On the translation of a Kripke model into a Beth model with constant domain

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Abstract

We revisit the fact that a first-order Kripke model can be translated into a first-order Beth model with constant domain, and provide a detailed proof of this fact.

Keywords: Beth model, Kripke model, predicate logic

1 Introduction

Kripke showed in [2] that a first-order Kripke model can be translated into a first-order Beth model with constant domain by a sophisticated transformation such that the same sentences are valid in both models. In [4, Section 13.1.6], Troelstra and van Dalen revisited the argument in their context. Unfortunately, however, their construction of a Beth model from a given Kripke model seems to contain a misreading of the argument in [2] (see Remark 3.5). In this article, we reformulate the development in [4, Section 13.1.6] and give a detailed proof of the above-mentioned fact (see Theorem 3.9). The underlying idea of our construction of a Beth model from a given Kripke model is basically the same as that in [2, Section 1.2].

2 Preliminaries

In this article, we fix a standard language $\mathcal{L}$ of first-order predicate logic containing the logical connectives $\wedge, \vee, \to, \exists, \forall, \perp$. For simplicity, we assume that there are no function symbols in our language $\mathcal{L}$.

Notation. A finite list $d_1, \dots, d_n$ (resp. $x_1, \dots, x_n$) of elements (resp. variables) is often abbreviated as $\vec{d}$ (resp. $\vec{x}$) if it does not cause confusion. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^*$ be the set of finite sequences of natural numbers. The letters σ, τ etc. range over $\mathbb{N}^*$ and $\langle\,\rangle$ denotes the empty sequence. The length of $\sigma \in \mathbb{N}^*$ is denoted by $|\sigma|$, and if $i < |\sigma|$, then σ_i denotes the i-th entry of σ. For $k \in \mathbb{N}$, $\langle k \rangle$ denotes the sequence of length 1 consisting only of k. The concatenation of finite sequences σ and τ is denoted by $\sigma * \tau$. The letters α, β etc. range over functions from $\mathbb{N}$ to $\mathbb{N}$, which are identified with infinite sequences of natural numbers. For $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$, $\overline{\alpha}j$ ($\in \mathbb{N}^*$) denotes the initial segment of α of length j, especially, $\overline{\alpha}0 = \langle\,\rangle$. For $\sigma \in \mathbb{N}^*$ such that $|\sigma| > 0$, let σ_l denote $\sigma_{|\sigma|-1}$ and $\sigma * \lambda i.\sigma_l$ denote the function $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha(i) = \sigma_i$ for $i < |\sigma|$ and $\alpha(i) = \sigma_l$ for $i \geq |\sigma|$.

We first recall the definition of a first-order Beth model in [4, Section 13.1.1], which is based on [6]. For the history and the variations of Beth models, see e.g. [5], [2] and [4, Section 13.1].

Definition 2.1 (cf. [4, Section 13.1.1]). A Beth model is a quadruple $\mathcal{B} := (K, \preceq, D, \Vdash)$ satisfying the following:

1. $(K, \preceq)$ is a spread with the initial segment relation $\preceq$ among the branches in K (thus $(K, \preceq)$ satisfies that $K \subseteq \mathbb{N}^*$, $\langle\,\rangle \in K$, for all $\sigma \in \mathbb{N}^*$, either $\sigma \in K$ or $\sigma \notin K$, K is downward closed under $\preceq$, and for all $\sigma \in K$, there exists $m \in \mathbb{N}$ such that $\sigma * \langle m \rangle \in K$);

2. the domain function D assigns inhabited (non-empty) sets to elements $\sigma \in K$ such that for all $\sigma, \sigma' \in K$, if $\sigma \preceq \sigma'$, then $D(\sigma) \subseteq D(\sigma')$;

3. for each constant symbol c in $\mathcal{L}$, its interpretation (name) $\tilde{c}$ is in $D(\sigma)$ for all $\sigma \in K$;

4. the valuation $\Vdash$ is a binary relation between K and prime $\mathcal{L}(\widetilde{D})$-sentences $P(d_1, \ldots, d_n)$ (where P is a predicate symbol of arity n) with $d_1, \ldots, d_n$ in $\widetilde{D} := \bigcup_{\sigma \in K} D(\sigma)$ such that for all $\sigma \in K$,

   B0: $\sigma \nVdash \perp$;

   B1: $\sigma \Vdash P(d_1, \ldots, d_n)$ if and only if $d_1, \ldots, d_n \in D(\sigma)$ and for all $\alpha \in \sigma$, there exists $m \in \mathbb{N}$ such that $\overline{\alpha}m \Vdash P(d_1, \ldots, d_n)$, where $\alpha \in \sigma$ means that α is an infinite branch of the spread $(K, \preceq)$ (namely, $\overline{\alpha}j \in K$ for all $j \in \mathbb{N}$) and is an extension of σ (namely, $\alpha(i) = \sigma_i$ for all $i < |\sigma|$).

The valuation $\Vdash$ is extended to compound $\mathcal{L}(\widetilde{D})$-sentences by the following clauses (where $\alpha \in \sigma$ means the same as in B1):

B2: $\sigma \Vdash A \wedge B :\Leftrightarrow \sigma \Vdash A$ and $\sigma \Vdash B$;

B3: $\sigma \Vdash A \vee B :\Leftrightarrow$ for all $\alpha \in \sigma$, there exists $n \in \mathbb{N}$ such that $\overline{\alpha}n \Vdash A$ or $\overline{\alpha}n \Vdash B$;

B4: $\sigma \Vdash A \to B :\Leftrightarrow$ for all $\sigma' \succeq \sigma$, if $\sigma' \Vdash A$, then $\sigma' \Vdash B$.

B5: $\sigma \Vdash \exists x A(x) :\Leftrightarrow$ for all $\alpha \in \sigma$, there exist $n \in \mathbb{N}$ and $d \in D(\overline{\alpha}n)$ such that $\overline{\alpha}n \Vdash A(d)$;

B6: $\sigma \Vdash \forall x A(x) :\Leftrightarrow$ for all $\sigma' \succeq \sigma$ and $d \in D(\sigma')$, $\sigma' \Vdash A(d)$.

For a $\mathcal{L}$-sentence A, A is said to be valid in $\mathcal{B}$ if $\langle\,\rangle \Vdash A$ (where each constant symbol c is interpreted by its name $\tilde{c}$), and we then write $\mathcal{B} \Vdash A$.

Remark 2.2. In Beth's original definition [1], the domain function is constant, namely, $D(\sigma) = \widetilde{D}$ for all $\sigma \in K$. If the domain function is constant, then $\sigma \Vdash \forall x A(x)$ if and only if $\sigma \Vdash A(d)$ for all $d \in \widetilde{D}$.

Lemma 2.3. *Let $(K, \preceq, D, \Vdash)$ be a Beth model. For a prime formula $P(\vec{x})$ whose free variables are exactly those in $\vec{x}$, $\sigma, \sigma' \in K$ such that $\sigma' \succeq \sigma$, and $\vec{d} \in \widetilde{D}$, if $\sigma \Vdash P(\vec{d})$, then $\sigma' \Vdash P(\vec{d})$.*

Proof. By $\sigma \Vdash P(\vec{d})$ and $\sigma' \succeq \sigma$, we have $\vec{d} \in D(\sigma')$. For each $\alpha \in \sigma'$, since $\alpha \in \sigma$, there exists $m \in \mathbb{N}$ such that $\overline{\alpha}m \Vdash P(\vec{d})$, by our assumption $\sigma \Vdash P(\vec{d})$. Thus we have $\sigma' \Vdash P(\vec{d})$. □

Proposition 2.4 (Monotonicity)**.** *Let $(K, \preceq, D, \Vdash)$ be a Beth model. For any formula $A(\vec{x})$ whose free variables are exactly those in $\vec{x}$, $\sigma, \sigma' \in K$ such that $\sigma' \succeq \sigma$, and $\vec{d} \in \widetilde{D}$, if $\sigma \Vdash A(\vec{d})$, then $\sigma' \Vdash A(\vec{d})$.*

Proof. By induction on the structure of formulas. For prime formulas, we are done by Lemma 2.3. The induction step is verified in a straightforward way by the definition of the valuation for compound formulas. □

Proposition 2.5 (Covering property)**.** *Let $(K, \preceq, D, \Vdash)$ be a Beth model. For any formula $A(\vec{x})$ whose free variables are exactly those in $\vec{x}$, $\sigma \in K$ and $\vec{d} \in D(\sigma)$, if for all $\alpha \in \sigma$, there exists $n \in \mathbb{N}$ such that $\overline{\alpha}n \Vdash A(\vec{d})$, then $\sigma \Vdash A(\vec{d})$.*

Proof. By induction on the structure of formulas. For prime formulas, we are done by condition (B1) in Definition 2.1. For the induction step, assume that the assertion holds for $A(\vec{x})$ and $B(\vec{x})$.

Suppose that $\sigma \in K$, $\vec{d} \in D(\sigma)$ and for all $\alpha \in \sigma$, there exists $n \in \mathbb{N}$ such that $\overline{\alpha}n \Vdash A(\vec{d}) \wedge B(\vec{d})$. Then, by using the induction hypothesis (we write I.H. hereafter), we have $\sigma \Vdash A(\vec{d})$ and $\sigma \Vdash B(\vec{d})$, and hence, $\sigma \Vdash A(\vec{d}) \wedge B(\vec{d})$.

Suppose that $\sigma \in K$, $\vec{d} \in D(\sigma)$ and for all $\alpha \in \sigma$, there exists $n \in \mathbb{N}$ such that $\overline{\alpha}n \Vdash A(\vec{d}) \vee B(\vec{d})$. By the definition of $\overline{\beta}n \Vdash A(\vec{d}) \vee B(\vec{d})$, for each $\beta \in \sigma$, there exists $m \in \mathbb{N}$ such that $\overline{\beta}m \Vdash A(\vec{d})$ or $\overline{\beta}m \Vdash B(\vec{d})$. Thus we have $\sigma \Vdash A(\vec{d}) \vee B(\vec{d})$.

Suppose that $\sigma \in K$, $\vec{d} \in D(\sigma)$ and for all $\alpha \in \sigma$, there exists $n \in \mathbb{N}$ such that $\overline{\alpha}n \Vdash A(\vec{d}) \to B(\vec{d})$. To show $\sigma \Vdash A(\vec{d}) \to B(\vec{d})$, fix $\sigma' \in K$ such that $\sigma' \succeq \sigma$ and suppose also $\sigma' \Vdash A(\vec{d})$. We claim that for all $\beta \in \sigma'$, there exists $n' \in \mathbb{N}$ such that $\overline{\beta}n' \Vdash B(\vec{d})$. Fix $\beta \in \sigma'$. Since $\beta \in \sigma$, by our supposition, there exists $m \in \mathbb{N}$ such that $\overline{\beta}m \Vdash A(\vec{d}) \to B(\vec{d})$. Now $m \leq |\sigma'|$ or $m > |\sigma'|$. In the former case, since $\sigma' = \overline{\beta}|\sigma'|$, we have $\overline{\beta}|\sigma'| \Vdash B(\vec{d})$. In the latter case, by Proposition 2.4, we have $\overline{\beta}m \Vdash A(\vec{d})$, and hence, $\overline{\beta}m \Vdash B(\vec{d})$. Thus we have shown our claim. Then, by I.H. for $B(\vec{d})$, we have $\sigma' \Vdash B(\vec{d})$.

Suppose that $\sigma \in K$, $\vec{d} \in D(\sigma)$ and for all $\alpha \in \sigma$, there exists $n \in \mathbb{N}$ such that $\overline{\alpha}n \Vdash \exists z A(z, \vec{d})$. To show $\sigma \Vdash \exists z A(z, \vec{d})$, fix $\beta \in \sigma$. By our supposition, there exists $m \in \mathbb{N}$ such that $\overline{\beta}m \Vdash \exists z A(z, \vec{d})$. Since $\beta \in \overline{\beta}m$, there exists $l \in \mathbb{N}$ and $d \in D(\overline{\beta}l)$ such that $\overline{\beta}l \Vdash A(d, \vec{d})$. Thus we have shown $\sigma \Vdash \exists z A(z, \vec{d})$.

Suppose that $\sigma \in K$, $\vec{d} \in D(\sigma)$ and for all $\alpha \in \sigma$, there exists $n \in \mathbb{N}$ such that $\overline{\alpha}n \Vdash \forall z A(z, \vec{d})$. To show $\sigma \Vdash \forall z A(z, \vec{d})$, fix $\sigma' \in K$ such that $\sigma' \succeq \sigma$ and $d \in D(\sigma')$. We claim that for all $\beta \in \sigma'$, there exists $m \in \mathbb{N}$ such that $\overline{\beta}m \Vdash A(d, \vec{d})$. Fix $\beta \in \sigma'$. Since $\beta \in \sigma$, by our supposition, there exists $n' \in \mathbb{N}$ such that $\overline{\beta}n' \Vdash \forall z A(z, \vec{d})$. Put $m := \max\{n', |\sigma'|\}$. Since $\overline{\beta}|\sigma'| = \sigma'$, we have $d \in D(\overline{\beta}|\sigma'|) \subseteq D(\overline{\beta}m)$. Then, by the definition of $\overline{\beta}n' \Vdash \forall z A(z, \vec{d})$, we have $\overline{\beta}m \Vdash A(d, \vec{d})$. Thus we have shown our claim. Then, by I.H., we have $\sigma' \Vdash A(d, \vec{d})$. □

3 Translation of a Kripke model into a Beth model with constant domain

We first recall the definition of a first-order Kripke model in our language $\mathcal{L}$ (which has no function symbols).

Definition 3.1 (cf. [3, Section 3.5.7]). A Kripke model is a tuple $\mathcal{K} := (K, \preceq, D, \Vdash)$ satisfying the following:

1. $(K, \preceq)$ is an inhabited partially ordered countable set;
2. the domain function D assigns inhabited countable sets to elements $k \in K$ such that for all $k, k' \in K$, if $k \preceq k'$, then $D(k) \subseteq D(k')$;
3. for each constant symbol c in $\mathcal{L}$, its interpretation $\tilde{c}$ is in $D(k)$ for all $k \in K$;
4. the valuation $\Vdash$ is a binary relation between K and prime $\mathcal{L}(\widetilde{D})$-sentences $P(d_1, \dots, d_n)$ (where P is a predicate symbol of arity n) with $d_1, \dots, d_n$ in $\widetilde{D} := \bigcup_{k \in K} D(k)$ such that for all $k, k' \in K$,
   (a) $k \nVdash \perp$;
   (b) if $k \Vdash P(d_1, \dots, d_n)$, then $d_1, \dots, d_n \in D(k)$;
   (c) if $k \Vdash P(d_1, \dots, d_n)$ and $k \preceq k'$, then $k' \Vdash P(d_1, \dots, d_n)$.

The valuation $\Vdash$ is extended to compound $\mathcal{L}(\widetilde{D})$-sentences by the following clauses:

- $k \Vdash A \wedge B :\Leftrightarrow k \Vdash A$ and $k \Vdash B$;
- $k \Vdash A \vee B :\Leftrightarrow k \Vdash A$ or $k \Vdash B$;
- $k \Vdash A \to B :\Leftrightarrow$ for all $k' \succeq k$, if $k' \Vdash A$, then $k' \Vdash B$;
- $k \Vdash \exists x A(x) :\Leftrightarrow$ there exist $d \in D(k)$ such that $k \Vdash A(d)$;
- $k \Vdash \forall x A(x) :\Leftrightarrow$ for all $k' \succeq k$ and $d \in D(k')$, $k' \Vdash A(d)$.

For a $\mathcal{L}$-sentence A, A is said to be valid in $\mathcal{K}$ if $k \Vdash A$ (where each constant symbol c is interpreted by its name $\tilde{c}$) for all $k \in K$, and we then write $\mathcal{K} \Vdash A$.

Remark 3.2. The monotonicity holds for Kripke models: for any Kripke model $\mathcal{K} := (K, \preceq, D, \Vdash)$, $\mathcal{L}$-formula $A(\vec{x})$ whose free variables are exactly those in $\vec{x}$, and $k, k' \in K$, if $\vec{d} \in D(k)$, $k \Vdash A(\vec{d})$, and $k \preceq k'$, then $k' \Vdash A(\vec{d})$ holds.

Remark 3.3. For a Kripke model $\mathcal{K} := (K, \preceq, D, \Vdash)$, since K is countable, one can identify K with a subset of $\mathbb{N}$.

In the rest of this article, for a Kripke model $\mathcal{K} := (K, \preceq, D, \Vdash)$, we assume that K is a subset of $\mathbb{N}$ and also that for all $k \in \mathbb{N}$, either $k \in K$ or $k \notin K$.

Definition 3.4 (Translation of a Kripke model into a Beth model with constant domain)**.** Let $\mathcal{K} := (K, \preceq, D, \Vdash)$ be a Kripke model. We construct a quadruple $(K', \preceq', D', \Vdash')$ as follows. Let K' be a set of finite sequences σ of elements in K such that

$$\forall i, j < |\sigma| (i < j \Rightarrow \sigma_i \preceq \sigma_j).$$

Note that K' contains the empty sequence $\langle\,\rangle$. Let $\sigma \preceq' \sigma'$ be a relation that σ is an initial segment of σ'. Now, let $\widetilde{C}$ be the set of the interpretations (names) of constant symbols in $\mathcal{L}$ (in the Kripke model $\mathcal{K}$), where we assume $\widetilde{C} \cap \mathbb{N} = \emptyset$ without loss of generality. For all $\sigma \in K'$, let $D'(\sigma) := \widetilde{C} \cup \mathbb{N}$. For the definition of valuation $\Vdash'$, fix a decomposition of $\mathbb{N}$ into infinite sets $\{\mathbb{N}_i \mid i \in \mathbb{N}\}$ such that $\mathbb{N}_i \cap \mathbb{N}_j = \emptyset$ for all $i, j \in \mathbb{N}$ such that $i \neq j$ and $\mathbb{N} = \bigcup_{i \in \mathbb{N}} \mathbb{N}_i$. Let $M_i := \widetilde{C} \cup \bigcup_{j<i} \mathbb{N}_j$. In addition, fix a coding function $\psi_{(\cdot)}$ such that for each $\sigma \in K'$,

1. ψ_σ is a function from $M_{|\sigma|}$ to $\widetilde{C} \cup \bigcup_{i<|\sigma|} D(\sigma_i)$;
2. $\psi_\sigma(m) = \psi_{\sigma'}(m)$ for all $m \in M_{|\sigma|}$ and $\sigma' \succeq' \sigma$;
3. ψ_σ maps $\mathbb{N}_{|\sigma|-1}$ onto $D(\sigma_l)$ for $\sigma \neq \langle\,\rangle$;
4. $\psi_{\langle\,\rangle}$ is the identity map over $\widetilde{C}$.

Then we define the valuation $\Vdash'$ as follows: For each $\sigma \in K'$, a prime formula $P(x_1, \dots, x_n)$ whose free variables are exactly those listed, and $d_1, \dots, d_n \in \widetilde{C} \cup \mathbb{N}$, $\sigma \Vdash' P(d_1, \dots, d_n)$ is defined as $\tau_l \Vdash P(\psi_\tau(d_1), \dots, \psi_\tau(d_n))$ for all $\tau \succ' \sigma$ (namely, $\tau \in K'$, $\tau \succeq' \sigma$ and $|\tau| > |\sigma|$) such that $d_1, \dots, d_n \in M_{|\tau|-1}$ (note $\tau_l = \tau_{|\tau|-1}$).

Remark 3.5. A quadruple constructed from a Kripke model in [4, Section 13.1.6] may not be a Beth model: In [4, Section 13.1.6], for a Kripke model $(K, \preceq, D, \Vdash)$, it is written that for each $\sigma \in K'$ and a prime formula $P(x_1, \dots, x_n)$,

$$\sigma \Vdash' P(d_1, \dots, d_n) :\Leftrightarrow \sigma_l \Vdash P(\psi_\sigma(d_1), \dots, \psi_\sigma(d_n)). \tag{1}$$

Since $\psi_\sigma(d_i)$ is not defined when $d_i \notin M_{|\sigma|}$, one should understand that the right-hand side of (1) requires $d_1, \dots, d_n \in M_{|\sigma|}$. However, this additional requirement collapses condition (B1) in Definition 2.1: Let $k \in K$ and $d \in \mathbb{N} \setminus \mathbb{N}_0$. Then, for a prime formula $P(x)$ such that $\forall \alpha \in \langle k \rangle\, \exists m \in \mathbb{N}\, (\overline{\alpha}m \Vdash' P(d))$ (e.g., a tautological formula $x = x$), we have $\langle k \rangle \nVdash' P(d)$ since $d \notin M_{|\langle k \rangle|}$.

Proposition 3.6. *For a Kripke model $\mathcal{K} := (K, \preceq, D, \Vdash)$, the quadruple $\mathcal{B}_{\mathcal{K},\psi} := (K', \preceq', D', \Vdash')$ constructed in Definition 3.4 is a Beth model with constant domain $\widetilde{C} \cup \mathbb{N}$.*

Proof. It suffices to show that $\mathcal{B}_{\mathcal{K},\psi}$ satisfies condition (B1) in Definition 2.1. Fix $\sigma \in K'$. Suppose that $d_1, \dots, d_n \in \widetilde{C} \cup \mathbb{N}$ and for all $\alpha \in \sigma$, there exists $m \in \mathbb{N}$ such that $\overline{\alpha}m \Vdash' P(d_1, \dots, d_n)$. To show $\sigma \Vdash' P(d_1, \dots, d_n)$, fix $\tau \in K'$ such that $\tau \succ' \sigma$ and $d_1, \dots, d_n \in M_{|\tau|-1}$. Put $\alpha' := \tau * \lambda i.\tau_l$. Then α' is an infinite branch of $(K', \preceq')$ extending σ. Therefore, by our supposition, there exists $m' \in \mathbb{N}$ such that $\overline{\alpha'}m' \Vdash' P(d_1, \dots, d_n)$. Put $j := \max\{m' + 1, |\tau|\}$. Then $(\overline{\alpha'}j)_l = \tau_l$ and $\psi_{\overline{\alpha'}j}(d_i) = \psi_\tau(d_i)$ for all $i \in \{1, \dots, n\}$. Since $\overline{\alpha'}m' \Vdash' P(d_1, \dots, d_n)$ and $\overline{\alpha'}j \succ' \overline{\alpha'}m'$, we have $\tau_l \Vdash P(\psi_\tau(d_1), \dots, \psi_\tau(d_n))$.

The converse direction is trivial. □

By Proposition 3.6, for a Kripke model $\mathcal{K} := (K, \preceq, D, \Vdash)$, we call the quadruple $\mathcal{B}_{\mathcal{K},\psi} := (K', \preceq', D', \Vdash')$ constructed in Definition 3.4 the **induced Beth model** of $\mathcal{K}$ (by a coding function ψ).

Remark 3.7. The underlying idea of our construction of the induced Beth model from a given Kripke model in Definition 3.4 is basically the same as that by Kripke in [2, pp. 112-113], which was pointed out by Naosuke Matsuda.

Lemma 3.8. *For any Kripke model $\mathcal{K} := (K, \preceq, D, \Vdash)$ and its induced Beth model $\mathcal{B}_{\mathcal{K},\psi} := (K', \preceq', D', \Vdash')$ by a coding function ψ, for any $\mathcal{L}$-formula $A(x_1, \ldots, x_n)$ whose free variables are exactly those listed, $\sigma \in K'$, and $d_1, \ldots, d_n \in M_{|\sigma|}$, the following hold:*

(i) *if $\sigma = \langle\,\rangle$, then*

$$\sigma \Vdash' A(d_1, \ldots, d_n) \Leftrightarrow k \Vdash A(d_1, \ldots, d_n) \text{ for all } k \in K;$$

(ii) *if $\sigma \neq \langle\,\rangle$, then*

$$\sigma \Vdash' A(d_1, \ldots, d_n) \Leftrightarrow \sigma_l \Vdash A(\psi_\sigma(d_1), \ldots, \psi_\sigma(d_n)).$$

Proof. By induction on the structure of formulas. In what follows, for $\tau \in \mathbb{N}^*$ and $\vec{d} = d_1, \ldots, d_n$, we loosely use the notation $\psi_\tau(\vec{d})$ to mean $\psi_\tau(d_1), \ldots, \psi_\tau(d_n)$.

For the base step, fix a prime formula $P(\vec{x})$, $\sigma \in K'$ and $\vec{d} \in M_{|\sigma|}$.

(i) Let $\sigma = \langle\,\rangle$. Since $M_{|\langle\rangle|} = \widetilde{C}$ and $\psi_\tau(\vec{d}) = \psi_{\langle\rangle}(\vec{d}) = \vec{d}$ for all $\tau \succ' \sigma$, it is straightforward to see that $\sigma \Vdash' P(\vec{d})$ is equivalent to that $k \Vdash P(\vec{d})$ for all $k \in K$.

(ii) Let $\sigma \neq \langle\,\rangle$. First, suppose $\sigma \Vdash' P(\vec{d})$. Since $\sigma * \langle \sigma_l \rangle \succ' \sigma$, by the definition of $\sigma \Vdash' P(\vec{d})$, we have $\sigma_l \Vdash P(\psi_\sigma(\vec{d}))$. Next, suppose $\sigma_l \Vdash P(\psi_\sigma(\vec{d}))$. Fix $\tau \succ' \sigma$ such that $\vec{d} \in M_{|\tau|-1}$. Since $\tau_l \succeq \sigma_l$, by the monotonicity of $\Vdash$ and the property that $\psi_\sigma(\vec{d}) = \psi_\tau(\vec{d})$, we have $\tau_l \Vdash P(\psi_\tau(\vec{d}))$. Thus we have shown $\sigma \Vdash' P(\vec{d})$.

For the induction step, assume that the assertion holds for $A(\vec{x})$ and $B(\vec{x})$.

Case of $(A \wedge B)(\vec{x})$: Fix $\sigma \in K'$ and $\vec{d} \in M_{|\sigma|}$.

(i) Let $\sigma = \langle\,\rangle$. Then

$$\begin{array}{cl} & \sigma \Vdash' (A \wedge B)(\vec{d}) \\ \Longleftrightarrow & \langle\,\rangle \Vdash' A(\vec{d}) \text{ and } \langle\,\rangle \Vdash' B(\vec{d}) \\ \underset{\text{(i) of I.H.}}{\Longleftrightarrow} & k \Vdash A(\vec{d}) \text{ for all } k \in K \text{ and } k \Vdash B(\vec{d}) \text{ for all } k \in K \\ \Longleftrightarrow & k \Vdash (A \wedge B)(\vec{d}) \text{ for all } k \in K. \end{array}$$

(ii) Let $\sigma \neq \langle\,\rangle$. Then

$$\begin{array}{cl} & \sigma \Vdash' (A \wedge B)(\vec{d}) \\ \Longleftrightarrow & \sigma \Vdash' A(\vec{d}) \text{ and } \sigma \Vdash' B(\vec{d}) \\ \underset{\text{(ii) of I.H.}}{\Longleftrightarrow} & \sigma_l \Vdash A(\psi_\sigma(\vec{d})) \text{ and } \sigma_l \Vdash B(\psi_\sigma(\vec{d})) \\ \Longleftrightarrow & \sigma_l \Vdash (A \wedge B)(\psi_\sigma(\vec{d})). \end{array}$$

Case of $(A \vee B)(\vec{x})$: Fix $\sigma \in K'$ and $\vec{d} \in M_{|\sigma|}$. By the monotonicity of $\Vdash'$, we have that $\sigma \Vdash' (A \vee B)(\vec{d})$ is equivalent to that $(*)_1$: for all $\alpha \in \sigma$, there exists $n > 0$ such that $\overline{\alpha}n \Vdash' A(\vec{d})$ or $\overline{\alpha}n \Vdash' B(\vec{d})$.

(i) Let $\sigma = \langle\,\rangle$. In the following, we show that $(*)_1$ is equivalent to that $k \Vdash (A \vee B)(\vec{d})$ for all $k \in K$. First, suppose that $(*)_1$ holds. Then we have that for all $k \in K$, there exists $n > 0$ such that $\overline{(\lambda i.k)}n \Vdash' A(\vec{d})$ or $\overline{(\lambda i.k)}n \Vdash' B(\vec{d})$ (note that $\lambda i.k$ is an infinite branch of $(K', \preceq')$). Then, by (ii) of I.H., we have that for all $k \in K$, there exists $n > 0$ such that $k \Vdash A(\psi_{\overline{(\lambda i.k)}n}(\vec{d}))$ or $k \Vdash B(\psi_{\overline{(\lambda i.k)}n}(\vec{d}))$. Since $\vec{d} \in M_{|\langle\,\rangle|}$, the latter assertion is equivalent to that $k \Vdash (A \vee B)(\vec{d})$ for all $k \in K$. For the converse direction, suppose that $k \Vdash (A \vee B)(\vec{d})$ for all $k \in K$. Fix an infinite branch α of $(K', \preceq')$. By our supposition, we have $\alpha(0) \Vdash (A \vee B)(\vec{d})$, and hence, $\alpha(0) \Vdash A(\vec{d})$ or $\alpha(0) \Vdash B(\vec{d})$. Then we have $\langle\alpha(0)\rangle_l \Vdash A(\psi_{\langle\alpha(0)\rangle})(\vec{d})$ or $\langle\alpha(0)\rangle_l \Vdash B(\psi_{\langle\alpha(0)\rangle})(\vec{d})$. By (ii) of I.H., we have that $\overline{\alpha}1 \Vdash' A(\vec{d})$ or $\overline{\alpha}1 \Vdash' B(\vec{d})$.

(ii) Let $\sigma \neq \langle\,\rangle$. In the following, we show that $(*)_1$ is equivalent to that $\sigma_l \Vdash (A \vee B)(\psi_\sigma(\vec{d}))$. First, suppose that $(*)_1$ holds. Then there exists $n > 0$ such that $\overline{(\sigma * \lambda i.\sigma_l)}n \Vdash' A(\vec{d})$ or $\overline{(\sigma * \lambda i.\sigma_l)}n \Vdash' B(\vec{d})$ (note that $\sigma * \lambda i.\sigma_l$ is an infinite branch of $(K', \preceq')$). By the monotonicity of $\Vdash'$, we may assume $n \geq |\sigma|$. By (ii) of I.H., we have that $\sigma_l \Vdash A(\psi_\sigma(\vec{d}))$ or $\sigma_l \Vdash B(\psi_\sigma(\vec{d}))$, which is equivalent to $\sigma_l \Vdash (A \vee B)(\psi_\sigma(\vec{d}))$. For the converse direction, suppose $\sigma_l \Vdash (A \vee B)(\psi_\sigma(\vec{d}))$. Then, for any infinite branch α of $(K', \preceq')$ extending σ, by (ii) of I.H., we have that $\overline{\alpha}|\sigma| \Vdash' A(\vec{d})$ or $\overline{\alpha}|\sigma| \Vdash' B(\vec{d})$. Thus we have $(*)_1$.

Case of $(A \to B)(\vec{x})$: Fix $\sigma \in K'$ and $\vec{d} \in M_{|\sigma|}$. Then $\sigma \Vdash' (A \to B)(\vec{d})$ is equivalent to $(*)_2$: for all $\sigma' \in K'$ such that $\sigma' \succeq' \sigma$, if $\sigma' \Vdash' A(\vec{d})$, then $\sigma' \Vdash' B(\vec{d})$.

(i) Let $\sigma = \langle\,\rangle$. In the following, we show that $(*)_2$ is equivalent to that for all $k \in K$, if $k \Vdash A(\vec{d})$, then $k \Vdash B(\vec{d})$, which is equivalent to that $k \Vdash (A \to B)(\vec{d})$ for all $k \in K$. First, suppose that $(*)_2$ holds. Fix $k \in K$. Suppose $k \Vdash A(\vec{d})$. Then, by (ii) of I.H., we have $\langle k\rangle \Vdash' A(\psi_{\langle k\rangle}(\vec{d}))$. Note that $\psi_{\langle k\rangle}(\vec{d}) = \vec{d}$ since $\vec{d} \in M_{|\langle\,\rangle|}$. Since $\langle k\rangle \in K'$, by $(*)_2$, we have $\langle k\rangle \Vdash' B(\psi_{\langle k\rangle}(\vec{d}))$. Again by (ii) of I.H., we have $k \Vdash B(\vec{d})$. For the converse direction, fix $\sigma' \in K'$ and suppose $\sigma' \Vdash' A(\vec{d})$. Then, by using I.H. as in the above argument, we have $\sigma' \Vdash' B(\vec{d})$. Here we use (i) of I.H. in the case of $\sigma' = \langle\,\rangle$ and (ii) of I.H. in the case of $\sigma' \neq \langle\,\rangle$.

(ii) Let $\sigma \neq \langle\,\rangle$. In the following, we show that $(*)_2$ is equivalent to $\sigma_l \Vdash (A \to B)(\psi_\sigma(\vec{d}))$. First, suppose that $(*)_2$ holds. Fix $k \in K$ such that $k \succeq \sigma_l$. Note that $\psi_\sigma(\vec{d}) = \psi_{\sigma*\langle k\rangle}(\vec{d})$ since $\vec{d} \in M_\sigma$. Suppose $k \Vdash A(\psi_\sigma(\vec{d}))$, equivalently, $k \Vdash A(\psi_{\sigma*\langle k\rangle}(\vec{d}))$. By (ii) of I.H., we have $\sigma * \langle k\rangle \Vdash' A(\vec{d})$. Since $\sigma * \langle k\rangle \in K'$ and $\sigma * \langle k\rangle \succeq \sigma$, by $(*)_2$, we have $\sigma * \langle k\rangle \Vdash' B(\vec{d})$. Again by (ii) of I.H., we have $k \Vdash B(\psi_{\sigma*\langle k\rangle}(\vec{d}))$, equivalently, $k \Vdash B(\psi_\sigma(\vec{d}))$. For the converse direction, suppose $\sigma_l \Vdash (A \to B)(\psi_\sigma(\vec{d}))$. Fix $\sigma' \in K'$ such that $\sigma' \succeq \sigma$. Suppose $\sigma' \Vdash A(\vec{d})$. Now $\sigma' \neq \langle\,\rangle$ since $\sigma \neq \langle\,\rangle$. By (ii) of I.H., we have $\sigma'_l \Vdash A(\psi_{\sigma'}(\vec{d}))$. Note that $\psi_{\sigma'}(\vec{d}) = \psi_\sigma(\vec{d})$ since $\vec{d} \in M_\sigma$. Since $\sigma'_l \succeq \sigma_l$, we have $\sigma'_l \Vdash B(\psi_{\sigma'}(\vec{d}))$ by our supposition and the monotonicity of $\Vdash$. Again by (ii) of I.H., we have $\sigma' \Vdash' B(\vec{d})$.

Case of $\exists y A(y, \vec{x})$: Fix $\sigma \in K'$ and $\vec{d} \in M_{|\sigma|}$.

(i) Let $\sigma = \langle\rangle$. By using the monotonicity of $\Vdash'$, we have that $\sigma \Vdash' \exists y A(y, \vec{x})$ is equivalent to $(*)_3$: for all $\alpha \in \sigma$, there exists $n \in \mathbb{N}$ and $m \in M_n$ such that $\overline{\alpha}n \Vdash' A(m, \vec{d})$. In the following, we show that $(*)_3$ is equivalent to that for all $k \in K$, there exists $d_0 \in D(k)$ such that $k \Vdash A(d_0, \vec{d})$, which is equivalent to $k \Vdash \exists y A(y, \vec{d})$. First, suppose that $(*)_3$ holds. Fix $k \in K$. Since $\lambda i.k$ is an infinite branch of $(K', \preceq')$, by $(*)_3$, there exists $n \in \mathbb{N}$ and $m \in M_n$ such that $\overline{(\lambda i.k)}n \Vdash' A(m, \vec{d})$. By the monotonicity of $\Vdash'$, we may assume $n > 0$. By (ii) of I.H., we have $k \Vdash A(\psi_{\overline{(\lambda i.k)}n}(m), \psi_{\overline{(\lambda i.k)}n}(\vec{d}))$. Note that $\psi_{\overline{(\lambda i.k)}n}(\vec{d}) = \vec{d}$ since $\vec{d} \in M_{|\langle\rangle|}$. In addition, we have $\psi_{\overline{(\lambda i.k)}n}(m) \in D(k)$ by the definition of ψ. Then we have that there exists $d_0 \in D(k)$ such that $k \Vdash A(d_0, \vec{d})$. For the converse direction, suppose that for all $k \in K$, there exists $d_0 \in D(k)$ such that $k \Vdash A(d_0, \vec{d})$. Fix an infinite branch α of $(K', \preceq')$. Then $\alpha(0) \in K$. By our supposition, there exists $d_0 \in D(\alpha(0))$ such that $\alpha(0) \Vdash A(d_0, \vec{d})$. Since $\psi_{\langle\alpha(0)\rangle} : M_1 \to D(\alpha(0))$ and $\psi_{\langle\alpha(0)\rangle}$ maps $\mathbb{N}_0$ onto $D(\alpha(0))$, there exists $m \in M_1$ such that $\psi_{\langle\alpha(0)\rangle}(m) = d_0$. Since $m, \vec{d} \in M_{|\overline{\alpha}1|}$, we have $(\overline{\alpha}1)_l \Vdash A(\psi_{\overline{\alpha}1}(m), \psi_{\overline{\alpha}1}(\vec{d}))$. By (ii) of I.H., we have $\overline{\alpha}1 \Vdash' A(m, \vec{d})$ for this $m \in M_1$. Thus we have $(*)_3$.

(ii) Let $\sigma \neq \langle\rangle$. Note that $\sigma \Vdash' \exists y A(y, \vec{x})$ is equivalent to $(*)_4$: for all $\alpha \in \sigma$, there exists $j, n \in \mathbb{N}$ and $m \in M_n$ such that $\overline{\alpha}j \Vdash' A(m, \vec{d})$. In the following, we show that $(*)_4$ is equivalent to that $\sigma_l \Vdash \exists y A(y, \psi_\sigma(\vec{d}))$. First, suppose that $(*)_4$ holds. By the monotonicity of $\Vdash'$, $(*)_4$ is equivalent to for all $\alpha \in \sigma$, there exists $j, n \in \mathbb{N}$ and $m \in M_n$ such that $j \geq |\sigma| + n$ and $\overline{\alpha}j \Vdash' A(m, \vec{d})$, which implies $\alpha(j-1) \Vdash A(\psi_{\overline{\alpha}j}(m), \psi_{\overline{\alpha}j}(\vec{d}))$ by (ii) of I.H. (note $j \geq |\sigma| > 0$). Applying this to the function $\alpha_\sigma := \sigma * \lambda i.\sigma_l$, which is an infinite branch of $(K', \preceq')$, we have that there exists $j, n \in \mathbb{N}$ and $m \in M_n$ such that $j \geq |\sigma| + n$ and $\sigma_l \Vdash A(\psi_{\overline{\alpha_\sigma}j}(m), \psi_{\overline{\alpha_\sigma}j}(\vec{d}))$. Since $j \geq |\sigma| + n$, we have $\psi_{\overline{\alpha_\sigma}j}(m) \in D(\sigma_l)$ and $\psi_{\overline{\alpha_\sigma}j}(\vec{d}) = \psi_\sigma(\vec{d})$. Therefore we have $\sigma_l \Vdash \exists y A(y, \psi_\sigma(\vec{d}))$. For the converse direction, suppose $\sigma_l \Vdash \exists y A(y, \psi_\sigma(\vec{d}))$. Fix an infinite branch α of $(K', \preceq')$ extending σ. By our supposition, there exists $d_0 \in D(\sigma_l)$ such that $\sigma_l \Vdash A(d_0, \psi_\sigma(\vec{d}))$. Since ψ_σ maps $\mathbb{N}_{|\sigma|-1}$ onto $D(\sigma_l)$, there exists $m \in M_{|\sigma|}$ such that $d_0 = \psi_\sigma(m)$. Then we have $\sigma_l \Vdash A(\psi_\sigma(m), \psi_\sigma(\vec{d}))$. By (ii) of I.H., we have $\sigma \Vdash' A(m, \vec{d})$. Since $\sigma = \overline{\alpha}|\sigma|$, by taking j and n as $|\sigma|$, we have that $m \in M_n$ and $\overline{\alpha}j \Vdash' A(m, \vec{d})$. Thus we have $(*)_4$.

Case of $\forall y A(y, \vec{x})$: Fix $\sigma \in K'$ and $\vec{d} \in M_{|\sigma|}$. Note that $\sigma \Vdash' \forall y A(y, \vec{d})$ is equivalent to $(*)_5$: for all $\sigma' \in K'$ such that $\sigma' \succeq' \sigma$, $n \in \mathbb{N}$ and $m \in M_n$, $\sigma' \Vdash' A(m, \vec{d})$ holds.

(i) Let $\sigma = \langle\rangle$. In the following, we show that $(*)_5$ is equivalent to that for all $k \in K$, for all $k' \in K$ such that $k' \succeq k$, and for all $d_0 \in D(k')$, $k' \Vdash A(d_0, \vec{d})$ holds. Note that $(*)_5$ is equivalent to that $k \Vdash \forall y A(y, \vec{d})$ for all $k \in K$. First, suppose that $(*)_5$ holds. Fix $k, k' \in K$ such that $k' \succeq k$ and $d_0 \in D(k')$. Since $\psi_{\langle k'\rangle}$ maps $\mathbb{N}_0$ onto $D(k')$, there exists $m_0 \in M_1$ such that $d_0 = \psi_{\langle k'\rangle}(m_0)$. Therefore, by $(*)_5$ (note $\langle k'\rangle \in K'$), we have $\langle k'\rangle \Vdash' A(m_0, \vec{d})$. By (ii) of I.H., we have $k' \Vdash A(\psi_{\langle k'\rangle}(m_0), \psi_{\langle k'\rangle}(\vec{d}))$. Since $\psi_{\langle k'\rangle}(\vec{d}) = \psi_{\langle\rangle}(\vec{d}) = \vec{d}$, we have $k' \Vdash A(d_0, \vec{d})$. For the converse direction, suppose that for all $k \in K$, for all $k' \in K$ such that $k' \succeq k$, and for all $d_0 \in D(k')$, $k' \Vdash A(d_0, \vec{d})$ holds. Fix $\sigma' \in K', n \in \mathbb{N}$ and $m \in M_n$. By Proposition 2.5, it suffices to show that for all $\alpha \in \sigma'$, there exists $j \in \mathbb{N}$ such that $\overline{\alpha}j \Vdash' A(m, \vec{d})$. To show the latter, fix an infinite branch α of $(K', \preceq')$ extending σ'. Put $j := |\sigma'| + n + 1$. Take k' as $(\overline{\alpha}j)_l$. Since $\psi_{\overline{\alpha}j}$ is a map from M_j to $\bigcup_{i<j} D(\alpha(i))$, we have $\psi_{\overline{\alpha}j}(m) \in D(k')$. Then, by our supposition, we have $k' \Vdash A(\psi_{\overline{\alpha}j}(m), \vec{d})$. Since $\vec{d} = \psi_{\langle\rangle}(\vec{d}) = \psi_{\overline{\alpha}j}(\vec{d})$, by (ii), we have $\overline{\alpha}j \Vdash' A(m, \vec{d})$.

(ii) Let $\sigma \neq \langle\,\rangle$. In the following, we show that $(*)_5$ is equivalent to that for all $k \in K$ such that $k \succeq \sigma_l$, for all $d_0 \in D(k)$, $k \Vdash A(d_0, \psi_\sigma(\vec{d}))$ holds. Note that the latter assertion is equivalent to $\sigma_l \Vdash \forall y A(y, \psi_\sigma(\vec{d}))$. First, suppose that $(*)_5$ holds. Fix $k \in K$ such that $k \succeq \sigma_l$ and $d_0 \in D(k)$. Put $\sigma' := \sigma * \langle k \rangle$. Since $\psi_{\sigma'}$ maps $\mathbb{N}_{|\sigma|}$ onto $D(k)$, there exists $m \in M_{|\sigma'|}$ such that $\psi_{\sigma'}(m) = d_0$. Since $\sigma' \in K'$ and $\sigma' \succeq \sigma$, by $(*)_5$, we have $\sigma' \Vdash' A(m, \vec{d})$. By (ii) of I.H., we have $k \Vdash A(\psi_{\sigma'}(m), \psi_{\sigma'}(\vec{d}))$. Since $\psi_{\sigma'}(m) = d_0$ and $\psi_{\sigma'}(\vec{d}) = \psi_\sigma(\vec{d})$, we have $k \Vdash A(d_0, \psi_\sigma(\vec{d}))$. For the converse direction, suppose that for all $k \in K$ such that $k \succeq \sigma_l$, for all $d_0 \in D(k)$, $k \Vdash A(d_0, \psi_\sigma(\vec{d}))$. Fix $\sigma' \in K'$ such that $\sigma' \succeq' \sigma$, $n \in \mathbb{N}$ and $m \in M_n$. By Proposition 2.5, it suffices to show that for all $\alpha \in \sigma'$, there exists $j \in \mathbb{N}$ such that $\overline{\alpha}j \Vdash' A(m, \vec{d})$. To show the latter, fix an infinite branch α of $(K', \preceq')$ extending σ'. Put $j := |\sigma'| + n$. Note $j > 0$ now. Take k as $(\overline{\alpha}j)_l$. Since $\overline{\alpha}j \succeq' \sigma$, we have $(\overline{\alpha}j)_l \succeq \sigma_l$. Since $\psi_{\overline{\alpha}j}$ is a map from M_j to $\bigcup_{i<j} D(\alpha(i))$, we have $\psi_{\overline{\alpha}j}(m) \in D(k)$. Then, by our supposition, we have $(\overline{\alpha}j)_l \Vdash A(\psi_{\overline{\alpha}j}(m), \psi_\sigma(\vec{d}))$. Since $\psi_\sigma(\vec{d}) = \psi_{\overline{\alpha}j}(\vec{d})$, we have $k \Vdash A(\psi_{\overline{\alpha}j}(m), \psi_{\overline{\alpha}j}(\vec{d}))$. Then, by (ii) of I.H., we have $\overline{\alpha}j \Vdash' A(m, \vec{d})$. $\square$

Theorem 3.9. *For a given Kripke model $\mathcal{K} := (K, \preceq, D, \Vdash)$ and its induced Beth model $\mathcal{B}_{\mathcal{K},\psi} := (K', \preceq', D', \Vdash')$ by a coding function ψ,*

$$\mathcal{B}_{\mathcal{K},\psi} \Vdash' A \Leftrightarrow \mathcal{K} \Vdash A$$

holds for all $\mathcal{L}$-sentence A.

Proof. Immediate from (i) of Lemma 3.8. $\square$

Acknowledgements

The author thanks Kenji Miyamoto, Satoru Niki and Kojiro Higuchi for helpful discussions. He also thanks Naosuke Matsuda for his remark (see Remark 3.7).

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