A predicate extension of GL and its completeness with respect to constant domain Kripke models

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Abstract

We introduce the logic $\mathbf{QGL}^{\mathrm{r}}$, a predicate extension of the provability logic $\mathbf{GL}$, and prove its completeness with respect to the class of finite irreflexive and transitive Kripke frames with constant domains. $\mathbf{QGL}^{\mathrm{r}}$ is the fragment of the natural predicate extension of $\mathbf{GL}$ consisting of formulas in which no free variables occur within the scope of the modal operator. Two predicate extensions of $\mathbf{GL}$ closely related to $\mathbf{QGL}^{\mathrm{r}}$ are $\mathbf{QGL}^{\mathrm{b}}$, introduced by Yavorsky [5], and $\mathbf{ML}^3$, introduced by Schwartz and Tourlakis [3]. We prove that $\mathbf{QGL}^{\mathrm{r}}$ coincides with the set of normal formulas of $\mathbf{QGL}^{\mathrm{b}}$ and $\mathbf{ML}^3$.

1 Introduction

In this paper, we introduce the logic $\mathbf{QGL}^{\mathrm{r}}$, a predicate extension of the provability logic $\mathbf{GL}$, and prove its completeness with respect to the class of finite irreflexive and transitive Kripke frames with constant domains. $\mathbf{QGL}^{\mathrm{r}}$ is the fragment of the natural predicate extension of $\mathbf{GL}$, consisting of formulas in which no free variables occur within the scope of the modal operator. Two predicate extensions of $\mathbf{GL}$ closely related to $\mathbf{QGL}^{\mathrm{r}}$ are $\mathbf{QGL}^{\mathrm{b}}$, introduced by Yavorsky [5], and $\mathbf{ML}^3$, introduced by Schwartz and Tourlakis [3]. We prove that $\mathbf{QGL}^{\mathrm{r}}$ coincides with the set of normal formulas in $\mathbf{QGL}^{\mathrm{b}}$ and $\mathbf{ML}^3$.

Yavorsky [5] introduced the logic $\mathbf{QGL}^{\mathrm{b}}$ to study the fragment of a predicate extension of $\mathbf{GL}$ in which no free variable occurs in the scope of the modal operator. Such formulas are called *normal* in that paper. However, the definition of formula construction in that paper follows the standard approach. Therefore, the set of formulas includes formulas of the form $\Box\phi$ where ϕ contains free variables. To eliminate free variables from the scope of the modal operator, Yavorsky introduced the axiom schema

$$\Box\phi \supset \Box\forall x\phi. \tag{1}$$

Consequently, every formula in $\mathbf{QGL}^{\mathrm{b}}$ is equivalent to a normal formula. Yavorsky [5] proved that $\mathbf{QGL}^{\mathrm{b}}$ is arithmetically sound and complete, and that it is complete with respect to irreflexive and transitive finite Kripke frames.

Subsequently, Schwartz and Tourlakis [3] introduced the logic $\mathbf{ML}^3$ to develop a cut-free system for the same fragment of the predicate extension of $\mathbf{GL}$. As with $\mathbf{QGL}^{\mathrm{b}}$, $\mathbf{ML}^3$ employs standard formula construction and adopts the axiom schema (1). Schwartz and Tourlakis proved that $\mathbf{ML}^3$ is arithmetically sound, and that it is complete with respect to finite irreflexive and transitive Kripke frames. Subsequently, Hao and Tourlakis [2] established that $\mathbf{ML}^3$ is arithmetically complete and that it is complete with respect

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to irreflexive and transitive finite Kripke frames with constant domains. However, the proof of Kripke completeness of that paper relies on a lemma of Tourlakis and Kibedi [4] in which the constant domains condition is not assumed.

In this paper, we introduce the logic $\mathbf{QGL}^{\mathrm{r}}$, a predicate extension of $\mathbf{GL}$. Every formula in $\mathbf{QGL}^{\mathrm{r}}$ is constructed so that no free variable occurs within the scope of the modal operator, and the axiom schema (1) is not included. We prove that, for every normal formula ϕ, the statements $\phi \in \mathbf{QGL}^{\mathrm{b}}$, $\phi \in \mathbf{ML}^3$, and $\phi \in \mathbf{QGL}^{\mathrm{r}}$ are equivalent. This result shows that $\mathbf{QGL}^{\mathrm{b}}$ and $\mathbf{ML}^3$ are conservative extensions of $\mathbf{QGL}^{\mathrm{r}}$. We also provide a self-contained proof of the completeness of $\mathbf{QGL}^{\mathrm{r}}$ with respect to the class of finite irreflexive and transitive Kripke frames with constant domains.

The paper is organized as follows. In Section 2, we fix notation and terminology. In Section 3, we introduce the logic $\mathbf{QGL}^{\mathrm{r}}$. In Section 4, we prove the completeness of $\mathbf{QGL}^{\mathrm{r}}$ with respect to the class of finite irreflexive and transitive Kripke frames with constant domains.

2 Preliminaries

In this section, we fix notation and terminology.

Definition 2.1. The language of predicate modal logic is defined by the following symbols:

1. a countable set Var of variables;
2. $\top$ and $\bot$;
3. the logical connectives: $\wedge$, $\neg$;
4. the quantifier: $\forall$;
5. for each $n \in \omega$, a countable set $\mathsf{Pred}(n)$ of predicate symbols of arity n;
6. the modal operator: $\Box$.

A predicate symbol of arity 0 is called a *propositional variable.*

Definition 2.2. The set Φ of predicate modal formulas is the smallest set that satisfies:

1. $\top$ and $\bot$ are in Φ;
2. if $P \in \mathsf{Pred}(n)$ and $x_1, \dots, x_n \in \mathsf{Var}$ then $P(x_1, \dots, x_n) \in \Phi$, for each $n \in \omega$;
3. if ϕ and ψ are in Φ then $(\phi \wedge \psi) \in \Phi$;
4. if $\phi \in \Phi$ then $(\neg\phi)$ and $(\Box\phi)$ are in Φ;
5. if $\phi \in \Phi$ and $x \in \mathsf{Var}$ then $(\forall x\phi) \in \Phi$.

An *atomic formula* is a formula of the form $P(x_1, \dots, x_n)$ for some $n \in \omega$, $P \in \mathsf{Pred}(n)$, and $x_1, \dots, x_n \in \mathsf{Var}$. The symbols $\vee$, $\supset$, and $\exists$ are defined in the usual way. We use $\phi \equiv \psi$ and $\Diamond\phi$ as abbreviations for $(\phi \supset \psi) \wedge (\psi \supset \phi)$ and $\neg\Box\neg\phi$, respectively. Let $\bar{x} = x_1, \dots, x_n$ be a list of mutually distinct variables and $\bar{y} = y_1, \dots, y_n$ be a list of variables. For each formula ϕ, define $[\bar{y}/\bar{x}]\phi$ as the instance of substituting $y_1, \dots, y_n$ into the free occurrences of $x_1, \dots, x_n$ in ϕ, respectively. For each formula $\phi \in \Phi$, we write $\mathsf{FV}(\phi)$ for the set of all free variables in ϕ. A formula ϕ is said to be *closed* if $\mathsf{FV}(\phi) = \emptyset$. For each set Γ of formulas, we write $\Box\Gamma$ and $\Box^{-1}\Gamma$ to denote the sets $\{\Box\phi \mid \phi \in \Gamma\}$ and $\{\phi \mid \Box\phi \in \Gamma\}$, respectively.

Definition 2.3. A formula $\phi \in \Phi$ is *normal* if every subformula $\Box\psi$ of ϕ contains no free occurrences of variables. For each formula $\phi \in \Phi$, define a normal formula $n(\phi)$ recursively as follows:

1. $n(\top) = \top$ and $n(\bot) = \bot$;
2. $n(P(x_1, \ldots, x_n)) = P(x_1, \ldots, x_n)$ for each atomic formula;
3. $n(\phi \wedge \psi) = n(\phi) \wedge n(\psi)$;
4. $n(\neg\phi) = \neg n(\phi)$;
5. $n(\Box\phi) = \Box\forall x_1 \cdots \forall x_k n(\phi)$, where $x_1, \ldots, x_k$ is the list of all free variables occurring in $n(\phi)$;
6. $n(\forall x\phi) = \forall x\, n(\phi)$.

Definition 2.4. A *Kripke frame* is a pair $\langle W, R\rangle$, where W is a non-empty set and R is a binary relation on W. A *constant domain predicate Kripke frame* over a Kripke frame $F = \langle W, R\rangle$ is a triple $\langle W, R, D\rangle$, where D is a non-empty set called the *domain*.

Definition 2.5. A *constant domain Kripke model* is a four-tuple $\langle W, R, D, I\rangle$, where $\langle W, R, D\rangle$ is a constant domain predicate Kripke frame and I is a mapping called an *interpretation* which maps each pair (w, P), where w is a member of W and P is an n-ary predicate symbol, to an n-ary relation $I(w, P) \subseteq D^n$ over D. An *assignment $\mathcal{A}$ for M* is a mapping from the set of variables to D. The relation $\models$ among a constant domain Kripke model $M = \langle W, R, D, I\rangle$, a world $w \in W$, an assignment $\mathcal{A}$, and a formula ϕ is defined recursively as follows:

1. $M, w \models_{\mathcal{A}} \top$ and $M, w \not\models_{\mathcal{A}} \bot$;
2. for any predicate P of arity n,
   $M, w \models_{\mathcal{A}} P(x_1, \ldots, x_n) \;\Leftrightarrow\; (\mathcal{A}(x_1), \ldots, \mathcal{A}(x_n)) \in I(w, P)$;
3. $M, w \models_{\mathcal{A}} \phi \wedge \psi \;\Leftrightarrow\; M, w \models_{\mathcal{A}} \phi$ and $M, w \models_{\mathcal{A}} \psi$;
4. $M, w \models_{\mathcal{A}} \neg\phi \;\Leftrightarrow\; M, w \not\models_{\mathcal{A}} \phi$;
5. $M, w \models_{\mathcal{A}} \forall x\phi \;\Leftrightarrow\; M, w \models_{\mathcal{A}'} \phi$ for every assignment $\mathcal{A}'$ such that $\mathcal{A}(y) = \mathcal{A}'(y)$ for any $y \neq x$;
6. $M, w \models_{\mathcal{A}} \Box\phi \;\Leftrightarrow\; (w, w') \in R$ implies $M, w' \models_{\mathcal{A}} \phi$ for any w' in W.

Let $M = \langle W, R, D, I\rangle$ be a constant domain Kripke model. For any $\phi \in \Phi$ and $w \in W$, we write $M, w \models \phi$, if $M, w \models_{\mathcal{A}} \phi$ for any assignment $\mathcal{A}$. Let ϕ be any formula and let $\mathcal{A}$ and $\mathcal{A}'$ be any assignments for M. If $\mathcal{A}(x) = \mathcal{A}'(x)$ for every free variable x of ϕ, then $M, w \models_{\mathcal{A}} \phi$ if and only if $M, w \models_{\mathcal{A}'} \phi$. Hence, if ϕ is a closed formula, then $M, w \models_{\mathcal{A}} \phi$ if and only if $M, w \models \phi$, for any assignment $\mathcal{A}$. We write $M \models \phi$, if $M, w \models \phi$ for every $w \in W$.

Let $F = \langle W, R\rangle$ be a Kripke frame. We write $F \models \phi$ if, for every domain D and any interpretation I, the model $M = \langle W, R, D, I\rangle$ satisfies $M \models \phi$. Let Γ be a set of formulas. If $F \models \phi$ for any $\phi \in \Gamma$, we write $F \models \Gamma$. Let $\mathcal{C}$ be a class of Kripke frames. We write $\mathcal{C} \models \phi$ if $F \models \phi$ for every $F \in \mathcal{C}$, and write $\mathcal{C} \models \Gamma$ if $\mathcal{C} \models \phi$ for every $\phi \in \Gamma$. A logic $\mathbf{L}$ is *sound* with respect to $\mathcal{C}$, if $\phi \in \mathbf{L}$ implies $\mathcal{C} \models \phi$, and said to be complete with respect to $\mathcal{C}$ if the converse holds.

3 Predicate extensions of GL

In this section, we introduce four predicate extensions of **GL**: **QGL**, $\mathbf{QGL}^{\mathrm{b}}$, $\mathbf{ML}^3$, and $\mathbf{QGL}^{\mathrm{r}}$. We define a logic by a set of formulas closed under certain conditions, rather than as a proof system in which the formulas of the logic are derivable.

First, we introduce the logic **QGL**, a natural extension of **GL**.

Definition 3.1. The logic **QGL** is the least subset of Φ that contains

1. all formulas derivable in classical predicate logic;

2. $\Box(\phi \supset \psi) \supset (\Box\phi \supset \Box\psi)$ for any ϕ in Φ;

3. $\Box(\Box\phi \supset \phi) \supset \Box\phi$ for any ϕ in Φ,

and closed under

1. substitution of formulas of Φ into atomic formulas (see [1]);

2. generalization, that is, for any $\phi \in \Phi$ and $x \in \mathsf{Var}$, if $\phi \in \mathbf{QGL}$ then $\forall x\phi \in \mathbf{QGL}$;

3. necessitation, that is, for any ϕ in Φ, if $\phi \in \mathbf{QGL}$ then $\Box\phi \in \mathbf{QGL}$.

Next, we introduce the logic $\mathbf{QGL}^{\mathrm{b}}$, which was given by Yavorsky [5].

Definition 3.2. The logic $\mathbf{QGL}^{\mathrm{b}}$ is the least subset of Φ that contains all the formulas 1-3 in Definition 3.1 and

$$\Box\phi \supset \Box\forall x\phi \tag{2}$$

for any $\phi \in \Phi$, and is closed under substitution, generalization, and necessitation.

The following lemma is immediate:

Lemma 3.3. *(Yavorsky [5]). For every $\phi \in \Phi$, we have $\phi \equiv n(\phi) \in \mathbf{QGL}^{\mathrm{b}}$.*

Next, we introduce the logic $\mathbf{ML}^3$ given by Schwartz and Tourlakis [3].

Definition 3.4. The logic $\mathbf{ML}^3$ is the least subset of Φ that contains $\Lambda \cup \Box\Lambda$, where Λ is the set of all formulas 1-3 of Definition 3.1 and (2), and is closed under substitution and generalization.

Again, the following lemma is immediate:

Lemma 3.5. *$\phi \equiv n(\phi) \in \mathbf{ML}^3$ for any $\phi \in \Phi$.*

Schwartz and Tourlakis [3] proved that $\phi \in \mathbf{ML}^3$ implies $\Box\phi \in \mathbf{ML}^3$ for all $\phi \in \Phi$. Therefore, we immediately have the following:

Theorem 3.6. $\mathbf{QGL}^{\mathrm{b}} = \mathbf{ML}^3$.

Finally, we define the logic $\mathbf{QGL}^{\mathrm{r}}$, which is a subset of the following set Φ^{r} of formulas.

Definition 3.7. The set Φ^{r} is the least subset of Φ that satisfies the following:

1. $\top$ and $\bot$ are in Φ^{r};

2. if $P \in \mathsf{Pred}(n)$ and $x_1, \dots, x_n \in \mathsf{Var}$ then $P(x_1, \dots, x_n) \in \Phi^{\mathrm{r}}$, for each $n \in \omega$;

3. if ϕ and ψ are in Φ^{r} then $(\phi \wedge \psi) \in \Phi^{\mathrm{r}}$;

4. if $\phi \in \Phi^{\mathrm{r}}$ then $(\neg\phi) \in \Phi^{\mathrm{r}}$;

5. if $\phi \in \Phi^{\mathrm{r}}$ is a closed formula then $(\Box\phi) \in \Phi^{\mathrm{r}}$;

6. if $\phi \in \Phi^{\mathrm{r}}$ and $x \in \mathsf{Var}$ then $(\forall x\phi) \in \Phi^{\mathrm{r}}$.

That is, Φ^{r} is the set of all formulas that have no free variables in the scope of the modal operator.

Definition 3.8. The logic $\mathbf{QGL}^{\mathrm{r}}$ is the least subset of Φ^{r} that contains

1. all formulas derivable in classical logic;

2. $\Box(\phi \supset \psi) \supset (\Box\phi \supset \Box\psi)$ for any ϕ in Φ^{r};

3. $\Box(\Box\phi \supset \phi) \supset \Box\phi$ for any ϕ in Φ^{r},

and closed under

1. substitution of formulas of Φ^{r} into atomic formulas if the substitution instances are in Φ^{r};

2. generalization: for any $\phi \in \Phi^{\mathrm{r}}$ and $x \in \mathsf{Var}$, if $\phi \in \mathbf{QGL}^{\mathrm{r}}$ then $\forall x\phi \in \mathbf{QGL}^{\mathrm{r}}$;

3. restricted necessitation: for any ϕ in Φ^{r}, if $\phi \in \mathbf{QGL}^{\mathrm{r}}$ and ϕ is closed, then $\Box\phi \in \mathbf{QGL}^{\mathrm{r}}$.

Clearly, $\mathbf{QGL}^{\mathrm{r}} \subseteq \mathbf{QGL}^{\mathrm{b}}$. $\mathbf{QGL}^{\mathrm{b}}$ and $\mathbf{QGL}^{\mathrm{r}}$ are equivalent in the following sense.

Theorem 3.9. *For any $\phi \in \Phi$, $\phi \in \mathbf{QGL}^{\mathrm{b}}$, if and only if $n(\phi) \in \mathbf{QGL}^{\mathrm{r}}$.*

Proof. ($\Rightarrow$): By induction on the closure conditions of $\mathbf{QGL}^{\mathrm{b}}$. ($\Leftarrow$): Suppose that $n(\phi) \in \mathbf{QGL}^{\mathrm{r}}$. Then, $n(\phi) \in \mathbf{QGL}^{\mathrm{b}}$. Since $\phi \equiv n(\phi) \in \mathbf{QGL}^{\mathrm{b}}$, we have $\phi \in \mathbf{QGL}^{\mathrm{b}}$. $\Box$

Theorem 3.10. $\mathbf{QGL} \cap \Phi^{\mathrm{r}} = \mathbf{QGL}^{\mathrm{b}} \cap \Phi^{\mathrm{r}} = \mathbf{ML}^3 \cap \Phi^{\mathrm{r}} = \mathbf{QGL}^{\mathrm{r}}$.

Proof. We prove only that $\mathbf{QGL}^{\mathrm{b}} \cap \Phi^{\mathrm{r}} \subseteq \mathbf{QGL}^{\mathrm{r}}$. Suppose $\phi \in \Phi^{\mathrm{r}}$ and $\phi \in \mathbf{QGL}^{\mathrm{b}}$. Then, $n(\phi) \in \mathbf{QGL}^{\mathrm{r}}$, by Theorem 3.9. Hence, $\phi \in \mathbf{QGL}^{\mathrm{r}}$, since $\phi \equiv n(\phi) \in \mathbf{QGL}^{\mathrm{r}}$. $\Box$

4 Completeness of $\mathbf{QGL}^{\mathrm{r}}$

In this section, we show that $\mathbf{QGL}^{\mathrm{r}}$ is complete with respect to the class of finite irreflexive and transitive Kripke frames with constant domains.

Let U be a subset of the set Var of all variables. We denote by $\Phi^{\mathrm{r}}(U)$ the set of formulas of Φ^{r} whose free variables are contained in U.

Definition 4.1. Let $S \subseteq \Phi^{\mathrm{r}}$ and $\phi \in \Phi^{\mathrm{r}}$. We write $S \vdash \phi$ if there exists a finite subset $T \subseteq S$ such that $\bigwedge T \supset \phi \in \mathbf{QGL}^{\mathrm{r}}$. A set $S \subseteq \Phi^{\mathrm{r}}$ is *consistent* if $S \nvdash \bot$.

Lemma 4.2. *Let S be a consistent set. For any formula $\phi \in \Phi^{\mathrm{r}}$, either $S \cup \{\phi\}$ or $S \cup \{\neg\phi\}$ is consistent.*

Proof. Suppose that both $S \cup \{\phi\}$ and $S \cup \{\neg\phi\}$ are not consistent. Then, there exists a finite subset T of S such that $\bigwedge T \wedge \phi \supset \bot \in \mathbf{QGL}^{\mathrm{r}}$ and $\bigwedge T \wedge \neg\phi \supset \bot \in \mathbf{QGL}^{\mathrm{r}}$. Hence, $\bigwedge T \supset \bot \in \mathbf{QGL}^{\mathrm{r}}$, which is a contradiction. □

Lemma 4.3. *Let S be a consistent set. For any formulas ψ_1 and ψ_2 in Φ^{r}, if $S \cup \{\psi_1 \wedge \psi_2\}$ is consistent, then so is $S \cup \{\psi_1 \wedge \psi_2, \psi_1, \psi_2\}$; if $S \cup \{\neg(\psi_1 \wedge \psi_2)\}$ is consistent, then $S \cup \{\neg(\psi_1 \wedge \psi_2), \neg\psi_1\}$ or $S \cup \{\neg(\psi_1 \wedge \psi_2), \neg\psi_2\}$ is consistent.*

Proof. Suppose that $S \cup \{\psi_1 \wedge \psi_2\}$ is consistent. If $S \cup \{\psi_1 \wedge \psi_2, \psi_1, \psi_2\}$ is not consistent, then, there exists a finite subset T of S such that

$$T \wedge (\psi_1 \wedge \psi_2) \supset \bot \in \mathbf{QGL}^{\mathrm{r}}.$$

This is a contradiction. Suppose that $S \cup \{\neg(\psi_1 \wedge \psi_2)\}$ is consistent. If $S \cup \{\neg(\psi_1 \wedge \psi_2), \neg\psi_1\}$ and $S \cup \{\neg(\psi_1 \wedge \psi_2), \neg\psi_2\}$ are not consistent, then there exists a finite subset $T \subseteq S$ such that

$$T \wedge \neg(\psi_1 \wedge \psi_2) \wedge \neg\psi_1 \supset \bot \in \mathbf{QGL}^{\mathrm{r}} \text{ and } T \wedge \neg(\psi_1 \wedge \psi_2) \wedge \neg\psi_2 \supset \bot \in \mathbf{QGL}^{\mathrm{r}}.$$

Then,

$$T \wedge \neg(\psi_1 \wedge \psi_2) \supset \bot \in \mathbf{QGL}^{\mathrm{r}},$$

which is a contradiction. □

Lemma 4.4. *Let U be a finite set of variables and S be a consistent subset of $\Phi^{\mathrm{r}}(U)$. For any formula ψ in Φ^{r}, if $S \cup \{\forall x\psi\}$ is consistent, then so is*

$$S \cup \{\forall x\psi\} \cup \{[y/x]\psi \mid y \in U\}; \tag{3}$$

if $S \cup \{\neg\forall x\psi\}$ is consistent, then $S \cup \{\neg\forall x\psi, \neg[y/x]\psi\}$ is consistent for any variable $y \notin U$.

Proof. Suppose that $S \cup \{\forall x\psi\}$ is consistent. If (3) is not consistent, then there exists a finite subset T of S such that

$$\bigwedge (T \cup \{\forall x\psi\} \cup \{[y/x]\psi \mid y \in U\}) \supset \bot \in \mathbf{QGL}^{\mathrm{r}}.$$

Then,

$$\bigwedge (T \cup \{\forall x\psi\}) \supset \bot \in \mathbf{QGL}^{\mathrm{r}},$$

which is a contradiction. Suppose that $S \cup \{\neg\forall x\psi\}$ is consistent and $y \notin U$. If $S \cup \{\neg\forall x\psi, \neg[y/x]\psi\}$ is not consistent, then there exists a finite subset T of S such that

$$\bigwedge (T \cup \{\neg\forall x\psi\} \cup \{\neg[y/x]\psi\}) \supset \bot \in \mathbf{QGL}^{\mathrm{r}}.$$

Hence,

$$\bigwedge (T \cup \{\neg\forall x\psi\}) \supset [y/x]\psi \in \mathbf{QGL}^{\mathrm{r}}.$$

Since there y does not occur free in $T \cup \{\neg\forall x\psi\}$,

$$\bigwedge (T \cup \{\neg\forall x\psi\}) \supset \forall x\psi \in \mathbf{QGL}^{\mathrm{r}}.$$

Hence,

$$\bigwedge (T \cup \{\neg\forall x\psi\}) \supset \bot \in \mathbf{QGL}^{\mathrm{r}},$$

which is a contradiction. □

Definition 4.5. A consistent set S is said to be *saturated* if it satisfies the following conditions:

1. for all $\phi \in \Phi^{\mathrm{r}}$, either $\phi \in S$ or $\neg\phi \in S$;
2. for all ϕ and ψ in Φ^{r}, $\phi \wedge \psi \in S$ if and only if $\phi \in S$ and $\psi \in S$;
3. for all ϕ in Φ^{r}, $\forall x\phi \in S$ if and only if $[y/x]\phi \in S$ for every $y \in \mathsf{Var}$.

Lemma 4.6. *Let U be a finite subset of* Var *and S be a consistent subset of $\Phi^{\mathrm{r}}(U)$. Then, there exists a saturated set S' such that $S \subseteq S'$.*

Proof. Let $(\phi_i)_{i\in\omega}$ be an enumeration of all formulas in Φ^{r}. For each $n \in \omega$, define a list l_n of formulas of Φ^{r} by

$$l_n = \phi_0, \ldots, \phi_n.$$

Then, define the list $(\gamma_i)_{i\in\omega}$ of formulas in Φ^{r} by

$$l_0, l_1, l_2, \ldots.$$

Note that each formula $\phi \in \Phi^{\mathrm{r}}$ occurs infinitely many times in the list $(\gamma_i)_{i\in\omega}$. Define a sequence $(S_i, U_i)_{i\in\omega}$ of pairs, where each S_i is a consistent subset of $\Phi^{\mathrm{r}}(U_i)$ and each U_i is a finite set of variables, as follows: $(S_0, U_0) = (S, U)$. Assume (S_k, U_k) has been defined. Then, define (S_{k+1}, U_{k+1}) according to the construction of γ_k as follows:
Case $\gamma_k = \top$: $S_{k+1} = S_k \cup \{\top\}$, $U_{k+1} = U_k$.
Case $\gamma_k = \bot$: $S_{k+1} = S_k \cup \{\neg\bot\}$, $U_{k+1} = U_k$.
Case $\gamma_k = P(x_1, \ldots, x_n)$: If $S_k \cup \{\gamma_k\}$ is consistent, then $S_{k+1} = S_k \cup \{\gamma_k\}$. If not, $S_{k+1} = S_k \cup \{\neg\gamma_k\}$. $U_{k+1} = U_k \cup \{x_1, \ldots, x_n\}$.
Case $\gamma_k = \psi_1 \wedge \psi_2$: If $S_k \cup \{\gamma_k\}$ is consistent, then $S_{k+1} = S_k \cup \{\gamma_k, \psi_1, \psi_2\}$. If not, $S_{k+1} = S_k \cup \{\neg\gamma_k, \psi_i\}$, where i is 1 or 2 such that S_{k+1} is consistent. $U_{k+1} = U_k \cup \mathsf{FV}(\gamma_k)$.
Case $\gamma_k = \neg\psi$ or $\gamma_k = \Box\psi$: If $S_k \cup \{\gamma_k\}$ is consistent, then $S_{k+1} = S_k \cup \{\gamma_k\}$. If not, $S_{k+1} = S_k \cup \{\neg\gamma_k\}$. $U_{k+1} = U_k \cup \mathsf{FV}(\gamma_k)$.
Case $\gamma_k = \forall x\psi$: If $S_k \cup \{\gamma_k\}$ is consistent, then $S_{k+1} = S_k \cup \{\gamma_k\} \cup \{[y/x]\psi \mid y \in U_k\}$. $U_{k+1} = U_k \cup \mathsf{FV}(\gamma_k)$. If not, $S \cup \{\neg\gamma_k, \neg[y/x]\psi\}$, where y is a variable such that $y \notin U_k$. $U_{k+1} = U_k \cup \mathsf{FV}(\gamma_k) \cup \{y\}$.

By Lemma 4.2, Lemma 4.3, and Lemma 4.4, the sequence $(S_i, U_i)_{i\in\omega}$ is well defined and satisfies the required conditions. Define $S = \bigcup_{k\in\omega} S_k$. We claim that S is saturated. We verify only that if $\forall\psi \in S$, then $[y/x]\psi \in S$ for every $y \in \mathsf{Var}$. Suppose that $\forall x\psi \in S$. Take any $y \in \mathsf{Var}$. There exists $k \in \omega$ such that $\forall x\psi \in S_k$. By definition of $(\gamma_i)_{i\in\omega}$, there exists $l > k$ such that $\gamma_l = [y/x]\psi$. Since $S \cup \{\neg\gamma_l\}$ is not consistent,

$$[y/x]\psi = \gamma_l \in S_l \subseteq S.$$

□

Lemma 4.7. *Let $W \subseteq \mathcal{P}(\Phi^{\mathrm{r}})$ be the set of all saturated sets of formulas. Define the binary relation R on W as follows: For any S and T in W, (S, T) in R if*

1. *$\Box^{-1}S \cup \Box\Box^{-1}S \subseteq T$. That is, for any $\phi \in \Phi^{\mathrm{r}}$, if $\Box\phi \in S$ then $\Box\phi \in T$ and $\phi \in T$;*
2. *there exists $\Box\phi \in T$ such that $\Box\phi \notin S$.*

Then R is transitive and irreflexive.

Proof. First, we show transitivity. Suppose that (S_1, S_2) and (S_2, S_3) are in R. Take any $\Box\phi \in S_1$. Then, $\Box\phi \in S_2$, since $(S_1, S_2) \in R$. Hence, $\Box\phi \in S_3$ and $\phi \in S_3$, since $(S_2, S_3) \in R$. Next, we show irreflexivity. Suppose that $(S, S) \in R$. Then there exists $\Box\phi$ such that $\Box\phi \in S$ and $\Box\phi \notin S$. This is a contradiction. □

Lemma 4.8. *Let $\langle W, R\rangle$ be the Kripke frame given in Lemma 4.7. Suppose $S \in W$ and $\Box\gamma \notin S$. Then there exists $T \in W$ such that*

$$(S, T) \in R,\ \Box\gamma \in T,\ \text{and } \gamma \notin T\ . \tag{4}$$

Proof. Suppose that $\Box\gamma \notin S$. We claim that

$$S' = \Box^{-1}S \cup \Box\Box^{-1}S \cup \{\neg(\Box\gamma \supset \gamma)\}$$

is consistent. If not, there exists finite sets $\Gamma_1 \subseteq \Box^{-1}S$ and $\Gamma_2 \subseteq \Box\Box^{-1}S$ such that

$$\bigwedge(\Gamma_1 \cup \Gamma_2) \supset (\Box\gamma \supset \gamma) \in \mathbf{QGL}^{\mathrm{r}}.$$

Hence,

$$\bigwedge(\Box\Gamma_1 \cup \Box\Gamma_2) \supset \Box(\Box\gamma \supset \gamma) \in \mathbf{QGL}^{\mathrm{r}}.$$

By definition, $\Box\Gamma_1 \subseteq S$, and since $\Box\phi \supset \Box\Box\phi \in \mathbf{QGL}^{\mathrm{r}}$ for any $\phi \in \Phi^{\mathrm{r}}$, $\Box\Gamma_2 \subseteq S$. Therefore, $\Box(\Box\gamma \supset \gamma) \in S$. By $\Box(\Box\gamma \supset \gamma) \supset \Box\gamma \in \mathbf{QGL}^{\mathrm{r}}$, we have $\Box\gamma \in S$. This is a contradiction, and we conclude the proof of the claim. By definition of Φ^{r}, $\mathsf{FV}(S') = \emptyset$. Hence, there exists a saturated set T such that $S' \subseteq T$, by Lemma 4.6. It is clear that $\Box^{-1}S \cup \Box\Box^{-1}S \subseteq T$. Since $\Box\gamma \wedge \neg\gamma \equiv \neg(\Box\gamma \supset \gamma) \in T$, $\Box\gamma$ and $\neg\gamma$ are in T. □

Lemma 4.9. *Let $\langle W, R\rangle$ be the Kripke frame given in Lemma 4.7. Define $M = \langle W, R, D, I\rangle$ by $D = \mathsf{Var}$ and for any $S \in W$ and any n-ary predicate P,*

$$I(S, P) = \{(x_1, \ldots, x_n) \mid P(x_1, \ldots, x_n) \in S\}.$$

Define $\mathcal{A}$ by the identity mapping on Var. Then, for any $\phi \in \Phi^{\mathrm{r}}$ and $S \in W$, $M, S \models_{\mathcal{A}} \phi$ if and only if $\phi \in S$.

Proof. Induction on ϕ. The case for atomic formulas follows from the definition of M. The cases for negation, conjunction, and universal quantifier follow from Definition 4.5. The case for the modal operator follows from Lemma 4.8. □

Lemma 4.10. *Let M be the Kripke model given in Lemma 4.9. Let $\phi \in \Phi^{\mathrm{r}}$. If $\phi \notin \mathbf{QGL}^{\mathrm{r}}$, then $M \not\models \phi$.*

Proof. Suppose that $\phi \notin \mathbf{QGL}^{\mathrm{r}}$. Then $\{\neg\phi\}$ is consistent. Hence, there exists a saturated set S such that $\neg\phi \in S$ by Lemma 4.6. Hence, $M, S \not\models_{\mathcal{A}} \phi$ by Lemma 4.9. □

Theorem 4.11. $\mathbf{QGL}^{\mathrm{r}}$ *is sound and complete with respect to the class $\mathcal{FI}$ of finite irreflexive and transitive Kripke frames with constant domains.*

Proof. We prove only completeness. Suppose that $\phi \notin \mathbf{QGL}^{\mathrm{r}}$. It is sufficient to show that there exists a finite irreflexive and transitive constant domain Kripke model which refutes ϕ. Let $M = \langle W, R, D, I\rangle$ be the constant domain Kripke model and $\mathcal{A}$ be the assignment given in Lemma 4.9. Then, by Lemma 4.10, there exists $S_0 \in W$ such that $M, S_0 \not\models_{\mathcal{A}} \phi$. We construct a finite submodel M' of M such that

$M' \not\models \phi$. Let Σ be the set of all subformulas of ϕ of the form $\Box\gamma$. Here, the set of subformulas of $\forall x\psi$ is $\{[y/x]\psi \mid y \in \mathsf{Var}\}$. It is clear that for each finite set U of variables, the set of subformulas of ϕ in $\Phi^{\mathrm{r}}(U)$ is finite. Therefore, Σ is a finite set, since $\Sigma \subseteq \Phi^{\mathrm{r}}(\emptyset)$. For each $S \in W$, let Θ_S be $\{\Box\gamma \in \Sigma \mid M, S \not\models_{\mathcal{A}} \Box\gamma\}$. We define a finite subset W' of W by induction. Put $W_0 = \{S_0\}$. Suppose that $W_n = \{S_0, \ldots, S_m\}$ has been constructed. For each S_i with $\Theta_{S_i} \neq \emptyset$ and each $\Box\gamma \in \Theta_{S_i}$, there exists a saturated set T that satisfies (4) in Lemma 4.8. We add T to W_n, if W_n does not contain a saturated set that satisfies (4). Since R is irreflexive and transitive, Θ_T is a proper subset of Θ_{S_i}. Therefore, the construction terminates.

Define a finite model $M' = \langle W', R', D', I' \rangle$, where $D' = \mathsf{Var}$, and R' and I' are the restrictions of R and I on W', respectively. Then, for any subformula ψ of ϕ and $S \in W'$,

$$M', S \models_{\mathcal{A}} \psi \iff M, S \models_{\mathcal{A}} \psi.$$

Hence $M' \not\models \phi$. $\Box$

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