On computable real numbers uniformly distributed modulo 1 with respect to Koksma sequences

Kakeru Yokoyama

Abstract

This paper effectively constructs computable real numbers with uniform distribution properties using algorithmic randomness theory. Although classical uniform distribution theory shows that many sequences are uniformly distributed modulo 1 for almost all real numbers, providing concrete examples has been challenging. By utilizing a total Solovay test, we prove that for any effective K-Koksma sequence $u = (u_n)_{n\in\mathbb{N}}$, a computable real number x can be effectively constructed such that $(u_n(x))$ is uniformly distributed and well-distributed modulo 1. We also show that the discrepancy of this constructed real number satisfies $D_N(u_{|N}(x)) < \mathcal{O}\left(\frac{\log N}{\sqrt[12]{N}}\right)$.

1 Introduction

The classical theory of uniform distribution modulo 1 studies the asymptotic behavior of the fractional parts of real numbers within the unit interval $(0, 1)$. In many contexts, the theory of uniform distribution is developed within a real-analytic framework, where sequences and functions are analyzed from a measure-theoretic perspective. As a result, many classical theorems assert that a property holds for "almost all" real numbers, without identifying individual instances or providing explicit constructions of such numbers. For instance, a theorem of Koksma [4] establishes that for certain function sequences, most notably $(x^n)_{n\in\mathbb{N}}$, the sequence is uniformly distributed modulo 1 for almost all real numbers $x > 1$. However, this metric result does not specify which concrete values of $x > 1$ yield a uniformly distributed sequence. Furthermore, the behavior of such sequences for specific, individual parameters is seldom addressed in the literature, leaving many related questions unresolved.

One major obstacle to analyzing concrete real numbers is that almost all real numbers are uncomputable. Since it is inherently challenging to explicitly construct or manipulate uncomputable objects, classical real analysis typically circumvents this difficulty by focusing on properties that hold for "almost all" real numbers in a measure-theoretic sense. A similar challenge arises in the theory of uniform distribution modulo 1; constructing concrete real numbers that exhibit specific uniform distribution properties remains a difficult task using purely analytical methods.

Algorithmic randomness theory provides a suitable framework for analyzing individual instances of objects that possess generic properties (i.e., properties holding almost everywhere). Indeed, under various formal definitions of algorithmic randomness, almost all sequences (in the measure-theoretic sense) are classified as random. This theory provides effective tools for analyzing random sequences and offers methods to construct sequences with specific computational or random properties. For instance, Chaitin's constant Ω, which represents the halting probability of a universal prefix-free Turing machine, is a well-known example of an uncomputable real number whose binary expansion forms an algorithmically random sequence.

Therefore, it is natural to investigate the theory of uniform distribution modulo 1 from the perspective of algorithmic randomness. For instance, Avigad [1] introduced the concept of "UD random" real numbers, which generate uniformly distributed sequences when multiplied by a sequence of distinct integers, and situated this class within the hierarchy of algorithmic randomness. Furthermore, Becher and Grigorieff [2] demonstrated that membership in specific randomness classes guarantees corresponding uniform distribution properties. As a concrete implication of this result, one can deduce, for example, that the sequence $((1+\Omega)^n)_{n\in\mathbb{N}}$ is uniformly distributed modulo 1. Real numbers exhibiting these distribution properties can be constructed using several

different approaches (such as those in [8]), without necessarily focusing on the computability or effective constructibility of the resulting numbers.

This naturally raises the question of whether there exists a computable or effectively constructible real number that yields a uniformly distributed sequence modulo 1 when substituted into a function sequence such as $(x^n)_{n\in\mathbb{N}}$. The existence of such computable real numbers has been studied by Levin [8] and Lerma [7] using different methodologies. In this paper, by extending the techniques developed by Becher and Grigorieff [2], we provide a complementary, alternative construction of computable real numbers possessing such uniform distribution properties.

2 Prerequisites

In this section, we introduce the fundamental concepts that serve as prerequisites for the Main theorem.

2.1 Martin-Löf randomness

A fundamental property of random sequences is that they "do not possess any special properties." In other words, sequences with special properties—such as the frequency of 1s being higher than 0s, or a 0 always being followed by a 1—are not random sequences. A random sequence can be defined as one that completely lacks such properties.

The generality or specificity of a property is represented by the size of the set of elements satisfying that property. That is, "a sequence has a more special property" means that "the sequence is contained in a set with a smaller measure." From this, saying that $\alpha \in 2^\omega$ is non-random can be expressed as α being contained in some set of small measure. Conversely, being random means not being contained in any set of small measure. The formal definition is as follows.

Definition 1.

For a computable measure μ on the Cantor space, a uniformly c.e. sequence U_i (where $U_i \subset 2^\omega$) satisfying $\mu(U_i) \leq 2^{-i}$ is called a Martin-Löf test.

A sequence $\alpha \in 2^\omega$ is said to be Martin-Löf random if $\alpha \notin \bigcap_i U_i$ holds for any Martin-Löf test U_i.

2.1.1 random test

By slightly modifying the definition of the Martin-Löf test, we can consider different definitions of random sequences.

Definition 2.

For a computable measure μ on the Cantor space, a uniformly c.e. sequence $U_i \subset [0,1]$ satisfying $\mu(U_i) \leq 2^{-i}$ such that each $\mu(U_i)$ is uniformly computable is called a Schnorr test.

A real number $\alpha \in [0,1]$ is said to be Schnorr random if $\alpha \notin \bigcap_i U_i$ holds for any Schnorr test (U_i).

Since Schnorr tests are subsumed by Martin-Löf tests, it naturally holds for the random sequences defined outside them that Martin-Löf random ⇒ Schnorr random.

We can also consider the following definition.

Definition 3.

A uniformly c.e. sequence U_i such that $\sum_i \mu(U_i)$ converges to a finite value is called a Solovay test. Furthermore, if it converges to a computable real number, it is called a total Solovay test.

Belonging to at most finitely many U_i for any Solovay test is equivalent to being Martin-Löf random. Also, belonging to at most finitely many U_i for any total Solovay test is equivalent to being Schnorr random.

2.2 u.d. mod 1

Definition 4.
A sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ *is said to be* uniformly distributed modulo 1 if it satisfies the following:

$$\forall [a,b] \subset [0,1] \ \lim_{N\to\infty} \frac{1}{N} \#\{n < N : x_n - \lfloor x_n \rfloor \in [a,b]\} = b - a$$

Hereafter, it is abbreviated as u.d. mod 1.

represents the counting measure. That is, $\frac{1}{N}\#\{n < N : x_n - \lfloor x_n \rfloor \in [a,b]\}$ represents the proportion of the first N terms of x_n whose fractional part falls within $[a,b]$. The fact that the limit of this proportion equals the length of the interval itself, for any given interval, implies that the distribution is uniform.

Let us consider the following sequences as x_n:

- $b^n x$
- $a_n x$ (a_n: distinct integers)
- $(1+x)^n$

For almost all real numbers x on $[0,1]$, these are u.d. mod 1. However, it is difficult to give a specific, concrete example of such an x. In fact, if a Martin-Löf random real or a Schnorr random real is taken as x, all three of these sequences become u.d. mod 1 [2, Theorem 1].

A random test was defined as the intersubsection of a sequence of sets with decreasing measures, i.e., a null set. Since random sequences or random reals are elements that do not belong to these sets, it can be said that almost all real numbers on $[0,1]$ are random reals. Through this concept of "almost everywhere," randomness and u.d. mod 1 can be connected. For instance, we can consider the following concept:

Definition 5.
A real number $x \in [0,1]$ *such that* $(a_n x)$ *is* u.d. mod 1 for any sequence of distinct integers a_n is called UD random.

Since $(a_n x)$ is u.d. mod 1 for any Schnorr random real x, the implication Martin-Löf random $\Rightarrow$ Schnorr random $\Rightarrow$ UD random holds.

2.3 Weyl's Criterion

Theorem 6 (Weyl's Criterion)**.** *[5, Theorem 2.1]*
The following are equivalent:

- (x_n) *is* u.d. mod 1
- $\forall h \in \mathbb{Z}\backslash\{0\} \ \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \exp(2i\pi h x_j) = 0$

Intuitively, this property is interpreted as the equivalence between "being uniformly distributed on the interval $[0,1]$" and "the average position of points on the unit circle being the origin."

2.4 well-distributed modulo 1

Definition 7.
A sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ *is said to be* well-distributed modulo 1 if it satisfies the following:

$$\forall k \in \mathbb{N}, \forall [a,b] \subset [0,1] \ \lim_{N\to\infty} \frac{1}{N} \#\{n + k < N : x_{n+k} - \lfloor x_{n+k} \rfloor \in [a,b]\} = b - a$$

Hereafter, it is abbreviated as w.d. mod 1.

Intuitively, w.d. mod 1 is interpreted to mean that the sequence remains u.d. mod 1 even if any finite number of initial terms are discarded. Weyl's Criterion also holds for w.d. mod 1.

Theorem 8 (Weyl's Criterion for w.d. mod 1)**.**
The following are equivalent:

- (x_n) *is* w.d. mod 1
- $\forall k \in \mathbb{N}, \forall h \in \mathbb{Z}\backslash\{0\} \lim_{N\to\infty} \frac{1}{N} \sum_{j=k}^{k+N-1} \exp(2i\pi h x_j) = 0$

2.5 discrepancy

Definition 9. *For an* N bit prefix $(x_{|N})$ of a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$, the discrepancy $D_N(x_{|N})$ is defined as follows:

$$D_N(x_{|N}) = \sup_{0\le a<b\le 1} \left| \frac{\#\{n < N : x_n - \lfloor x_n \rfloor \in [a,b]\}}{N} - (b-a) \right|$$

Intuitively, discrepancy is interpreted as the degree of bias in the interval with the largest deviation within the N-bit prefix. Naturally, when (x_n) is u.d. mod 1, $\lim_{N\to\infty} D_N(x_{|N}) = 0$ holds. It is also known that $D_N(x_{|N}) > \mathcal{O}(\frac{\log N}{N})$ generally holds as a lower bound for the rate of convergence.

2.6 koksma sequence

We have already shown that x such that $b^n x$, $a_n x$, $(1+x)^n$, etc., are u.d. mod 1 applies to almost all real numbers on $[0,1]$, including Martin-Löf random reals and Schnorr random reals. The three types of sequences given as examples here can be further generalized.

Definition 10.
For $K > 0$*, a sequence of functions* $u_n : [0,1] \to \mathbb{R}$ *is called a* K-koksma sequence if it satisfies the following:

- All u_n are continuously differentiable.
- For any n, m, $u'_m(x) - u'_n(x)$ is a monotonic function on $[0,1]$.
- For any $n \neq m$ and $x \in [0,1]$, $|u'_m(x) - u'_n(x)| \geq K$.

When all u_n, u'_n are computable, it is called an effective K-koksma sequence.

The sequence of real numbers obtained by substituting a Martin-Löf random real or a Schnorr random real into any effective koksma sequence is u.d. mod 1 [2, Theorem 1]

3 Main theorem

The following is the main theorem of this paper.

Theorem 11.
For any effective K-koksma sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$, a computable real x such that $(u_n(x))$ is u.d. mod 1 can be effectively constructed.

The following are lemmas for proving this theorem.

Lemma 12. *[2, Lemma 27]*
Let $e(x) = e^{2i\pi x}$*. For a sequence of real numbers* $\mathbf{x} = (x_n)_{n\geq 1}$*, let* $S_N(\mathbf{x}) = \frac{1}{N}\sum_{j=1}^{N} e(x_j)$*. Also, let* $h\mathbf{x}$ *denote the sequence of real numbers* $(hx_n)_{n\in\mathbb{N}}$*. The following conditions are equivalent:*

1. *The sequence* $(x_n)_{n\geq 1}$ *is u.d. mod 1.*

2. *For all* $h \in \mathbb{Z} \setminus \{0\}$, $\lim_{N\to\infty} S_N(h\mathbf{x}) = 0$.

3. *There exists a strictly monotonically increasing sequence of positive integers* $(M_k)_{k\in\mathbb{N}}$ *such that* $M_{k+1} - M_k = o(M_k)$, *and* $\lim_{k\to\infty} S_{M_k}(h\mathbf{x}) = 0$ *holds for all* $h \in \mathbb{Z} \setminus \{0\}$.

Proof.

The equivalence of (1) and (2) follows from Weyl's Criterion.

By setting $M_k = k$, (2) $\Rightarrow$ (3) is trivial.

We prove the converse, that is, (3) $\Rightarrow$ (2).

Note that since $|e(x)| = 1$ for all x, the absolute value of the sum of p such exponentials is bounded by p. Therefore, if $M_k \le N < M_{k+1}$,

$$
\begin{aligned}
|S_N(hx) - S_{M_k}(hx)| &= \left| \left(\frac{1}{N} - \frac{1}{M_k} \right) \sum_{l=1}^{M_k} e(hx_l) + \frac{1}{N} \sum_{l=M_k+1}^{N} e(hx_l) \right| \\
&\le \frac{N - M_k}{NM_k} \left| \sum_{l=1}^{M_k} e(hx_l) \right| + \frac{1}{N} \left| \sum_{l=M_k+1}^{N} e(hx_l) \right| \\
&\le \frac{N - M_k}{NM_k} M_k + \frac{N - M_k}{N} \\
&= \frac{N - M_k}{N} + \frac{N - M_k}{N} = \frac{2(N - M_k)}{N} \\
&\le \frac{2(M_{k+1} - M_k)}{M_{k+1}}
\end{aligned}
$$

The last inequality holds because the function $\frac{t-M_k}{t}$ is monotonically increasing with respect to t for $t \ge M_k$.

By the triangle inequality,

$$
|S_N(h\mathbf{x})| \le |S_{M_k}(h\mathbf{x})| + |S_N(h\mathbf{x}) - S_{M_k}(h\mathbf{x})| \le |S_{M_k}(h\mathbf{x})| + \frac{2(M_{k+1} - M_k)}{M_{k+1}}
$$

By assumption, $|S_{M_k}(h\mathbf{x})| \to 0$ as $k \to \infty$. Also, $M_{k+1} - M_k = o(M_k)$ implies $\frac{M_{k+1}-M_k}{M_k} \to 0$, which implies $\frac{M_{k+1}}{M_k} \to 1$, so $\frac{M_{k+1}-M_k}{M_{k+1}} \to 0$ as well. Therefore, both terms on the right-hand side of the above inequality converge to 0. Thus, $|S_N(h\mathbf{x})| \to 0$ as $N \to \infty$. □

Lemma 13.

If B_k *is a* total Solovay test, then among the elements belonging to at most finitely many B_k, there exists a computable one, and it can be effectively constructed.

Proof.

Let L be a natural number satisfying $\mu\left(\bigcup_{k>L} B_k\right) < \frac{1}{4}$.

Hereafter, let $\alpha_{|n}$ denote the n-bit prefix of a finite or infinite sequence α, let $[x]$ denote the set of all infinite sequences having a finite sequence x as a prefix, and let xy denote the concatenation of finite sequences x and y. Also, we identify real numbers with infinite 0-1 sequences.

We determine the 0-th bit as follows. Let $c_0 = \frac{1}{4}$, $\varepsilon_n = 2^{-2n-3}$, $c_{n+1} = \frac{1}{2}c_n + \varepsilon_n$.

$$
\mu\left([0] \cap \bigcup_{k>L} B_k\right),\ \mu\left([1] \cap \bigcup_{k>L} B_k\right)
$$

We calculate these in parallel. When it is determined that either value is strictly less than c_1, we set the 0-th bit to 0 if it is the left one, and 1 if it is the right one. (If simultaneous, setting it to 0 is fine.) Since $\mu\left(\bigcup_{k>L} B_k\right)$ is computable and strictly less than $\frac{1}{4}$, this operation can be effectively executed.

When the desired sequence α has been constructed up to the n-th bit ($n \geq 0$), we determine the $(n+1)$-th bit as follows.

$$\mu\left([\alpha_{|n+1}0] \cap \bigcup_{k>L} B_k\right),\ \mu\left([\alpha_{|n+1}1] \cap \bigcup_{k>L} B_k\right)$$

We calculate these in parallel. When it is determined that either value is strictly less than c_{n+1}, we set the $(n+1)$-th bit to 0 if it is the left one, and 1 if it is the right one. (If simultaneous, setting it to 0 is fine.) Since $\mu\left([\alpha_{|n+1}] \cap \bigcup_{k>L} B_k\right)$ is computable and strictly less than c_n, this operation can be effectively executed. The sequence α constructed in this way is a computable sequence.

We show that α belongs to only finitely many B_k. Assume that α belongs to infinitely many B_k. Since B_k are open sets, for a sufficiently large n, $[\alpha_{|n}] \subseteq \bigcup_{k>L} B_k$ holds. Therefore,

$$\mu\left([\alpha_{|n}] \cap \bigcup_{k>L} B_k\right) = \mu\left([\alpha_{|n}]\right) = 2^{-n}$$

On the other hand, from the construction method of α,

$$\mu\left([\alpha_{|n}] \cap \bigcup_{k>L} B_k\right) < c_n$$

holds. Since the general term of c_n is $c_n = \frac{3}{4} \cdot 2^{-n} - 2^{-2n-1} = 2^{-n}\left(\frac{3}{4} - 2^{-(n+1)}\right) < 2^{-n}$, a contradiction arises.

From the above, α is a computable sequence that belongs to only finitely many B_k. □

We now present the proof of Theorem 11.

Proof.

Fix a K-koksma sequence $\mathbf{u}$. We define

$$S_{N,h}(\mathbf{u}(x)) = \frac{1}{N}\sum_{j=1}^{N} \exp(2i\pi h u_j(x))$$

We define

$$A^{\varepsilon}_{N,h} = \{x \in (0,1) : |S_{N,h}(\mathbf{u}(x))| > \varepsilon\}$$

Here, there exists a set $B^{\varepsilon}_{N^2,h}$ such that $A^{2\varepsilon}_{N^2,h} \subseteq B^{\varepsilon}_{N^2,h} \subseteq A^{\varepsilon}_{N^2,h}$, its measure $\mu(B^{\varepsilon}_{N^2,h})$ is computable, and it satisfies $\sum_{N>L} \mu(B^{\varepsilon}_{N^2,h}) < C\frac{\ln L + 1}{L}$ (where C is a constant depending on h and ε) [2, 4.4].

We define

$$B_k = \bigcup_{\substack{0<|h|\leq k \\ h\in\mathbb{Z}}} \bigcup_{N>k^3} B^{\frac{1}{k}}_{N^4,h}$$

By Lemma 12, the condition that $(u_n(x))$ is not u.d. is equivalent to the existence of an increasing sequence of positive integers M_N with $M_{N+1} - M_N = o(M_N)$ and some $h \in \mathbb{Z}_{\neq 0}$ such that $\lim_{N\to\infty} S_{M_N,h}(\mathbf{u}(x)) \neq 0$ holds.

Since $M_N = N^4$ satisfies the condition for M_N, when $(u_n(x))$ is not u.d., there exist infinitely many $A^{\frac{1}{2k}}_{N^4,h}$ containing x for some h and a sufficiently large k. Therefore, for a sufficiently large k, $x \in B_k$ holds.

If $(u_n(x))$ is not u.d., then x is contained in infinitely many B_k; thus, when x is contained in only finitely many B_k, $(u_n(x))$ is u.d.

By Lemma 13, if B_k is a total Solovay test, then among the elements contained in only finitely many B_k, there exists a computable one, and this is precisely the computable real for which $(u_n(x))$ is u.d. mod 1.

Below, we show that B_k is a total Solovay test (i.e., $\sum_{k\geq 1}\mu\left(B_k\right)$ is a computable real).

By assumption, $\mu(B^{\frac{1}{k}}_{N^4,h})$ is computable. Furthermore, $\mu\left(\bigcup_{N>\sqrt{L}} B^{\frac{1}{k}}_{N^4,h}\right) < \sum_{N>\sqrt{L}}\mu\left(B^{\frac{1}{k}}_{N^4,h}\right) < \sum_{N>L}\mu\left(B^{\frac{1}{k}}_{N^2,h}\right) < C\dfrac{\ln(L)+1}{L}$ holds (where C is a constant depending on h and k).

$\mu\left(\bigcup_{\sqrt{L}\geq N>k^3} B^{\frac{1}{k}}_{N^4,h}\right)$ is the measure of a finite union of sets with computable measures, and is therefore computable. Therefore, for a given precision δ, if we choose L such that $C\dfrac{\ln(L)+1}{L} < \delta$ is satisfied, we have

$$\mu\left(\bigcup_{N>k^3} B^{\frac{1}{k}}_{N^4,h}\right) \leq \mu\left(\bigcup_{\sqrt{L}\geq N>k^3} B^{\frac{1}{k}}_{N^4,h}\right) + C\frac{\ln(L)+1}{L} \leq \mu\left(\bigcup_{N>k^3} B^{\frac{1}{k}}_{N^4,h}\right) + \delta$$

from which we can compute $\mu\left(\bigcup_{N>k^3} B^{\frac{1}{k}}_{N^4,h}\right)$ to precision δ. In other words, $\mu\left(\bigcup_{N>k^3} B^{\frac{1}{k}}_{N^4,h}\right)$ is computable.

Furthermore, $\mu(B_k) = \mu\left\{\bigcup_{\substack{0<|h|\leq k\\ h\in\mathbb{Z}}}\bigcup_{N>k^3} B^{\frac{1}{k}}_{N^4,h}\right\}$ is the measure of a finite union of sets with computable measures, and is therefore computable.

$$\begin{aligned}
\mu(B_k) &\leq \sum_{\substack{0<|h|\leq k\\ h\in\mathbb{Z}}}\sum_{N>k^3}\mu(B^{\frac{1}{k}}_{N^4,h})\\
&\leq \sum_{\substack{0<|h|\leq k\\ h\in\mathbb{Z}}}\sum_{N>k^3}\mu(A^{\frac{1}{k}}_{N^4,h})\\
&\leq \sum_{\substack{0<|h|\leq k\\ h\in\mathbb{Z}}}\sum_{N>k^3} k^2\left(1+\frac{17}{K|h|}\right)\frac{\ln(N^4)}{N^4}\\
&\leq 4k^2\sum_{\substack{0<|h|\leq k\\ h\in\mathbb{Z}}}\left(1+\frac{17}{K|h|}\right)\sum_{N>k^3}\frac{\ln N}{N^4}\\
&\leq 4k^2\cdot\left(\sum_{\substack{0<|h|\leq k\\ h\in\mathbb{Z}}} 18\right)\cdot\left(\sum_{N>k^3}\frac{1}{N^3}\right) \quad (\text{assuming } K\geq 1, \ln N < N)\\
&\leq 4k^2\cdot(2k\cdot 18)\cdot\frac{1}{2k^6} = \frac{72}{k^3}
\end{aligned}$$

From this, $\sum_{k\geq 1}\mu(B_k)$ converges to a finite value.

Furthermore, $\sum_{L\geq k\geq 1}\mu(B_k)$ is computable, and for a given precision δ, if we take L such that

$\sum_{k>L} \mu(B_k) \leq \sum_{k>L} \frac{72}{k^3} \approx \frac{36}{L^2} < \delta$ is satisfied,

$$\sum_{k=1}^{L} \mu(B_k) \leq \sum_{k\geq 1} \mu(B_k) \leq \sum_{k=1}^{L} \mu(B_k) + \frac{36}{L^2} < \sum_{k=1}^{L} \mu(B_k) + \delta$$

we can compute $\sum_{k\geq 1} \mu(B_k)$ to precision δ. In other words, $\sum_{k\geq 1} \mu(B_k)$ is computable.

From the above, B_k is a total Solovay test. □

Corollary 14.

For a K-koksma sequence $\mathbf{u} = (u_n)_{n\in\mathbb{Z}}$, a computable real x such that $(u_n(x))$ is w.d. mod 1 exists and can be effectively constructed.

Proof.

It suffices to change a part of the proof. We define

$$S_{N,h,t}(\mathbf{u}(x)) = \frac{1}{N} \sum_{j=t}^{N+t} \exp(2i\pi h u_j(x))$$

We define

$$A^{\varepsilon}_{N,h,t} = \{x \in (0,1) : |S_{N,h,t}(\mathbf{u}(x))| > \varepsilon\}$$

There exists a set $B^{\varepsilon}_{N^2,h,t}$ such that $A^{2\varepsilon}_{N^2,h,t} \subseteq B^{\varepsilon}_{N^2,h,t} \subseteq A^{\varepsilon}_{N^2,h,t}$, its measure $\mu(B^{\varepsilon}_{N^2,h,t})$ is computable, and it satisfies

$$\sum_{N>L} \mu(B^{\varepsilon}_{N^2,h,t}) < C\frac{\ln L + 1}{L}$$

(where C is a constant depending on h, ε, t).

We define

$$B'_k = \bigcup_{\substack{0\leq t<k \\ t\in\mathbb{Z}}} \bigcup_{\substack{0<|h|\leq k \\ h\in\mathbb{Z}}} \bigcup_{N>k^3} B^{\frac{1}{k}}_{N^4,h,t}$$

The condition that $(u_n(x))$ is not w.d. is equivalent to the condition that

$$\exists t \in \mathbb{N},\ \exists h \in \mathbb{Z}_{\neq 0} \text{ s.t. } \lim_{N\to\infty} S_{N^4,h,t}(\mathbf{u}(x)) \neq 0$$

holds.

In other words, when $(u_n(x))$ is not w.d., there exist infinitely many $A^{\frac{1}{2k}}_{N^4,h,t}$ containing x for some h, t and a sufficiently large k. Therefore, for a sufficiently large k, $x \in B'_k$ holds.

Thereafter, the same proof can be carried out for B'_k. □

For the sequence of real numbers constructed in this theorem, we can provide an upper bound for the discrepancy.

Corollary 15.

For the computable real x constructed in Theorem 11, the discrepancy $D_N(u_{|N}(x)) < \mathcal{O}(\frac{\log N}{\sqrt[12]{N}})$ holds.

Proof.

The definition of the set B_k was as follows:

$$B_k = \bigcup_{\substack{0<|h|\leq k \\ h\in\mathbb{Z}}} \bigcup_{N>k^3} B^{\frac{1}{k}}_{N^4,h}$$

Also, from the relation $A^{2\varepsilon}_{N^2,h} \subseteq B^{\varepsilon}_{N^2,h}$, $x \notin B_k$ implies the following condition:

$$x \in \bigcap_{\substack{0<|h|\leq k \\ h\in\mathbb{Z}}} \bigcap_{N>k^3} \left(A^{\frac{2}{k}}_{N^4,h}\right)^c$$

Here, since the definition of the set $A^{\varepsilon}_{N,h}$ is $A^{\varepsilon}_{N,h} = \{x \in (0,1) : |S_{N,h}(\mathbf{u}(x))| > \varepsilon\}$, the above inclusion relation can be expressed by the following equivalent conditional expression:

$$0 < \forall |h| \leq k, \forall N > k^3, \quad |S_{N^4,h}(\mathbf{u}(x))| \leq \frac{2}{k}$$

From this expression, $|S_{N,h}(\mathbf{x})| < \frac{2}{\sqrt[12]{N}}$ is derived. Here, we apply the Erdős–Turán theorem [5, Theorem 2.5]. According to this theorem, for any finite sequence $\{x_1, \cdots, x_N\} = \mathbf{x}$ and any positive integer m, the following inequality holds:

$$D_N \leq \frac{6}{m+1} + \frac{4}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1} \right) |S_{N,h}(\mathbf{x})|$$

By substituting the previously derived $|S_{N,h}(\mathbf{x})| < \frac{2}{\sqrt[12]{N}}$ into this inequality, we finally obtain the following as the evaluation of the discrepancy $D_N(u_{|N}(x))$:

$$D_N(u_{|N}(x)) < \mathcal{O}\left(\frac{\log N}{\sqrt[12]{N}} \right)$$

□

As noted in subsection 3.4, for an N-bit prefix of a u.d. mod 1 sequence of real numbers (x_n), $D_N(x_{|N}) > \mathcal{O}(\frac{\log N}{N})$ generally holds.

4 Conclusion

Becher and Grigorieff previously constructed Solovay tests for the sets of sequences that do not satisfy Weyl's Criterion to demonstrate that random reals possess the property of uniform distribution. In this paper, by appropriately taking the union of all these tests, we constructed a single total Solovay test that encompasses all sequences failing to satisfy Weyl's Criterion. Furthermore, by constructing a computable element outside of this test, we successfully obtain a computable element that satisfies Weyl's Criterion.

References

[1] Jeremy Avigad. Uniform distribution and algorithmic randomness. *The Journal of Symbolic Logic*, 78(1):334–344, 2013.

[2] Verónica Becher and Serge Grigorieff. Randomness and uniform distribution modulo one. *Information and Computation*, 285:104857, 2022.

[3] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic Randomness and Complexity*. Theory and Applications of Computability. Springer, New York, 2010.

[4] J. F. Koksma. Ein mengentheoretischer satz über die gleichverteilung modulo eins. *Compositio Mathematica*, 2:250–258, 1935.

[5] Lauwerens Kuipers and Harald Niederreiter. *Uniform Distribution of Sequences*. Pure and Applied Mathematics. Wiley-Interscience, New York, 1974.

[6] Miguel A. Lerma. Distribution of powers modulo 1 and related topics. 1995. arXiv:1312.5996.

[7] Miguel A. Lerma. Construction of a number greater than one whose powers are uniformly distributed modulo one. Preprint, February 1996.

[8] M. B. Levin. Effektivizatsiya teoremy koksma. *Matematicheskie Zametki*, 47(1):163–166, 1990. In Russian.