 < a < 1$ we estimate

$$\begin{aligned}\|\mathcal{F}[U(-t)u](t)\|_{L^2(\mathbb{R}\setminus[-r,r])}^2 &\leq e^{-2ar^{1/s}}\|e^{a|\xi|^{1/s}}\mathcal{F}[U(-t)u](t)\|_{L^2(\mathbb{R}\setminus[-r,r])}^2\\ &\leq e^{-\log\log t}\|e^{a|\xi|^{1/s}}\mathcal{F}[U(-t)u](t)\|_{L^2}^2 \leq C(\log t)^{-1}\end{aligned} \tag{4.8}$$

for any $t \geq e^e$. Therefore (4.6) and (4.8) yield

$$\begin{aligned}\frac{d}{dt}\{(\log\log t)^{-s}(\log t)^2\|\mathcal{F}[U(-t)u](t)\|_{L^2}^2\} &\leq Ct^{-1} + C(\log\log t)^{-s}t^{-1} + Ct^{-1-\frac{\tilde{\gamma}-3\gamma}{2}}\\ &\leq Ct^{-1} + Ct^{-1-\frac{\tilde{\gamma}-3\gamma}{2}}\end{aligned} \tag{4.9}$$

for any $t \geq e^e$. By the dissipative effect of the solution, $\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$, we integrate (4.9) over $[e^e, t]$ and deduce that

$$\begin{aligned}(\log\log t)^{-s}(\log t)^2\|\mathcal{F}[U(-t)u](t)\|_{L^2}^2 &\leq (\log\log e^e)^{-s}(\log e^e)^2\|\mathcal{F}[U(-t)u](e^e)\|_{L^2}^2\\ &\quad + C\log t + C\\ &\leq C(1+\log t).\end{aligned} \tag{4.10}$$

Therefore we conclude

$$\|u(t)\|_{L^2} = \|\mathcal{F}[U(-t)u](t)\|_{L^2} \leq C(\log\log t)^{\frac{s}{2}}(\log t)^{-\frac{1}{2}}$$

for any $t \geq e^e$. ❑

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