Reverse Mathematics of Uniform Distribution Theory

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Abstract

This paper examines the fundamental properties of uniform distribution theory within the framework of reverse mathematics. In particular, focusing on the uniform distribution modulo 1 of real sequences, we analyze the logical strength required for its definition, equivalent characterizations, and basic consequences.

The discussion is carried out mainly over RCA_0, and aims to clarify"which properties of uniform distribution can be formalized within RCA_0." In conventional arguments, measure-theoretic reasoning and limit operations are assumed, and thus the required axiomatic strength has been implicitly presupposed; however, in this paper, we reconstruct the theory based on finite approximations and computable evaluations, and show that many properties can in fact be handled within RCA_0.

1 Introduction

The purpose of this paper is to analyze, within the framework of reverse mathematics, the degree of axiomatic strength required for the basic concepts and main theorems appearing in uniform distribution theory.

Reverse mathematics takes second-order arithmetic as a foundational system and clarifies the logical resources behind mathematical statements by investigating equivalences between theorems and axioms; it was proposed by Friedman and later systematized by Simpson and others [9]. In this paper, we take RCA_0, the most basic system, as the base system.

The theory of uniform distribution studies how the fractional parts x_n of a real sequence (x_n) are distributed over the unit interval $[0, 1)$, describing this distribution in terms of limiting frequencies [1], [5]. This paper deals with that notion.

Definition 1 (uniformly distributed modulo 1, [1] 1.1)**.** *Let $\{x\}$ denote the fractional part of x, and $\#$ the counting function.*

A real sequence (x_n) is said to be uniformly distributed modulo 1 if, for any interval $[a, b) \subset [0, 1)$,

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq n \leq N \mid \{x_n\} \in [a, b)\} = b - a$$

holds.

This classical definition does not directly use measure-theoretic assumptions, and since it is given by finite computations and limit operations, it can be naturally formalized even over RCA_0.

The main results of this paper can be summarized in the following three points.

First, we show that the definition of u.d. mod 1 and its equivalent characterizations (representation by characteristic functions, representation by uniformly continuous functions, and representation by periodic complex uniformly continuous functions) are all equivalent over RCA_0. The proofs are based on finite approximations by step functions and error estimates using moduli of uniform continuity.The classical results can be found in Chapter 1, Section 1 of [1].

Second, we show that fundamental theorems, including the Weyl criterion(Theorem 2.1 of [1]), are provable over RCA_0. In particular, by giving a finite construction of trigonometric polynomial approximations of periodic uniformly continuous functions, we clarify that Fourier coefficients can be handled in a computable manner.

Third, as concrete examples, we show that the uniform distribution of the sequence (nr) for an irrational number r, the almost everywhere uniform distribution of the sequence (x^n), and Koksma's general metric theorem are provable over RCA_0. We avoid Fatou's lemma used in classical proofs, and complete the argument by using a lemma on monotone convergence and evaluations of series as computable real numbers.The classical results can be found in Chapter 1, Section 4 of [1].

From the above, it becomes clear that the foundational part of uniform distribution theory does not depend on strong axioms, and can be developed within the framework of RCA_0, consisting only of finite and computable operations. This paper organizes the correspondence between the conceptual content of uniform distribution and its logical strength, and provides a foundation for further reverse mathematical analysis.

2 Preliminaries

2.1 Second-Order Arithmetic and the Arithmetical Hierarchy

In order to treat uniform distribution theory in analysis, we review the basic facts of second-order arithmetic in this section.For further details, see [6] and [9].

The language L_2 of second-order arithmetic is a two-sorted language with number variables and set variables. The definitions of arithmetical formulas Σ^0_n, Π^0_n, and Π^1_1 formulas follow the standard ones.

It is known that many concepts of countable mathematics can be formalized in one of the following five subsystems:

$$\mathrm{RCA}_0 < \mathrm{WKL}_0 < \mathrm{ACA}_0 < \mathrm{ATR}_0 < \Pi^1_1\text{-}\mathrm{CA}_0.$$

These are called Big Five.

In this paper, we consider the base system RCA_0. RCA_0 is a system equipped with Σ^0_1-induction and the Δ^0_1-comprehension scheme.

Definition 2 (Σ^0_1-induction)**.** *For any Σ^0_1 formula $\varphi(n)$, the statement*

$$(\varphi(0) \wedge \forall n\,(\varphi(n) \to \varphi(n+1))) \;\to\; \forall n\,\varphi(n)$$

is called the Σ^0_1-induction scheme.

Definition 3 (Δ^0_1-comprehension scheme)**.** *Let $\varphi(n)$ be a Σ^0_1 formula and $\psi(n)$ be a Π^0_1 formula. If $\forall n\,(\varphi(n) \leftrightarrow \psi(n))$, then*

$$\exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n)).$$

This is called the Δ^0_1-comprehension scheme.

The sets obtained by the Δ^0_1-comprehension scheme are limited to computable sets. Therefore, principles asserting the existence of non-computable sets require systems stronger than RCA_0.

2.2 Basic Definitions in Analysis

The basic building blocks of second-order arithmetic are natural numbers, and we define all concepts with the intention of coding them by natural numbers as much as possible. For example, integers and rational numbers are represented as pairs of natural numbers, and real numbers are represented as approximations by sequences of rational numbers. Functions are also considered via approximations by simple functions, so that they can be coded by natural numbers.

We now look at the definitions in detail. All of these definitions are carried out over RCA_0.The definitions are taken mainly from [2].

Definition 4 ([2] 1.1.2). *A real number is represented as a fast-converging Cauchy sequence of rational numbers. That is, it is represented by a sequence* $(q_n \mid n \in \mathbb{N})$ *s.t.* $(\forall n, q_n \in \mathbb{Q}) \wedge (\forall m, n\,(m < n \to |q_m - q_n| \leq 2^{-m}))$.

The set of real numbers is the completion of the countable dense sequence $\mathbb{Q}$.

Definition 5 ([2] 1.1.3). *A code of a complete separable metric space* $\hat{A}$ *is represented by a nonempty set* $A \subseteq \mathbb{N}$ *and a distance function* $d : A \times A \to \mathbb{R}$ *satisfying the following:*

For all $a, b, c \in A$,

$$(1)\ d(a,a) = 0 \quad (2)\ d(a,b) = d(b,a) \quad (3)\ d(a,b) + d(b,c) \geq d(a,c)$$

A complete separable metric space is the completion of a countable dense subset A, and elements of such a space are strongly Cauchy sequences of elements of A.

Definition 6 ([2] 1.1.8). *A compact metric space is a complete separable metric space* $\hat{A}$ *for which there exists a sequence of finite sequences of points of* $\hat{A}$, $\langle x_{ij} \mid i \leq n_j,\ j \in \mathbb{N}\rangle$, *satisfying the following property:*

$$\forall z \in \hat{A}\ \forall j \in \mathbb{N}\ \exists i \leq n_j \big(d(x_{ij}, z) < 2^{-j}\big).$$

The following definition concerns functions on a complete separable metric space. In order to develop measure theory, it is formulated using simple functions, and thus differs slightly from Definition 4.1.1 in [2].

Definition 7. *Let* $X = \hat{A}$ *and let* d *be the metric on* X. *Let* $B = A \times \mathbb{Q}^+$. *An element* $b = (a, r) \in B$ *represents a function* ϕ_b *as follows:*

$$\phi_b(x) = \begin{cases} 1 & d(a,x) \leq r \\ 0 & d(a,x) > r \end{cases}$$

Define $C = \mathbb{Q} \times B$. *If* $c = \{(q_1, b_1), \ldots (q_n, b_n)\}$ *is a finite subset of* C, *define* $\phi_c : X \to \mathbb{R}$ *by* $\phi_c(x) = \sum_{k=1}^{n} q_k \phi_{b_k}(x)$.

Let $S(X) = \{c \mid c \text{ is a finite subset of } C\}$. *Elements of* $S(X)$ *are called simple functions. Finally, define* $C(X) = \hat{S}(X)$ *with the sup norm* $\|c\| = \sup_{x \in X} |\phi_c(x)|$.

That is, a function $f \in C(X)$ *means that there exists a sequence of simple functions* $\langle f_n \rangle_{n \in \mathbb{N}} \subseteq S(X)$ *such that* $\|f_m - f_n\|_1 \leq 2^{-m}(n > m)$.

Definition 8. *A continuous function* $f : \hat{A} \to \hat{B}$ *between two metric spaces is said to be uniformly continuous if* $\forall \varepsilon > 0, \exists \delta > 0$ *s.t.* $d(x,y) < \delta \to d(f(x), f(y)) < \varepsilon$. *If there exists a function* $h : \mathbb{N} \to \mathbb{R}$ *such that* $d(x,y) < h(n) \to d(f(x), f(y)) < 2^{-n}$, *then* h *is called a modulus of uniform continuity.*

Remark 1. *In general, when using* lim, *we assume that a modulus of convergence is given for sequences and functions. In particular, points in* $C([0,1])$ *are uniformly continuous functions with a modulus, and when considering uniformly continuous functions, we assume that a modulus is given.*

2.3 Definitions in Functional Analysis

Definition 9. *We define a relation concerning the measure of open sets. Suppose an open set $U \subseteq \mathbb{R}$ is given by a sequence of open intervals $(I_n)_{n\in\mathbb{N}}$ as*

$$U = \bigcup_{n=1}^{\infty} I_n.$$

For $r \in \mathbb{R}^+$, define the following relation $\mu(U, r)$:

$$\mu(U, r) \iff \forall n \in \mathbb{N}\ \mu(\bigcup_{m<n} I_m) \leq r.$$

When attempting to define the measure of open sets, since we consider supremum and infimum, it does not necessarily exist. In fact, the statement that every open set has a real number as its measure is equivalent to the arithmetical comprehension axiom (ACA).

Definition 10. *If there exists a sequence of open sets (U_n) such that $\forall n, (\mu(U_n, 2^{-n}) \wedge A \subseteq U_n)$, then A is a null set.*

Remark 2. *The countable additivity of null sets in this definition can be shown over* RCA_0.

Proof. Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of null sets. For each i, by the definition of null sets, there exists a sequence of open sets $(U_i^n)_{n\in\mathbb{N}}$ such that

$$\forall n \in \mathbb{N}, \mu(U_i^n, 2^{-n}) \wedge A_i \subseteq U_i^n$$

holds.

Define, for each n,

$$V_n = \bigcup_{i=1}^{\infty} U_i^{n+i}.$$

Then, for any i, since $A_i \subseteq U_i^{n+i} \subseteq V_n$, we have

$$\bigcup_{i=1}^{\infty} A_i \subseteq V_n.$$

On the other hand, by subadditivity of measure,

$$\mu(V_n) \leq \sum_{i=1}^{\infty} \mu(U_i^{n+i}) \leq \sum_{i=1}^{\infty} 2^{-(n+i)} = 2^{-n}.$$

Hence $(V_n)_{n\in\mathbb{N}}$ is a sequence of open sets covering $\bigcup_{i\in\mathbb{N}} A_i$, and

$$\forall n \in \mathbb{N}, \mu(V_n, 2^{-n})$$

holds.

Therefore $\bigcup_{i=1}^{\infty} A_i$ is a null set. □

Definition 11 ([2] 4.1.4)**.** *Let $p \geq 1$ be a real number, and let X be a compact metric space. Define the separable Banach space $L_p(X)$ as the completion of $S(X)$ with respect to the L_p norm*

$$\|f\|_p = \mu\big(|f|^p\big)^{1/p}.$$

That is, a function $f \in L_p(X)$ means that there exists a sequence of simple functions $\langle f_n\rangle_{n\in\mathbb{N}} \subseteq S(X)$ such that

$$\|f_m - f_n\|_p \leq 2^{-m} \qquad (n > m)$$

holds.

In particular, when $X = [a, b]$, the L_1 norm is given by $\int_a^b |f(x)|dx$.

Definition 12 ([3] 2.5.1)**.** *Let I be a compact interval, and let $f, g\colon I \to \mathbb{R}$ be continuous functions. If*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in I, d(x, y) \leq \delta \text{ s.t. } \left|f(y) - f(x) - g(x)(y - x)\right| \leq \varepsilon |y - x|$$

holds, then f is said to be differentiable on I, and g is called the derivative of f on I.

If there exists a function $h : \mathbb{N} \to \mathbb{R}$ such that $d(x, y) < h(n) \to \left|f(y) - f(x) - g(x)(y - x)\right| \leq 2^{-n}|y - x|$, then h is called a modulus of differentiability.

3 uniformly distributed modulo 1

In this chapter, we consider uniform distribution modulo 1 in uniform distribution theory, and examine the behavior of various related concepts from the viewpoint of reverse mathematics.

3.1 Definition and properties of u.d. mod 1

For a real number x, let $[x]$ denote the integer part of x, and $\{x\} = x - [x]$ the fractional part of x. Let the unit interval be $I = [0, 1)$.

Let (x_n), $n = 1, 2, \ldots$, be a given real sequence. For a positive integer N and a subset E of I, define the counting function $A(E; N; (x_n))$ as the number of terms among $1 \leq n \leq N$ satisfying $\{x_n\} \in E$.

That is, for the sequence (x_n), it represents the proportion of those x_n up to the N-th term whose fractional parts belong to E. Instead of $A(E; N; (x_n))$, it is often written as $A(E; N)$.

We take this as the basic definition.

Definition 13. *The following definition is given over* RCA_0*.*

A real sequence (x_n), $n = 1, 2, \ldots$, is said to be uniformly distributed modulo 1 (abbreviated as u.d. mod 1) if (x_n) satisfies:

$$\forall a, b \in \mathbb{R},\ 0 \leq a < b \leq 1,\ \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \left|\frac{A([a, b); N; (x_n))}{N} - (b - a)\right| < \varepsilon \tag{1}$$

Also, it is expressed using the characteristic function as follows:

$$\forall a, b \in \mathbb{R},\ 0 \leq a < b \leq 1,\ \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \left|\frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)}(\{x_n\}) - \int_0^1 \chi_{[a,b)}(x)\, dx\right| < \varepsilon \tag{2}$$

Theorem 1. *The following statement is provable over* RCA_0*.*

(x_n) is u.d. mod 1 $\iff$ the following holds for every uniformly continuous function f:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \left|\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) - \int_0^1 f(x)dx\right| < \varepsilon \tag{3}$$

For the classical result, see Theorem 1.1 of [1].

Proof. Assume that (x_n) is u.d. mod 1.

First consider the case where f is a simple function. Let $f = \sum_{i=0}^{k-1} d_i \chi_{[a_i, a_{i+1})}(x)$ on $\overline{I} = [0, 1)$, where $0 = a_0 < a_1 < \cdots < a_k = 1$. Then $f \in S(X)$, and for any given ε, a suitable N can be obtained effectively.

Next we extend the argument to uniformly continuous functions.

In the classical formulation, one considers continuous functions. Here we instead restrict attention to uniformly continuous functions. This allows us to work with a modulus of uniform continuity and to obtain effective bounds for Riemann integrable functions.

Fix $\varepsilon > 0$. Since f is uniformly continuous and Riemann integrable, there exist simple functions $f_1, f_2 \in S(X)$ such that

$$((\forall x \in \overline{I})\, f_1(x) \le f(x) \le f_2(x)) \wedge (\int_0^1 (f_2(x) - f_1(x))\, dx \le \varepsilon)$$

By the assumption, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned}\int_0^1 f(x)\, dx - \varepsilon \le \int_0^1 f_1(x)\, dx - \frac{\varepsilon}{4} \le \frac{1}{N}\sum_{n=1}^{N} f_1(\{x_n\}) \le \frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) \\ \le \frac{1}{N}\sum_{n=1}^{N} f_2(\{x_n\}) \le \int_0^1 f_2(x)\, dx + \frac{\varepsilon}{4} \le \int_0^1 f(x)\, dx + \varepsilon.\end{aligned}$$

Thus, such an N is obtained effectively.

Conversely, suppose that (x_n) satisfies equation (3). Let $[a,b) \subset \overline{I}$ be arbitrary.

Fix $\varepsilon > 0$. Choose simple functions $g_1, g_2 \in S(X)$ such that

$$(g_1(x) \le \chi_{[a,b)}(x) \le g_2(x)) \wedge (\int_0^1 (g_2(x) - g_1(x))\, dx \le \varepsilon)$$

By the assumption, there exists N such that

$$\begin{aligned}b - a - \varepsilon \le \int_0^1 g_2(x)\, dx - \varepsilon \le \int_0^1 g_1(x)\, dx - \frac{\varepsilon}{4} \le \frac{1}{N}\sum_{n=1}^{N} g_1(\{x_n\}) \\ \le \frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)}(\{x_n\}) \le \frac{1}{N}\sum_{n=1}^{N} g_2(\{x_n\}) \le \int_0^1 g_2(x)\, dx + \frac{\varepsilon}{4} \\ \le \int_0^1 g_1(x)\, dx + \varepsilon \le b - a + \varepsilon.\end{aligned}$$

This establishes the claim. □

Corollary 1. *The following statement is provable over* RCA_0*.*

(x_n) is u.d. mod 1 $\iff$ equation (3) holds for every Riemann integrable function f on $\overline{I}$.

Corollary 2. *The following statement is provable over* RCA_0*.*

(x_n) is u.d. mod 1 $\iff$ the following holds for every complex uniformly continuous function f on $\mathbb{R}$ with period 1:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \left| \frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)dx \right| < \varepsilon \tag{4}$$

Lemma 1. *The following statement is provable over* RCA_0*.*

(x_n) is u.d. mod 1 $\implies \forall \alpha$, $(x_n + \alpha)$ is also u.d. mod 1

Proof. Assume that (x_n) is u.d. mod 1. By Definition 13, take arbitrary $0 \le a < b \le 1$ and $\varepsilon > 0$.

First, by the property of fractional parts,

$$\{x_n+\alpha\} \in [a,b) \iff \begin{cases} \{x_n\} \in [\{a-\alpha\}, \{b-\alpha\}), & \text{if } \{a-\alpha\} \le a, \\ \{x_n\} \in [0, \{b-\alpha\}) \cup [\{a-\alpha\}, 1), & \text{if } \{a-\alpha\} > a. \end{cases}$$

The above interval can be expressed, on the unit interval $I = [0,1)$, as the union of at most two half-open intervals. That is,

$$\{x_n+\alpha\} \in J_1 \cup J_2$$

for some $J_1, J_2 \subset I$, and their lengths satisfy

$$|J_1| + |J_2| = b - a.$$

(In the case $\{a-\alpha\} \le a$, it suffices to consider $\frac{\{a-\alpha\}+\{b-\alpha\}}{2}$.)

Therefore, for any $N \in \mathbb{N}$,

$$A([a,b); N; (x_n+\alpha)) = A(J_1; N; (x_n)) + A(J_2; N; (x_n))$$

holds.

The rest follows by evaluating using $\frac{\varepsilon}{2}$ from the fact that (x_n) is u.d. mod 1. □

Theorem 2. *The following statement is provable over* RCA_0.

(x_n) is u.d. mod 1, and (y_n) satisfies $\forall\varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N, |(x_n - y_n) - \alpha| < \varepsilon \implies$ (y_n) is also u.d. mod 1

For the classical result, see Theorem 1.2 of [1].

Proof. By Lemma 1, we may fix $\alpha = 0$.

Define $\varepsilon_n = x_n - y_n$. Let $0 < a < b < 1$ and choose ε from $0 < \varepsilon < \min(a, 1-b, \frac{b-a}{2})$.

$$\begin{gathered} \exists N_0 = N_0(\varepsilon) \text{ s.t. } \forall n \ge N_0, -\varepsilon \le \varepsilon_n \le \varepsilon \\ n \ge N_0 \to (a+\varepsilon \le \{x_n\} < b-\varepsilon \to a \le \{y_n\} < b) \\ \wedge(a \le \{y_n\} < b \to a-\varepsilon \le \{x_n\} < b+\varepsilon) \end{gathered}$$

At this time, there exists N_0 such that the following inequality holds:

$$\begin{aligned} \exists N_0(b-a-2\varepsilon = &\frac{A([a+\varepsilon, b-\varepsilon); N_0; (x_n))}{N_0} \le \frac{A([a,b); N_0; (y_n))}{N_0} \le \\ &\frac{A([a-\varepsilon, b+\varepsilon); N_0; (x_n))}{N_0} = b-a+2\varepsilon) \end{aligned}$$

Thus, an N satisfying the condition is obtained. □

3.2 Weyl criterion

3.2.1 Restricted Fourier transform over RCA_0

In the proof of the Weyl criterion, which will be used later to characterize u.d. mod 1 , we require a Fourier-analytic treatment of periodic uniformly continuous functions.

For this reason, we develop the argument over RCA_0 in a conditional form. The resulting proof is necessarily somewhat involved. The difficulty is that periodicity and uniform continuity alone do not provide sufficiently explicit information about the structure of a function f.

To overcome this, we approximate f by simpler functions while preserving a prescribed level of accuracy. At the same time, we modify the rough shape of f so that its structure becomes more tractable. Based on this approximation, we then construct its Fourier series.

Definition 14 (modulus of integrability, [4] Definition 2). *The following definition is carried out in* RCA_0.

Let f *be a continuous function from* $[a,b]$ *to* $\mathbb{R}$. *A modulus of integrability of* f *on* $[a,b]$ *is a function* h *from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that for all* $n \in \mathbb{N}$ *and all partitions* Δ_1, Δ_2 *of* $[a,b]$,

$$|\Delta_1| < \frac{2^{-h(n)}}{b-a} \wedge |\Delta_2| < \frac{2^{-h(n)}}{b-a} \to |S^{\Delta_1}_{[a,b]}(f) - S^{\Delta_2}_{[a,b]}(f)| < 2^{-n}$$

holds.

When f *has a modulus of integrability,* f *is said to be effectively integrable.*

Lemma 2 (from simple functions to piecewise linear functions). *The following statement is provable over* RCA_0.

Suppose $f_n \in S([0,1])$ *satisfies*

$$\forall x \ |f(x) - f_n(x)| \leq 2^{-n}$$

Then there exists a piecewise linear uniformly continuous function g_n *such that*

$$\forall x \in [0,1] \ |f_n(x) - g_n(x)| \leq 2^{-n}$$

holds.

Proof. f_n is represented by a finite partition

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

and a sequence of real numbers $c_0, \ldots, c_{m-1}$, and on each (t_i, t_{i+1}),

$$f_n(x) = c_i$$

holds.

Take interior points of each interval

$$t_i^{\mathrm{L}} = t_i + \tfrac{1}{4}(t_{i+1} - t_i), \qquad t_i^{\mathrm{R}} = t_{i+1} - \tfrac{1}{4}(t_{i+1} - t_i)$$

On the central part $[t_i^{\mathrm{L}}, t_i^{\mathrm{R}}]$, set $g_n = c_i$, and near the endpoints, linearly interpolate adjacent values. That is, define by the line segments connecting t_{i-1}^{R} and t_i^{L}, and t_i^{R} and t_{i+1}^{L}. Namely,

$$g_n(x) = \begin{cases} c_i, & x \in [t_i^{\mathrm{L}},\, t_i^{\mathrm{R}}], \\ c_{i-1} + \dfrac{c_i - c_{i-1}}{t_i^{\mathrm{L}} - t_{i-1}^{\mathrm{R}}}\big(x - t_{i-1}^{\mathrm{R}}\big), & x \in [t_i,\, t_i^{\mathrm{L}}], \\ c_i + \dfrac{c_{i+1} - c_i}{t_{i+1}^{\mathrm{L}} - t_i^{\mathrm{R}}}\big(x - t_i^{\mathrm{R}}\big), & x \in [t_i^{\mathrm{R}},\, t_{i+1}]. \end{cases}$$

Then g_n is a piecewise linear function consisting of finitely many linear functions, and therefore is uniformly continuous.

The error occurs only on the intervals where interpolation by slopes is performed. Its value is the difference of adjacent constant values, so

$$|f_n(x) - g_n(x)| \leq \max_i |c_{i-1} - c_i|.$$

From the construction of f_n, $\max_i |c_{i-1} - c_i| \leq 2^{-n}$ holds, so the claim follows. $\square$

Lemma 3 (smoothing by circular arcs)**.** *The following statement is provable over* RCA_0*.*

For every piecewise linear function g_n *and every* $\varepsilon > 0$*, there exists a* C^1 *function* $\tilde{g}_n$ *such that*

$$\forall x \; |g_n(x) - \tilde{g}_n(x)| < \varepsilon.$$

Moreover, both $\tilde{g}_n$ *and its derivative* $\tilde{g}_n'$ *are effectively integrable in* RCA_0*.*

Proof. We construct $\tilde{g}_n$ by modifying g_n only near its corner points.

Fix a corner point t_i^{L} (the argument for t_i^{R} is identical). On a neighborhood of width 2^{-k-l} around t_i^{L}, we replace the graph of g_n by a circular arc. The center of this circle is chosen on the angle bisector so that the arc is tangent to the two adjacent line segments. Outside these neighborhoods, we keep $\tilde{g}_n = g_n$. By construction, the resulting function $\tilde{g}_n$ is of class C^1.

We next estimate the approximation error. The inserted circular arc lies between the original two line segments and the line segment connecting the points$g_n\big(t_i^{\mathrm{L}} \pm 2^{-k-l}\big)$.

Hence the maximum deviation at t_i^{L} is bounded by

$$|l(t_i^{\mathrm{L}}) - g_n(t_i^{\mathrm{L}})| = \frac{|c_i - c_{i-1}|}{2^l}.$$

By the assumption $\max_i |c_i - c_{i-1}| \leq 2^{-n}$, we can make this bound smaller than ε by choosing l sufficiently large. The same estimate holds for t_i^{R}.

Finally, we verify effective integrability. The function $\tilde{g}_n$ is given explicitly as a finite combination of linear functions and circular arcs. For each component, both the integral and the derivative are computable in RCA_0. Therefore, $\tilde{g}_n$ and $\tilde{g}_n'$ are effectively integrable. □

Lemma 4 (uniform approximation by Fourier series)**.** *The following statement is provable over* RCA_0*.*

If $\tilde{g}_n$ *is a* C^1 *function (extended periodically) and* $\tilde{g}_n$ *and* $\tilde{g}_n'$ *are effectively integrable, then its Fourier series* $S_M^{(n)}$ *converges uniformly to* $\tilde{g}_n$*.*

Proof. It follows from Lemma 4 of [4]. Map $\tilde{g}_n$ to a function of period 2π from period 1 by the linear transformation $x \mapsto 2\pi x$, and use the periodic boundary condition $\tilde{g}_n(0) = \tilde{g}_n(1)$.

Therefore, for any $\varepsilon > 0$, for sufficiently large M,

$$\forall x \in [0,1], |\tilde{g}_n(x) - S_M^{(n)}(x)| < \varepsilon$$

holds. □

Theorem 3 (approximation theorem by trigonometric functions)**.** *The following tstatement is provable over* RCA_0*.*

Let $f : [0,1] \to \mathbb{R}$ *be a periodic uniformly continuous function, and suppose that a sequence of simple functions* $(f_n)_{n\in\mathbb{N}}$ *satisfies*

$$|f(x) - f_n(x)| < 2^{-n}$$

Then for any $\varepsilon > 0$*, there exist* N *and coefficients* $(a_0, \ldots, a_N, b_1, \ldots, b_N)$ *such that*

$$\forall x \in [0,1], \quad |f(x) - S_N(x)| < \varepsilon$$

holds, where

$$S_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos(2\pi k x) + b_k \sin(2\pi k x))$$

Proof. Fix $\varepsilon > 0$.

1. Take n sufficiently large so that $|f(x) - f_n(x)| < \varepsilon/4$
2. There exists a piecewise linear function g_n such that $|f_n(x) - g_n(x)| < \varepsilon/4$
3. There exists a C^1 function $\tilde{g}_n$ such that $|g_n(x) - \tilde{g}_n(x)| < \varepsilon/4$, and $\tilde{g}_n, \tilde{g}'_n$ are effectively integrable
4. There exists M such that $|\tilde{g}_n(x) - S_M^{(n)}(x)| < \varepsilon/4$

Therefore

$$|f(x) - S_M^{(n)}(x)| < \varepsilon$$

Taking $N = M$, we obtain the conclusion. □

3.2.2 Weyl criterion

Define the function $f(x) = e^{2\pi i h x}$ for $h \in \mathbb{Z}\setminus\{0\}$ (integers excluding 0). Since this f is a complex uniformly continuous function, it satisfies the condition of Corollary 2. Therefore, (x_n) being u.d. mod 1 implies that f satisfies equation (4).

Theorem 4 (Weyl criterion)**.** *The following theorem is provable over* RCA_0*.*
(x_n) *is u.d. mod 1* $\Leftrightarrow$

$$\forall h \in \mathbb{Z} \setminus \{0\}, \forall \varepsilon > 0, \exists N \in \mathbb{N}, \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| < \varepsilon \tag{5}$$

For the classical result, see Theorem 2.1 of [1].

Proof. $(\Rightarrow)$ was shown above.

$(\Leftarrow)$ Suppose (x_n) satisfies equation (5). Then we show that equation (4) holds for all complex continuous functions f on $\mathbb{R}$ with period 1. Let ε be any positive number. By Theorem 3, a trigonometric polynomial Ψ, which is a finite linear combination of functions of the form $e^{2\pi i h x}$ $(h \in \mathbb{Z} \setminus \{0\})$, is computably given and satisfies

$$\forall \varepsilon > 0, \exists \Psi, \forall x \in [0,1)\, |f(x) - \Psi(x)| < \varepsilon$$

For sufficiently large N satisfying equation (3.6), the following holds:

$$\begin{aligned}
&\left| \int_0^1 f(x)dx - \frac{1}{N}\sum_{n=1}^{N} f(x_n) \right| \\
&\leq \left| \int_0^1 (f(x) - \Psi(x))dx \right| + \left| \int_0^1 \Psi(x)dx - \frac{1}{N}\sum_{n=1}^{N} \Psi(x_n) \right| + \left| \frac{1}{N}\sum_{n=1}^{N} (\Psi(x_n) - f(x_n)) \right| \\
&\leq \left| \int_0^1 \varepsilon dx \right| + \left| 0 - \frac{1}{N}\sum_{n=1}^{N} \Psi(x_n) \right| + \left| \frac{1}{N}\sum_{n=1}^{N} \varepsilon \right| \\
&\leq 3\varepsilon
\end{aligned}$$

Therefore, it is computably shown that a sequence (x_n) satisfying equation (5) satisfies equation (4). Hence, by Corollary 2, it is provable over RCA_0.

In the classical proof, this trigonometric polynomial Ψ is given by the Weierstrass approximation theorem. While it can be confirmed in Chapter 4, Section 5 of [3] that the Weierstrass approximation theorem is provable over RCA_0, that proof only provides approximation by polynomial functions, and thus we prepared Theorem 3 in this paper.

□

Using the Weyl criterion, the following statement can be shown.

Corollary 3. *The following statement is provable over* RCA_0.

For an irrational number r, let $(x_n) = nr$. Then (x_n) is u.d. mod 1.

4 u.d. mod 1 for each sequence

This chapter refers to Section 4 of Chapter 1 in [1].

First, in this chapter, in order to deal with the concept of u.d. mod 1 a.a. x, we introduce its definition.

Definition 15. *The following definition is carried out over* RCA_0.

For all x in J, let $(u_n(x))$ be a sequence of real numbers. We say that $(u_n(x))$ is u.d. mod 1 for almost all x (u.d. mod 1 a.a. x) if, for all x belonging to J, except for a null set, $(u_n(x))$ is u.d. mod 1.

Remark 3. *In this chapter, proofs are carried out over* RCA_0*, but in order to give meaning to "almost everywhere,"* WWKL *is required. Indeed, since the whole set of real numbers can be taken as a null set, in order to ensure that there always exists an element outside a null set, one must assume* WWKL*, which is an axiom corresponding to this requirement. Therefore, although the statements themselves are shown over* RCA_0 *in this chapter, they do not necessarily always have meaning.*

For further details, see [7] and [8].

4.1 u.d. mod 1 for sequences of distinct integers

In this section, we study u.d. mod 1 for sequences of distinct integers.

In the classical proof of this result, Fatou's lemma is used. However, this argument only establishes the statement over ACA_0.

In the present setting, the situation is more restrictive. The sequence of functions under consideration is monotone increasing, and the expression corresponding to the upper bound in Fatou's lemma can in fact be realized as a real number.

Exploiting these additional features, we replace Fatou's lemma with a weaker principle adapted to this setting. This allows us to carry out the proof within RCA_0.

Lemma 5. *The following statement is provable over* RCA_0.

Let $\{f_n\}$ be a sequence of measurable monotone increasing functions. Furthermore, suppose that there exists $M \in \mathbb{R}$ such that $\lim_{n\to\infty} \int f_n d\mu = M$.

Then $\lim_{n\to\infty} f_n \in L_1$ exists and $\int \lim_{n\to\infty} f_n d\mu = M$.

Proof. For each f_n, approximate it by a simple function f_n^k so that $\int f_n - f_n^k d\mu < 2^{-k}$.

For each n, since f_n is a measurable function and an element of $C(X)$, such an approximating sequence can be constructed in RCA_0.

At this time, $\forall n, M - \int f_n d\mu < 2^{-n}$ holds.

Also, define $g_j = \max_{a,b\le j} f_b^a$ pointwise.

Then

$$0 \le M - \int f_n^n d\mu \le \int M - f_n d\mu + \int f_n - f_n^n d\mu < 2^{-n} + 2^{-n} = 2^{-n+1}$$

Since $g_j \geq f_j^j$, we have $M - \int g_j d\mu < 2^{-j+1}$, and by assumption $\int g_j d\mu \leq M$.
We verify that g_j is a Cauchy sequence in L_1.
Since g_j is monotone increasing, for $k > j$,

$$\begin{aligned} |g_k - g_j|_1 &= \int |g_k - g_j| d\mu = \int |g_k - M + M - g_j| d\mu \\ &\leq \int (M - g_k) d\mu + \int (M - g_j) d\mu \leq 2^{-k+1} + 2^{-j+1} \leq 2^{-j+2} \end{aligned}$$

Therefore, since completeness of the L_1 space is guaranteed by its definition over RCA_0, there exists $g \in L_1$ such that $\lim_{j\to\infty} g_j = g$.
Since $\forall n, g \geq f_n$ and $\int g - f_n d\mu \leq 2^{-n}$,
it is shown that $\lim_{n\to\infty} f_n = g$ in L_1.
That is, it follows from $\int g d\mu = M$. □

In addition, there is one more statement that needs to be shown. In fact, in the above statement, it is necessary to obtain the limit M within RCA_0. In this statement, the Basel problem is actually obtained within RCA_0.

Lemma 6. *The following statement is provable over* RCA_0.
In RCA_0, *the equation of the Basel problem*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

is provable, and the infinite series on the left-hand side can be obtained exactly as a computable real number.

Proof. Consider the function $f(x) = x^2$ on the interval $[-\pi, \pi]$, and extend it to the whole $\mathbb{R}$ as a periodic function with period 2π. This function is a uniformly continuous periodic function. Therefore, f satisfies the condition of Theorem 3.
The Fourier series expansion of $f(x) = x^2$ is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

By Theorem 3, this Fourier series converges uniformly to f. Therefore, this uniform convergence is provable in RCA_0.
From uniform convergence, pointwise convergence at any point $x \in [-\pi, \pi]$ is guaranteed. In particular, substituting $x = \pi$ into the Fourier expansion of x^2, we obtain

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi)$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2 - \pi^2/3}{4} = \frac{2\pi^2/3}{4} = \frac{\pi^2}{6}$$

is obtained.
Thus, the equation of the Basel problem is proved in RCA_0. Furthermore, since each step of the proof consists only of computable operations, it is shown that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is exactly equal to $\pi^2/6$ as a computable real number. □

Using the above lemmas, we consider u.d. mod 1 for sequences satisfying $\forall m, n \in \mathbb{N}$ s.t. $a_n \neq a_m$.

Theorem 5. *The following statement is provable over* RCA_0.

Let (a_n) be a sequence of integers satisfying $\forall m, n \in \mathbb{N}$ s.t. $a_n \neq a_m$. Then for $x \in \mathbb{R}$, $(a_n x)$ is u.d. mod 1 a.a. x.

For the classical result, see Theorem 4.1 in [1].

Proof. It suffices to show that $(a_n x)$ is u.d. mod 1 a.a. x for $x \in I = [0, 1)$.

This is because for $k \in \mathbb{Z}$, $\{a_n(x + k)\} = \{a_n x\}$ holds.

Also, a countable union of translations of a null set is still a null set. This is guaranteed over RCA_0 by Remark 2.

We prove this using the Weyl criterion. First fix $h \in \mathbb{Z} \setminus \{0\}$ and define

$$S(N, x) = \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h a_n x} \quad (N \geq 1, 0 \leq x \leq 1)$$

$$|S(N, x)|^2 = S(N, x)\overline{S(N, x)} = \frac{1}{N^2} \sum_{m,n=1}^{N} e^{2\pi i h(a_m - a_n)x}$$

Thus,

$$\int_0^1 |S(N, x)|^2 dx = \frac{1}{N^2} \sum_{m,n=1}^{N} \int_0^1 e^{2\pi i h(a_m - a_n)x} dx = \frac{1}{N}$$

This is because the integrand is equal to 1 only when $a_m = a_n$, and otherwise the integral is 0.

In the classical proof, Fatou's lemma was used from here. In this case, we avoid it by using Lemma 5 and Lemma 6.

Let $f_N(x) = |S(N^2, x)|^2 \geq 0$. Then $\int_0^1 f_N(x)\, dx = \frac{1}{N^2}$, so by Lemma 6,

$$\sum_{N=1}^{\infty} \int_0^1 f_N(x)\, dx = \sum_{N=1}^{\infty} \frac{1}{N^2} = \frac{\pi^2}{6}$$

can be computed.

Let $F_k(x) = \sum_{N=1}^{k} f_N(x)$, then $\{F_k\}$ is a sequence of measurable monotone increasing functions, and

$$\lim_{k\to\infty} \int_0^1 F_k(x)\, dx = \sum_{N=1}^{\infty} \frac{1}{N^2} = \frac{\pi^2}{6} < \infty$$

holds.

Therefore, by Lemma 5,

$$\sum_{N=1}^{\infty} f_N(x) = \lim_{k\to\infty} F_k(x) \in L^1([0, 1])$$

exists, and

$$\int_0^1 \sum_{N=1}^{\infty} f_N(x)\, dx = \frac{\pi^2}{6}$$

holds computably.

In particular, $\sum_{N=1}^{\infty} |S(N^2, x)|^2 < \infty$ a.a. x, and hence $|S(N^2, x)| \to 0$ $(N \to \infty)$ a.a. x holds.

That is,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, |S(N^2, x)| < \varepsilon \quad \text{a.a. } x$$

holds.

Since $\forall N \in \mathbb{N}, \exists m \in \mathbb{N}$ s.t. $m^2 \leq N < (m+1)^2$,

$$|S(N, x)| \leq |S(m^2, x)| + \frac{2m}{N} \leq |S(m^2, x)| + \frac{2}{\sqrt{N}}$$

From this, it also follows that $\forall \varepsilon > 0, \exists N \in \mathbb{N}, |S(N, x)| < \varepsilon$ a.a. x.

Finally, since h ranges over nonzero integers, which is countable, the measure of the exceptional set remains 0. Denoting this set by A, it is shown over RCA_0 that for all $x \in [0, 1] \setminus A$, $(a_n x)$ is u.d. mod 1.

□

4.2 u.d. mod 1 for sequences satisfying Koksma's condition

In this section, we present a statement that becomes a powerful tool when considering u.d. mod 1 for certain characteristic sequences of functions. This theorem is called Koksma's general metric theorem, and it is required when considering sequences such as (x^n).

Proposition 1. *The following statement is provable over* RCA_0.

Let u_n be a convergent series consisting of positive terms.

$$r(n) = \sum_{j=n}^{\infty} u_j, \quad \lambda(n) = \left(\sqrt{r(n)} + \sqrt{r(n+1)}\right)^{-1}$$

Define as above.

1. *As $n \to \infty$, $\lambda(n) \to \infty$*
2. *$\sum_{n=1}^{\infty} u(n)\lambda(n)$ converges*

Theorem 6. *The following statement is provable over* RCA_0.

Let $u_n(x)$ be a sequence of measurable functions on $[a, b]$. For $h \neq 0, N \geq 1$, define

$$S_h(N, x) = \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h u_n(x)} \quad (a \leq x \leq b)$$

Furthermore, define $I_h(N) = \int_a^b |S_h(N, x)| dx$.

If $\sum_{N=1}^{\infty} \frac{I_h(N)}{N}$ converges for each integer $h \neq 0$, then $u_n(x)$ is u.d. mod 1 a.a. $x \in [a, b]$.

For the classical result, see Theorem 4.2 in [1].

Proof. Fix h. Since $\sum_{N=1}^{\infty} \frac{I_h(N)}{N}$ converges, there exists $\lambda(n)$ $(\lambda(n) \to \infty)$ such that $\sum_{N=1}^{\infty} \frac{I_h(N)\lambda(N)}{N}$ converges.

By Lemma 1, this $\lambda(N)$ can be constructed computably.

$$\exists M = (M_1 < M_2 < \cdots) \text{ s.t. } M_{r+1} = \left[\frac{\lambda(M_r)}{\lambda(M_r) - 1}\right] + 1 \quad (r \geq 1)$$

Consider a sequence M of positive integers as above.

If we take M_1 such that $\lambda(M_1) - 1 > 0$, the sequence can be constructed.

Let N_r be defined as the integer N with $M_r < N \leq M_{r+1}$ that minimizes $I_h(N)$, obtained computably.

Then

$$I_h(N_r) \leq \frac{1}{M_{r+1} - M_r} \sum_{N=M_r+1}^{M_{r+1}} I_h(N) \leq \frac{M_{r+1}}{M_{r+1} - M_r} \sum_{N=M_r+1}^{M_{r+1}} \frac{I_h(N)}{N}$$

By the definition of M_r,

$$\frac{M_{r+1}}{M_{r+1} - M_r} < \lambda(M_r)$$

Hence

$$I_h(N_r) \leq \sum_{N=M_r+1}^{M_{r+1}} \frac{I_h(N)\lambda(N)}{N}$$

By Lemma 1, $\sum_{r=1}^{\infty} I_h(N_r) < M < \infty$, so it converges.

To exchange the integral and the limit, we prove by contradiction.

Assume that there exists a non-null set $A \subset [a, b]$ on which $\sum_{r=1}^{\infty} |S_h(N_r, x)|^2$ diverges.

Then for sufficiently large R and a real number M,

$$\int_A \sum_{r=1}^{R} |S_h(N_r, x)|^2 d\mu > M$$

holds.

On the other hand,

$$M < \sum_{r=1}^{R} \int_A |S_h(N_r, x)|^2 d\mu \leq \sum_{r=1}^{\infty} \int_a^b |S_h(N_r, x)|^2 d\mu = \sum_{r=1}^{\infty} I_h(N_r) < M$$

Thus a contradiction arises, so such a set A does not exist.

As a consequence of the proof by contradiction, since $\sum_{r=1}^{\infty} |S_h(N_r, x)|^2$ converges on any non-null set, we obtain that $\forall \varepsilon > 0, \exists r, |S(N_r, x)| < \varepsilon$ a.a. $x \in [a, b]$.

From the definition of M_r, $\frac{M_{r+1}}{M_r} \to 1$ $(r \to \infty)$, hence $\frac{N_{r+1}}{N_r} \to 1 (r \to \infty)$.

If $N_r < N \leq N_{r+1}$, then $|S(N, x)| \leq |S(N_r, x)| + \frac{N_{r+1} - N_r}{N_r}$.

Therefore, $\forall \varepsilon, \exists N, |S(N, x)| < \varepsilon$ a.a. $x \in [a, b]$ is shown over RCA_0.

$\square$

In this proof, we exchange the integral and the limit by a proof by contradiction, rather than by applying Lemma 5 used in Theorem 5. This choice is motivated by considerations from reverse mathematics.

In Theorem 5, we were able to determine the actual limit of the series $\sum_{N=1}^{\infty} I_h(N)$. In the present situation, however, we only know that this series is bounded, and this difference necessitates a different method of proof.

More generally, the monotone convergence theorem is equivalent to ACA_0. Thus, if we wish to avoid an argument by contradiction, we must instead assume ACA_0.

Proposition 2 (Second mean value theorem)**.** *The following statement is provable over* RCA_0*.*

Let $g : [a, b] \to \mathbb{R}$ be a monotone function on the interval $[a, b]$, and let $H : [a, b] \to \mathbb{C}$ be a C^1-class function. Then there exists a point $x_0 \in [a, b]$ such that

$$\int_a^b g(x)\, dH(x) = g(a) \int_a^{x_0} dH(x) + g(b) \int_{x_0}^b dH(x)$$

holds.

Proposition 3. *The following statement is provable over* RCA_0.

Let f be a real-valued function that has a monotone continuous derivative f' on the interval $[a,b]$, and suppose that $|f'(x)| \geq \lambda > 0$ holds for all $x \in [a,b]$.

Then, letting

$$J = \int_a^b e^{2\pi i f(x)}\, dx$$

it follows that $|J| < \dfrac{1}{\lambda}$.

The proof uses Proposition 2.

Theorem 7 (Koksma's general metric theorem)**.** *The following statement is provable over* RCA_0.

Let $(u_n(x))$, $n = 1, 2, \dots$ be a sequence of real numbers defined for all x on the interval $[a,b]$. For all $n \geq 1$, $u_n(x)$ is differentiable on $[a,b]$, and $u_n'(x)$ is continuous. For any two positive integers $m \neq n$, assume that the following conditions hold.

1. *The function $u_m'(x) - u_n'(x)$ is monotone with respect to x.*
2. *$|u_m'(x) - u_n'(x)| \geq K > 0$ holds, where K is a positive constant independent of x, m, n.*

Then the sequence $(u_n(x))$ is u.d. mod 1 a.a. $x \in [a,b]$.

For the classical result, see Theorem 4.3 in [1].

Proof. For a fixed integer $h \neq 0$, define the exponential sum

$$S_h(N,x) = \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h u_n(x)} \quad (a \leq x \leq b)$$

Then,

$$\begin{aligned} I_h(N) &= \int_a^b |S_h(N,x)|^2\, dx \\ &= \frac{1}{N^2}\sum_{m,n=1}^{N} \int_a^b e^{2\pi i h(u_m(x)-u_n(x))}\, dx \\ &\leq \frac{1}{N^2}\sum_{m,n=1}^{N} \left|\int_a^b e^{2\pi i h(u_m(x)-u_n(x))}\, dx\right|. \\ &= \frac{b-a}{N} + \frac{2}{N^2}\sum_{m=2}^{N}\sum_{n=1}^{m-1} \left|\int_a^b e^{2\pi i h(u_m(x)-u_n(x))}\, dx\right|. \end{aligned}$$

Here, apply Proposition 3 to each integral.

Since $|u_m'(x) - u_n'(x)|$ attains its minimum at the endpoints,

$$\begin{aligned} &\left|\int_a^b e^{2\pi i h(u_m(x)-u_n(x))}\, dx\right| \\ &\leq \frac{1}{|h|}\max\left(\frac{1}{|u_m'(a)-u_n'(a)|}, \frac{1}{|u_m'(b)-u_n'(b)|}\right) \\ &\leq \frac{1}{|h|}\left(\frac{1}{|u_m'(a)-u_n'(a)|} + \frac{1}{|u_m'(b)-u_n'(b)|}\right) \end{aligned}$$

Therefore,

$$I_h(N) \le \frac{b-a}{N} + \frac{2}{|h|}\frac{1}{N^2}\sum_{m=2}^{N}\sum_{n=1}^{m-1}\left(\frac{1}{|u'_m(a)-u'_n(a)|} + \frac{1}{|u'_m(b)-u'_n(b)|}\right).$$

Next, evaluate the sum $\sum_{n=1}^{m-1}\frac{1}{|u'_m(x)-u'_n(x)|}$. For fixed $x \in [a,b]$ and $2 \le m \le N$, rearrange the numbers $u'_1(x), u'_2(x), \ldots, u'_m(x)$ in order of magnitude. In this new ordering, the difference between two consecutive values is at least K by assumption.

Therefore, the distance from $u'_m(x)$ to the other values increases at least as $K, 2K, 3K, \ldots$. Hence,

$$\sum_{n=1}^{m-1}\frac{1}{|u'_m(x)-u'_n(x)|} < 2\sum_{j=1}^{N}\frac{1}{jK} < \frac{2}{K}\log(3N)$$

Thus,

$$\begin{aligned} I_h(N) &\le \frac{b-a}{N} + \frac{2}{|h|}\frac{1}{N^2}\sum_{m=2}^{N}\left(\frac{2}{K}\log(3N) + \frac{2}{K}\log(3N)\right) \\ &= \frac{b-a}{N} + \frac{2}{|h|}\frac{1}{N^2}\cdot(N-1)\cdot\frac{4}{K}\log(3N) \\ &< \frac{b-a}{N} + \frac{8}{|h|K}\cdot\frac{\log(3N)}{N}. \end{aligned}$$

As $N \to \infty$, since $\frac{\log(3N)}{N} \to 0$, it follows that $I_h(N) \to 0$. By Theorem 6, it follows over RCA_0 that the sequence $(u_n(x))$ is u.d. mod 1 a.a. $x \in [a,b]$. □

Using Koksma's general metric theorem, the following statement can be shown.

Corollary 4. *The following statement is provable over* RCA_0.
The sequence (x^n) *is u.d. mod 1 a.a.* $x \ge 1$.

5 Conclusion

In this paper, we systematically investigated fundamental properties arising in the theory of uniform distribution, with particular attention to their formalizability and provability over RCA_0. We showed that many conditions that have classically been treated by distinct methods or auxiliary theorems can in fact be handled within the computable framework of RCA_0, and admit a uniform characterization within a single foundational system.

参考文献

[1] L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences, Wiley-Interscience, 1974.

[2] K. Simic, Aspects of Ergodic Theory in Subsystems of Second-order Arithmetic, PhD thesis, Carnegie Mellon University, 2004.

[3] E. Bishop and D. Bridges, Constructive Analysis, Springer, 1985.

[4] K. Yokoyama, Reverse Mathematics for Fourier Expansion, 2008.

[5] T. Kihara, 数理情報学 1・講義ノート, 名古屋大学情報学部 講義資料, 2017.

[6] D. D. Dzhafarov and C. Mummert, Reverse Mathematics, Springer, 2022.

[7] X. Yu and S. G. Simpson, Measure Theory and Weak König's Lemma, Archive for Mathematical Logic, vol. 30, pp. 171–180, 1990.

[8] D. K. Brown, M. Giusto, and S. G. Simpson, vitali's theorem and WWKL, Archive for Mathematical Logic, vol. 41, pp. 191–206, 2002.

[9] S. G. Simpson, Subsystems of Second Order Arithmetic, Cambridge University Press, 2009.