Complexity of the SAT problem for the Hybrid Logic

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The satisfiability problem (SAT problem) is a fundamental benchmark for measuring the computational tractability of logics. In complexity theory, the hardness of such problems is categorized into classes like P, NP, PSPACE, and EXPTIME, based on the growth of computational resources required relative to the input size.

The satisfiability problem for propositional classical logic is an example of the NP-complete class. In comparison, modal logic, which describes transitions over graph structures, is typically situated in the PSPACE-complete class, requiring polynomial memory (see [4, Theorem 6.47], for example). Moving to first-order logic, the problem becomes undecidable.

Despite its greater expressive power compared to basic modal logic, hybrid logic exhibits remarkable computational robustness. In this paper, we revisit a theorem by Areces, Blackburn, and Marx [1], who showed that the complexity of the SAT problem for the hybrid logic $\mathbf{K}(@)$ is PSAPCE-complete, the same as that of standard modal logic. That is:

Main Theorem. The SAT problem for the hybrid logic $\mathbf{K}(@)$ is PSPACE-complete.

Some results in computer science show that a PSPACE-completeness of a decision problem is equivalent to the existence of a winning strategy for a player in a corresponding two-player game with finite time and space. For hybrid logic as well, we can construct a game in which the satisfiability of a formula is equivalent to the existence of a winning strategy in the game.

Furthermore, in the final part of this article, we prove the finite model property of hybrid logic. While the finite model property of hybrid logic has already been proven using tableau calculus (see, for example, [6]), it can also be demonstrated by utilizing winning strategies in two-player games.

1 Preliminaries

1.1 Hybrid Logic

Hybrid logic is an extension of modal logic with special propositional variables called *nominals*. Each nominal is true in exactly one world of a Kripke frame. In

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other words, we can assume a nominal as the *name* of the corresponding possible world.

In this section, we give a brief introduction to the Kripke semantics for hybrid logic. For more details, see [2, 3, 9] for instance.

Definition 1. Fix two disjoint sets **Prop** of *propositional variables* and **Nom** of *nominals*, both countably infinite. The formulas of hybrid logic are defined by the following grammar:

$$\varphi ::= p \mid i \mid \neg\varphi \mid \varphi \land \varphi \mid \Diamond\varphi \mid @_i\varphi,$$

where $p \in$ **Prop** and $i \in$ **Nom**. Other logical connectives such as $\lor, \rightarrow, \Box$ are defined as usual.

Definition 2. A tuple $\mathcal{M} = (W, R, V)$ is a *Kripke model* if:

- W is a non-empty set of *worlds* or *states*.
- R is a binary relation over W.
- V : **Prop** $\cup$ **Nom** $\rightarrow \mathcal{P}(W)$ is a *valuation function* where $V(i)$ is a singleton for all $i \in$ **Nom**.

If $\mathcal{M} = (W, R, V)$ and $w \in W$, we write $w \in \mathcal{M}$ for simplicity. Also, we write i^V to indicate an element $w \in W$ such that $V(i) = \{w\}$.

Definition 3. Given a model $\mathcal{M}$, a state $w \in \mathcal{M}$, and a formula φ, the relation $\mathcal{M}, w \models \varphi$ is defined inductively as follows:

$$\begin{aligned}
\mathcal{M}, w \models p &\iff w \in V(p), \text{ where } p \in \textbf{Prop} \\
\mathcal{M}, w \models i &\iff w = i^V, \text{ where } i \in \textbf{Nom} \\
\mathcal{M}, w \models \neg\varphi &\iff \text{not } \mathcal{M}, w \models \varphi \ (\iff \mathcal{M}, w \not\models \varphi) \\
\mathcal{M}, w \models \varphi \land \psi &\iff \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \Diamond\varphi &\iff \text{there exists } v \in \mathcal{M} \text{ such that } wRv \text{ and } \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models @_i\varphi &\iff \mathcal{M}, i^V \models \varphi.
\end{aligned}$$

When the model $\mathcal{M}$ under consideration is clear from the context, we write $w \models \varphi$ instead of $\mathcal{M}, w \models \varphi$.

Definition 4. A formula φ is *satisfiable* if there is a model $\mathcal{M}$ and its world $w \in \mathcal{M}$ such that $\mathcal{M}, w \models \varphi$. Moreover, a set of formulas Γ is *satisfiable* if there is a model $\mathcal{M}$ and its world w such that $\mathcal{M}, w \models \gamma$ for all $\gamma \in \Gamma$. A formula φ is *valid* if $\neg\varphi$ is not satisfiable.

Given a formula φ, the *length* of φ, denoted by $\mathsf{len}(\varphi)$, is the number of occurrences of propositional variables, nominals, and operators in the formula (excluding parentheses). Note that for the satisfaction operator, the at-sign and the subscript nominal are counted separately; that is, $\mathsf{len}(@_i\varphi) = \mathsf{len}(\varphi) + 2$. For example, $\mathsf{len}(@_i\Diamond(p \land q)) = 6$.

We write $\mathsf{md}(\varphi)$ to mean the *modal depth* of φ, the deepest nesting of the modal operator $\Diamond$. The satisfaction operator does not change the modal depth; $\mathsf{md}(@_i\varphi) = \mathsf{md}(\varphi)$. Other cases remain the same as usual.

Moreover, $\mathsf{sub}(\varphi)$ denotes *the set of all subformulas* of φ. If we have a formula of the form $@_i\varphi$ as a subformula, then we treat not only φ but also i as a subformula. For example,

$$\mathsf{sub}(@_i\Diamond(p \wedge q)) = \{@_i\Diamond(p \wedge q), \Diamond(p \wedge q), p \wedge q, p, q, i\}.$$

Observe that the length of a formula limits the size of the set of subformulas.

Proposition 5. For any formula φ, we have $|\mathsf{sub}(\varphi)| \leq \mathsf{len}(\varphi)$.

Proof. By induction on the complexity of φ. We show, for instance, the case for the satisfaction operator as follows:

$$|\mathsf{sub}(@_i\psi)| = |\mathsf{sub}(\psi) \cup \{@_i\psi, i\}| \leq |\mathsf{sub}(\psi)| + 2 \leq \mathsf{len}(\psi) + 2 = \mathsf{len}(@_i\psi). \quad \blacksquare$$

The standard hybrid logic possesses a sound and complete Hilbert-style axiomatic system $\mathbf{K}(@)$. Since these axioms and inference rules do not appear in the proofs within this paper, we will not list their definitions here. However, it is convenient to introduce the name of the logic, so we will do so here. For more about the axiomatic system of hybrid logic, see [5].

1.2 Results for Complexity

The *satisfiability problem (SAT problem)* for a logic is the problem of whether there is an interpretation that makes φ true. As for the hybrid logic, the satisfiability problem is to decide whether there is a model and a world where φ is true.

To show the main theorem, we must show that the SAT problem for $\mathbf{K}(@)$ is in PSPACE and is PSPACE-hard. The latter is straightforwardly shown as follows.

Lemma 6. The SAT problem for $\mathbf{K}(@)$ is PSPACE-hard.

Proof. This follows from the PSPACE-completeness of the standard modal logic $\mathbf{K}$ [10], as $\mathbf{K}(@)$ is a conservative extension of $\mathbf{K}$: the Kripke models for hybrid logic validate the same formulas without nominals and satisfaction operators as $\mathbf{K}$, as hybrid models without nominals are modal models. $\blacksquare$

To show the other one, we rely on the following theorems from the literature.

Theorem 7 (Chlebus [8])**.** The decision problem of whether a player has a winning strategy in a 2-player game is computable by a PTIME alternating Turing machine, provided that the game ends in polynomial rounds and the board size is polynomial.

Theorem 8 (Chandra, Kozen, and Stockmeyer [7, Theorem 3.2])**.** If a problem is solved by a PTIME alternating Turing machine, then there is a PSPACE Turing machine that solves the problem.

Given these propositions, what we ultimately need to do is to construct a 2-player game with finite time and finite space that detects the satisfiability of a formula.

2 The Satisfaction Game

Before introducing the 2-player game, we define the *φ-Hintikka sets*, maximal consistent sets constructed from subformulas of φ together with their negations. Let us write $\mathsf{sub}^+(\varphi) = \mathsf{sub}(\varphi) \cup \{\neg\psi \mid \psi \in \mathsf{sub}(\varphi)\}$. Then, the φ-Hintikka set is defined precisely as follows.

Definition 9. Given a formula φ, a *φ-Hintikka set* Γ is a set satisfying following conditions:

(a) $\Gamma \subseteq \mathsf{sub}^+(\varphi)$.

(b) Γ is satisfiable.

(c) For any formula $\psi \in \mathsf{sub}(\varphi)$, either $\psi \in \Gamma$ or $\neg\psi \in \Gamma$.

Lemma 10. Given a formula φ, any φ-Hintikka set is of polynomial size in $\mathsf{len}(\varphi)$.

Proof. Straightforward from Proposition 5. ■

Now, a given formula φ, we define the satisfaction game $\mathbf{SAT}_\varphi$ for φ. We call the players of the game Eve (she/her) and Adam (he/him). Through this game, Eve attempts to construct a model that satisfies a given formula φ. Adam *challenges* her by pointing out formulas of the form $\Diamond\psi$ in the model, suggesting that the subsequent world might be inconsistent. Eve dismisses these challenges by asserting that the successor can be constructed without contradiction. Through this repeated process, Adam wins if he can derive a contradiction, and Eve wins if he cannot.

Formally, each round progresses in the following manner.

Round 0. Eve plays a collection $\mathcal{X}_0 = \{X_0, \dots, X_m\}$ of φ-Hintikka sets, satisfying the following conditions:

(a) X_0 contains φ, and for any $1 \leq k \leq m$, there is some nominal i occurring in φ such that $i \in X_k$.

(b) Each nominal occurring in φ is in some X_k with $0 \leq k \leq m$.

(c) There is no nominal i such that $i \in X_k \cap X_l$ with $0 \leq k < l \leq m$.

(d) For all $0 \leq k \leq m$ and $@_i\psi \in \mathsf{sub}(\varphi)$, $@_i\psi \in X_k$ if and only if $\{i, \psi\} \subseteq X_l$ for some $0 \leq l \leq m$.

Then, Adam picks $Y_0 \in \mathcal{X}_0$ and $\Diamond\psi_0 \in Y_0$.

Round $n + 1$. Suppose $\mathcal{X}_n$, Y_n, and $\Diamond\psi_n \in Y_n$ are defined. Eve makes a φ-Hintikka set Y_{n+1} such that:

(a) $\psi_n \in Y_{n+1}$, and $\chi \in Y_{n+1}$ implies $\Diamond\chi \in Y_n$ for all $\Diamond\chi \in \mathsf{sub}(\varphi)$.

(b) For all $@_i\chi \in \mathsf{sub}(\varphi)$, $@_i\chi \in Y_{n+1}$ if and only if $\{i, \chi\} \subseteq X_l$ for some $0 \leq l \leq m$.

(c) If $i \in Y_{n+1}$ for some nominal i, then Y_{n+1} is equal to a set in $\mathcal{X}_n$. If this case happens, Eve wins immediately.

Then, put $\mathcal{X}_{n+1} = \mathcal{X}_n \cup \{Y_{n+1}\}$. Adam now picks $\Diamond\psi_{n+1} \in Y_{n+1}$ such that $\mathsf{md}(\Diamond\psi_{n+1}) < \mathsf{md}(\varphi) - n$.

If any player cannot make a move, they lose the game. In particular, at the end of Round $\mathsf{md}(\varphi)$, Adam cannot play any formula $\Diamond\psi$, so Eve wins.

The original version of the proof in [1] requires Eve to insert relation R into $\mathcal{X}_0$ in Round 0 so that it satisfies the following constraints:

For all $0 \leq k, l \leq m$ and $\Diamond\psi \in \mathsf{sub}(\varphi)$, if we have $X_k R X_l$ and $\Diamond\psi \notin X_k$, then $\psi \notin X_l$.

This condition helps detect early collections that do not constitute a valid model. However, it is not indispensable for proving the main theorem. Therefore, we omit this condition.

We check that this game requires polynomial time to end and polynomial information.

Lemma 11. Given a formula φ, the game $\mathbf{SAT}_\varphi$ needs polynomial time in $\mathsf{len}(\varphi)$. Moreover, the information they play is of polynomial size in $\mathsf{len}(\varphi)$.

Proof. The first clause is straightforward from the fact that $\mathbf{SAT}_\varphi$ ends after at most Round $\mathsf{md}(\varphi)$.

For the second clause, recall that each φ-Hintikka set is of linear size in $\mathsf{len}(\varphi)$ (Lemma 10). Moreover, the number of φ-Hintikka sets they play is of linear size in $\mathsf{len}(\varphi)$. In Round 0, the number of φ-Hintikka sets Eve plays depends on the number of nominals that φ contains. In addition, Eve adds φ-Hintikka sets at most $\mathsf{md}(\varphi)$ times since the game ends at most Round $\mathsf{md}(\varphi)$. Eventually, the information on the game board has a size of the square of $\mathsf{len}(\varphi)$. ■

3 Correctness of the Game

Now, we prove the claim that justifies the correctness of the 2-player game defined above. We show two facts: if we have a model and a world satisfying φ, then Eve acquires the winning strategy from it; conversely, if Eve has a winning strategy, we construct a model for φ from that.

Lemma 12. φ is satisfiable if and only if Eve has a winning strategy in $\mathbf{SAT}_\varphi$.

Proof. To prove the left-to-right direction, suppose that φ is satisfiable, which means there is a model $\mathcal{M}$ and its world w. Then, we can define a winning strategy for Eve in $\mathbf{SAT}_\varphi$ based on $\mathcal{M}$ and w as follows:

Round 0. Eve plays the following collection $\mathcal{X}_0$ of φ-Hintikka sets:

$$\mathcal{X}_0 = \big\{\{\psi \in \mathsf{sub}^+(\varphi) \mid v \vDash \psi\} \mid v = w \text{ or } \exists i \in \mathsf{sub}(\varphi).\ V(i) = \{v\}\big\}.$$

Intuitively, they are the world w and the worlds in $\mathcal{M}$ that are named by nominals occurring in φ.

Round $n+1$. Suppose that Adam plays $\Diamond\psi_n \in Y_n$ in Round n. Then, there must be a world w_n such that $w_n \vDash \psi$ for all $\psi \in Y_n$. Now, Eve picks up a world v satisfying $w_n R v$, and plays the following set as Y_{n+1}:

$$Y_{n+1} = \{\psi \in \mathsf{sub}^{+}(\varphi) \mid v \vDash \psi\}.$$

If Eve picks the sets as indicated, they will always satisfy the game's requirements, and Adam eventually loses. Therefore, the action shown above is Eve's winning strategy.

For the other direction, suppose Eve has a winning strategy σ in $\mathbf{SAT}_\varphi$. We first define a collection of φ-Hintikka sets $\{W_n \mid n \leq \mathsf{md}(\varphi)\}$ as follows:

- W_0 is the collection $\{X_0, \dots, X_m\}$ of φ-Hintikka sets which Eve plays her initial move via σ.
- Given that W_k is defined, W_{k+1} collects all of the φ-Hintikka sets which σ may indicate in answer to Adam's plays. That is, W_{k+1} consists of all sets that Eve may play at Round $k+1$.

We then build a tuple $\mathcal{M}^\sigma = (W^\sigma, R^\sigma, V^\sigma)$ as follows:

- $W^\sigma = \bigcup_{0 \leq k \leq \mathsf{md}(\varphi)} W_k$
- $X R^\sigma Y$ if and only if for all $\Diamond\psi \in \mathsf{sub}(\varphi)$, $\psi \in Y$ implies $\Diamond\psi \in X$
- $V^\sigma(s) = \{X \in W \mid s \in X\}$, for all $s \in \mathbf{Prop} \cup \mathbf{Nom}$.

To verify that $\mathcal{M}^\sigma$ is really a model for hybrid logic, we check that $V^\sigma(i)$ is a singleton for all nominals i occurring in φ[1]. The existence of $V^\sigma(i)$ is guaranteed by the rule (b) in Round 0. Moreover, the uniqueness of $V^\sigma(i)$ is guaranteed by the rules (c) in Round 0 and (c) in Round $n+1$.

To show that $\mathcal{M}^\sigma$ satisfies φ, we need to show that, for all states $X \in W_k$ and all subformulas ψ of φ such that $\mathsf{md}(\psi) \leq \mathsf{md}(\varphi) - k$, we have

$$\mathcal{M}^\sigma, X \vDash \psi \iff \psi \in X.$$

We prove this claim by induction on the complexity of ψ. The only interesting case is the case for $\Diamond$, the other cases are straightforward.

First, suppose that $\Diamond\psi \in X$. Then, we have $\psi \in Y$ for some Y in W_{k+1} since Adam may play $\Diamond\psi$ and Eve may respond $Y \ni \psi$. By the rule (a) in Round $k+1$ and the definition of R^σ, we have $X R^\sigma Y$. Moreover, by the induction hypothesis, we have $Y \vDash \psi$. Therefore, $X \vDash \Diamond\psi$. Now, suppose $\Diamond\psi \notin X$. Then, for all Y satisfying XRY, we have $\psi \notin Y$. From this fact and the induction hypothesis, $X \nvDash \Diamond\psi$ holds.

We can now conclude that $\mathcal{M}^\sigma, X_0 \vDash \varphi$, and so φ is satisfiable. ■

[1]The other nominals are irrelevant for determining the truth value of φ. Therefore, it is sufficient to associate those nominals with an arbitrary world, such as X_0.

Now, we are ready to prove the main theorem.

Proof of the main theorem. The SAT problem for **K**(@) is equivalent to determining whether Eve has a winning strategy in $\mathbf{SAT}_\varphi$ by Lemma 12. Then, combining Proposition 7, Proposition 8, and Lemma 11, we obtain the conclusion. ■

4 Finite Model Property

The 2-player game introduced in this article is not limited to proving complexity. Using a winning strategy for $\mathbf{SAT}_\varphi$, we can construct a finite model for any satisfiable formula φ.

Theorem 13. The hybrid logic **K**(@) has the finite model property.

Proof. Suppose that φ is satisfiable. Then, Eve has a winning strategy σ in $\mathbf{SAT}_\varphi$ by Lemma 12. According to the proof of Lemma 12, we can construct a model $\mathcal{M}^\sigma$ in which there is a world $X_0 \in \mathcal{M}$ such that $\mathcal{M}^\sigma, X_0 \models \varphi$ via σ. Now, in every round, there are only finitely many possibilities for the set that Eve can play and the formula that Adam can play. Furthermore, it is guaranteed that this game ends in at most Round md(φ). Therefore, the model $\mathcal{M}^\sigma$ constructed from σ must be finite. ■

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