Real Seiberg–Witten Theory and Exotic P^2-Knots

Jin Miyazawa
Research Institute for Mathematical Sciences, Kyoto University

1 Introduction

In 4-dimensional topology, the following problem, which is called smooth unknotting conjecture, is known to be difficult and important:

Conjecture 1.1. *Let $f\colon S^2 \to S^4$ be a smooth embedding. If $\pi_1(S^4 \setminus f(S^2)) \cong \mathbb{Z}$, then f is smoothly isotopic to the unknot.*

An embedding of S^2 to S^4 is called a 2-knot. The unknot is a 2-knot which bounds an embedded D^3 in S^4. If the fundamental group of the complement of a 2-knot is isomorphic to $\mathbb{Z}$, then this is topologically isotopic to the unknot. This was proven by Freedman and Quinn [3, 11.7A].

While conjecture 1.1 is still open, if we consider embeddings of $\mathbb{R}P^2$ instead of S^2, we have the following result.

Theorem 1.2. *[12] There is an infinite family of embeddings $f_n\colon \mathbb{R}P^2 \to S^4$ for all $n \in \mathbb{N}$ such that*

- *f_0 is the standard one,*
- *$\{f_n\}_n$ are topologically ambient isotopic to each other,*
- *if $n \neq m$, then f_n and f_m are not smoothly isotopic.*

In this article, the sketch of the proof of theorem 1.2 is explained. To prove theorem 1.2, we use Real Seiberg–Witten theory, which is a variant of the ordinary Seiberg–Witten theory.

2 Embeddings of surfaces into S^4

In this section, we recall some previous works on exotic embeddings of surfaces into S^4. If two smooth embeddings

$$f, g\colon S \to S^4$$

of a closed surface S are topologically ambient isotopic and not smoothly isotopic, we call f and g are exotic embeddings.

There are no known examples of exotic embeddings of orientable surfaces. In the case of non-orientable surfaces, the first such examples are discovered by Finashin–Kreck–Viro[2], in 1987. They constructed an infinite family of exotic embeddings of $\#_{10}\mathbb{R}P^2$ s. After that Finashin gave examples of exotic embeddings of $\#_6\mathbb{R}P^2$. Havens constructed exotic embeddings of $\#_6, \#_7, \#_8, \#_9\mathbb{R}P^2$. Levine–Lidman–Piccirillo also gave an example of $\#_{10}$. Matić–Öztürk–Reyes–Stipsicz–Urzúa proved that there exists an exotic embedding of $\#_5\mathbb{R}P^2$ in S^4.

To prove the exoticness of these examples, there is a common strategy: taking double branched covers along the surfaces and proving that they are not diffeomorphic. However, if one try to prove the exoticness of the embeddings of theorem 1.2 in this strategy, one have to find exotic $\mathbb{C}P^2$. This is known to be difficult. Moreover, the following result is proved by Hughes–Kim–Miller.

Theorem 2.1. *[4] The double branched covers along $f_n(\mathbb{R}P^2)$ of S^4 are diffeomorphic to $\mathbb{C}P^2$, where f_n are embeddings given in theorem 1.2.*

As a corollary, we have an infinite family of exotic $\mathbb{Z}/2$-action on $\mathbb{C}P^2$.

To prove theorem 1.2, we use Real Seiberg–Witten theory. [1] From this variant of the Seiberg–Witten theory, we can construct invariants of pairs of a smooth 4-manifold and a smooth involution on it.

3 Real Seiberg–Witten theory

3.1 Recall ordinary Seiberg–Witten theory

To explain Real Seiberg–Witten theory, let us recall ordinary Seiberg–Witten theory. For simplicity, we only consider the invariants on closed 4-manifolds.

Let X be a closed 4-manifold. The input of this invariant is spin^c structure on X. This always exists and classified by $H^2(X, \mathbb{Z})$. Note that this identification is not canonical. If the rank $b^+(X)$ of maximal positive definite subspace of $H^2(X)$ is bigger than 1, we have an integer $SW(X, \mathfrak{s})$ for each spin^c structure $\mathfrak{s}$ on X.

In the case that $b^+(X) = 1$, we have to fix another data c called chamber and we have an invariant $SW(X, \mathfrak{s}, c)$ which depends on the chamber. There are two choices of chambers for each spin^c structure. If we fix a spin^c structure, the difference between the invariants for different choices of chambers is ± 1.

These invariants are given by counting the solutions of the Seiberg–Witten equations. More precisely, these are obtained by counting the solutions up to gauge transformations because the Seiberg–Witten equations have a symmetry of an action of the gauge group $\mathcal{G} = \mathrm{Map}(X, S^1)$. It is necessary to take the quotient of the set of the solutions by this action otherwise the set of the solutions is too large.

The reason why we need the condition $b^+(X) > 0$ is to avoid *reducible solution* on Seiberg–Witten equations. Reducible solutions are solutions whose isotoropy groups are

[1]We use capital R when we spell the Real Seiberg–Witten theory and "Real" variants of related notions. This comes from Atiyah's Real K-theory, which is denoted by $KR(X)$. This notion is related to Real Seiberg–Witten theory because we consider an anti-linear lift of the involution on a 4-manifold X to a spinor bundle on it.

the non-trivial. In the negative definite case, we cannot avoid reducible solutions so it is difficult to define a counting invariant of the solutions up to $\mathcal{G}$-action.

3.2 Real Seiberg–Witten theory

Let me explain the Real Seiberg–Witten theory. The input data are the following: A closed 4-manifold X with an orientation preserving involution $\iota\colon X \to X$, a spin^c structure on X, and an anti-linear lift I on the spinor bundle $S = S^+ \oplus S^-$ of $\mathfrak{s}$. We assume that I preserves the $\mathbb{Z}/2$-grading of S and satisfies the following relation: $\rho(\iota^* a)I(\phi) = I(\rho(a)\phi)$ for any $a \in \Omega^1(X)$ and $\phi \in S$, where ρ is the Clifford multiplication.

We call a pair $(\mathfrak{s}, I)$ a Real spin^c structure. Let $(\mathfrak{s}, I)$ and $(\mathfrak{s}', I')$ be Real spin^c structures. We say $f : \mathfrak{s} \to \mathfrak{s}'$ an isomorphism of Real spin^c structures when f is an isomorphism of spin^c structures and satisfying $f \circ I = I' \circ f$.

Note that Real spin^c structure may not exists in some cases. For example if the fixed points of ι is finite, then the anti-linear lift I of ι satisfies $I^2 = -1$ on any fixed points. If the fixed points of ι has codimension 2, $H^1(X,\mathbb{Z})^{-\iota^*} = 0$, and there is a spin^c structure $\mathfrak{s}$ satisfying $\iota^*\mathfrak{s} \cong \bar{\mathfrak{s}}$, then we have an anti-linear lift I of ι such that $I^2 = 1$ ([7, Lemma 2.9]).

If there is a Real spin^c structure on X, the set of the isomorphism class is identified by the set of isomorphism class of Real line bundles, which is a complex line bundle with anti-linear lift of ι. We can interpret this classification result in terms of equivariant cohomology with local coefficient. This corresponds to the fact that the ordinary spin^c structures is classified by $H^2(X,\mathbb{Z})$.

Definition 3.1. Let X be a space with an involution ι. Let us consider the following space:

$$E\mathbb{Z}/2 \times X/i \times \iota$$

where i is the free involution on $E\mathbb{Z}/2$. Then we have a local system

$$l := (E\mathbb{Z}/2 \times X \times \mathbb{Z})/i \times \iota \times (-1)$$

on $E\mathbb{Z}/2 \times X/i \times \iota$. Let us define $H^*_{\mathbb{Z}/2}(X, \mathbb{Z}_-)$ by

$$H^*_{\mathbb{Z}/2}(X, \mathbb{Z}_-) := H^*(E\mathbb{Z}/2 \times X/i \times \iota, l).$$

Proposition 3.2. *The Real line bundle on X is classified by $H^2_{\mathbb{Z}/2}(X, \mathbb{Z}_-)$. Especially, if there is a Real spin^c structure on X, isomorphism classes of Real spin^c structure is classified by $H^2_{\mathbb{Z}/2}(X, \mathbb{Z}_-)$.*

Remark 3.3. If the involution is free, then Real spin^c structures are coincide with spin^{c-} structure on X/ι. See [14, Section 2.(v)].

Let $(\mathfrak{s}, I)$ be a Real spin^c structure on (X^4, ι). There is an involution induced by I on the configuration space $\mathcal{C}$ of Seiberg–Witten equations for the spin^c structure $\mathfrak{s}$. By abuse of notation, let us denote this action by I. We want to consider the fixed point set $\mathcal{C}^I$ and solutions preserved by I. However, as we mentioned, the Seiberg–Witten equations have a large symmetry called the gauge transformation $\mathcal{G} = \mathrm{Map}(X, U(1))$. A gauge transformation u preserves the fixed point set $\mathcal{C}^I$ if and only if $u = IuI = \iota^* u^{-1}$. We denote the set of such gauge transformations by $\mathcal{G}^I$.

Here, let us point out the difference between the ordinary Seiberg–Witten equations and Real one. Assume the formal dimension of the moduli space is 0 and $H^1(X, \mathbb{Z}) = 0$.

In ordinary case, the framed moduli space, which is a quotient of the set of the solutions by a subgroup $\mathcal{G}_0$ of $\mathcal{G}$ such that it acts freely on $\mathcal{C}$, is consist of some copies of $U(1)$ and a reducible point $*$ if $b^+(X) = 0$. This is because $\mathcal{G}/\mathcal{G}_0 \cong U(1)$. Moreover, the number of the copies of $U(1)$ depends on the metric so that this is not an invariant of $(X, \mathfrak{s})$.

In Real case, the framed moduli space is some copies of $\mathbb{Z}/2 \cong \{\pm 1\}$ and a reducible point. This is because $\mathcal{G}^I/\mathcal{G}^I_0 \cong \{\pm 1\}$ where $\mathcal{G}^I_0$ is a subgroup acting freely on $\mathcal{C}^I$. In this case, the number of irreducible solutions is not a invariant as in the ordinary case. However, if we forget ± 1 action on the framed moduli space and just signed count of them, this is independent of the metric and is well defined up to overall sign. Let us denote the absolute value of this signed count by

$$|\deg(X, \iota, \mathfrak{s}, I)|. \tag{1}$$

Proposition 3.4. *Let X be a closed oriented smooth 4-manifold with an orientation preserving involution ι. Let $(\mathfrak{s}, I)$ be a Real spin^c structure. Assume that $H^1(X, \mathbb{Z})^{-\iota^*} = 0$ and the formal dimension*

$$d(\mathfrak{s}, I) = \frac{c_1^2(\mathfrak{s}) - \sigma(X)}{8} - b^+(X)^{-\iota^*} \tag{2}$$

of the Real Seiberg–Witten moduli space is 0, where $b^+(X)^{-\iota^*}$ is a dimension of $H^+(X, \mathbb{R})^{-\iota^*}$ and $\sigma(X)$ is the signature of X. Then we have an non-negative integer $|\deg(X, \iota, \mathfrak{s}, I)|$ which satisfies the following properties:

1. *If $X = S^4$ and ι_0 is given by $(x_0, x_1, x_2, x_3, x_4) \mapsto (-x_0, -x_1, x_2, x_3, x_4)$. Then*
$$|\deg(S^4, \iota_0, \mathfrak{s}_0, I_0)| = 1$$
for the unique Real spin^c structure $(\mathfrak{s}_0, I_0)$ on (S^4, ι_0).
2. *If $X = -\mathbb{C}P^2$ and ι_1 be the complex conjugation. Then we also have*
$$|\deg(-\mathbb{C}P^2, \iota_1, \mathfrak{s}_1, I_1)| = 1$$
for the unique Real spin^c structure $(\mathfrak{s}_1, I_1)$ on $(-\mathbb{C}P^2, \iota_1)$ such that the formal dimension is 0.
3. *If $(X_0, \iota_0, \mathfrak{s}_0, I_0)$ and $(X_1, \iota_1, \mathfrak{s}_1, I_1)$ satisfy the above conditions, then we see the glued Real spin^c 4-manifold $(X_0 \# X_1, \iota_0 \# \iota_1, \mathfrak{s}_0 \# \mathfrak{s}_1, I_0 \# I_1)$ satisfies the conditions and*
$$|\deg(X_0 \# X_1, \iota_0 \# \iota_1, \mathfrak{s}_0 \# \mathfrak{s}_1, I_0 \# I_1)| = |\deg(X_0, \iota_0, \mathfrak{s}_0, I_0)||\deg(X_1, \iota_1, \mathfrak{s}_1, I_1)|.$$
4. *Let X be a double branched cover along the 1-roll spun knot of $(-2, 3, 7)$-pretzel knot in S^4 and ι be the covering transformation. Then there is a unique Real spin^c structure $(\mathfrak{s}, I)$ and*
$$|\deg(X, \iota, \mathfrak{s}, I)| = 3.$$

The most difficult and non-trivial property in proposition 3.4 is item 4. We describe the sketch of the proof of proposition 3.4 in section 3.3.

Corollary 3.5. Let P_0 be a standard P^2-knot such that the normal euler number is 2. Let P_n be a P^2-knot given by

$$P_n = P_0 \# \#_n S$$

where S is the 1-roll spun knot of $(-2,3,7)$-pretzel knot. Then, we have unique Real spin^c structure on $\Sigma_2(S^4, P_n)$ with respect to the covering transformation such that the formal dimension is 0. The degree invariant for the Real spin^c structure satisfies

$$|\deg(\Sigma_2(S^4, P_n))| = 3^n.$$

Especially, (S^4, P_n) is not pairwise diffeomorphic to (S^4, P_m) if $n \neq m$.

Proof. From item 3 of proposition 3.4, we see that

$$|\deg(\Sigma_2(S^4, P_n))| = |\deg(\Sigma_2(S^4, P_0))||\deg(\Sigma_2(S^4, S))|^n.$$

Here we omitted Real spin^c structures for each components. Since $\Sigma_2(S^4, P_0) \cong -\mathbb{C}P^2$, we see that $|\deg(\Sigma_2(S^4, P_0))| = 1$ from item 2 in proposition 3.4. From item 4 in proposition 3.4, we have $|\deg(\Sigma_2(S^4, S))| = 3$. □

Sketch of the proof of theorem 1.2. From corollary 3.5, P_n are not smoothly isotopic to each other. One can calculate the fundamental group of the complement of P_n and they are isomorphic to $\mathbb{Z}/2$. From the theorem by Lawson [10, Corollary], they are topologically ambient isotopic. □

3.3 Sketch of the proof of proposition 3.4

In this section we describe the proof of proposition 3.4. The proofs of item 1 and item 2 are basically applications of vanishing theorem of irreducible solutions of Seiberg–Witten equations for 4-manifolds with positive scalar curvature.

Let us explain the sketch of the proof of item 4 in proposition 3.4. Firstly, let us describe the definition of k-twist l-roll spun knots.

Let K be a knot in S^3. Let P be a 4-manifold called Montesinos' twin, which is the exterior of the standard embedding of T^2 in S^4. One can see that P is also given by the two point plumbing of $S^2 \times D^2 \sqcup S^2 \times D^2$. Let us denote S by one of the core, which is an embedded S^2.

By considering the tubular neighborhood of unknotted S^2, we have a decomposition

$$S^4 = D^3 \times S^1 \cup S^2 \times D^2. \tag{3}$$

Let p be a point on K and remove a small ball B_p in S^3 centered at p. We identify $D^3 \cong S^3 \backslash B_p$ and remove a tubular neighborhood of $K \backslash K \cap B_p$. This tubular neighborhood is diffeomorphic to $[-1,1] \times D^2$. Let us paste $\{-1,1\} \times D^2$ to the other component $S^2 \times D^2$ of S^4. Note that the pasting map is naturally defined by the decomposition eq. (3). One can check that $[-1,1] \times D^2 \cup S^2 \times D^2 \cong P$. Then, we have the following decomposition:

$$S^4 = S^3 \setminus \nu(K) \times S^1 \cup P \tag{4}$$

where $\nu(K)$ be a tubular neighborhood of K.

Using the decomposition eq. (4), we have an embedding of S^2 to S^4 given by the image of S in $P \subset S^4$. This is called spun 2-knot of K. If we twist the pasting map

$$\partial S^3 \setminus \nu(K) \to \partial P \tag{5}$$

by $A \in SL(3,\mathbb{Z})$ we obtain a 4-manifold X_A. If we choose a specific family of $SL(3,\mathbb{Z})$, we see that $X_A \cong S^4$ again. But the image of $S \subset P$ is different, see [15]. Set $A_{k,l} \in SL(3,\mathbb{Z})$ is a matrix

$$A_{k,l} = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & l & 1 \end{pmatrix}. \tag{6}$$

Let us denote by $\tau^k\rho^l(K) \in S^4 \cong X_{A_{k,l}}$ the image of $S \subset P \subset X_{A_{k,l}}$ and we call it k-twist l-roll spun knot.

If k is even, the double branched cover of $\tau^k\rho^l(K) \in S^4$ have the following decomposition:

$$\Sigma_2(S^4, \tau^k\rho^l(K)) = \widetilde{S^3 \setminus \nu(K)}^2 \times S^1 \cup_{A_{k/2,l}} \Sigma_2(P,S) \tag{7}$$

where $\widetilde{S^3 \setminus \nu(K)}^2$ is a double cover with respect to the unique nontrivial homomorphism $\pi_1(S^3 \setminus K) \to \mathbb{Z}/2$.

We can prove that the only solution of Real Seiberg–Witten equations on T^3 with an involution $(x,y,z) \mapsto (-x,y,z) \in U(1)^3 = T^3$ is reducible solution up to gauge transformation. We see that the intersection form of each component $\widetilde{S^3 \setminus \nu(K)}^2 \times S^1$ and $\Sigma_2(P,S)$ is 0 and $H^1(\cdot,\mathbb{Z})^{-\iota^*} = 0$. Therefore, we can define relative degree invariant for each components.

Lemma 3.6. *Let $K = P(-2,3,7)$ be the $(-2,3,7)$-pretzel knot. There are two Real spin^c structures $\mathfrak{s}_0, \mathfrak{s}_1$ on $\widetilde{S^3 \setminus \nu(K)}^2 \times S^1$ and*

$$|\deg(\widetilde{S^3 \setminus \nu(K)}^2 \times S^1, \mathfrak{s}_0)| = 1,$$
$$|\deg(\widetilde{S^3 \setminus \nu(K)}^2 \times S^1, \mathfrak{s}_1)| = 3.$$

The relative degree invariant for $\Sigma_2(P,S)$ is 1 for all Real spin^c structure.

Sketch of proof. The calculation $|\deg(\widetilde{S^3 \setminus \nu(K)}^2 \times S^1, \mathfrak{s}_i)|$ follows from the calculation of degree invariant of $\Sigma_2(S^3,K) \times S^1$ since $\mathfrak{s}_i$ can extend to this 4-manifold with involution for $i = 0,1$. If $i = 1$, the extended Real spin^c structure on $\Sigma_2(S^3,K) \times S^1$ is the product of the unique Real spin structure on $\Sigma_2(S^3,K)$ and spin structure on S^1.

One can see that all the solution of Real Seiberg–Witten equations are given by the pullback of the solution on $\Sigma_2(S^3,K)$. This is the Brieskorn sphere $\Sigma(2,3,7)$ and we can explicitly describe the solutions on this space by using the Real lattice Floer homotopy type constructed by Kang–Park–Taniguchi [5].

If we take $\mathfrak{s}_0$, this is given by the mapping torus of the unique Real spin structure on $\Sigma_2(S^3, K)$ by the monodromy -1 on the spinor. We see that the only reducible solution is preserved by this monodromy. $\square$

By looking carefully the induced Real spin^c structures on the boundary of $\widetilde{S^3 \setminus \nu(K)}^2 \times S^1$, we see that if $k/2 + l$ is odd, the Real spin^c structure $\mathfrak{s}_1$ extends to $\Sigma_2(S^4, \tau^k \rho^l(K))$. Thus we have the following result:

Proposition 3.7. *Let $K = P(-2, 3, 7)$. Then if k is even and $k/2 + l$ is odd, then there is unique Real spin^c on* $|\deg(\Sigma_2(S^4, \tau^k \rho^l(K)))| = 3$.

This is the sketch of the proof of item 4 in proposition 3.4.

4 Further developments

After I submit the paper [12], there are many development in Real Seiberg–Witten theory. Let us explain some developments. This is not a complete list of the literatures on Real Seiberg–Witten theory.

Kuhrman [8] construct more examples of exotic P^2-knots which is given by connecting sum of the standard P^2-knot and 1-roll spun of Montesinos knot $K(2, 3, |6s + 1|)$ along which the double branched cover is the Brieskorn sphere $\Sigma(2, 3, |6s + 1|)$. He calculated both the fundamental group of the complements of the 2-knots and the degree invariants.

He also gave a way to calculate the degree invariant for a knots in S^3. As in the sketch of the proof of lemma 3.6, the counting of 3-dimensional Real Seiberg–Witten equations on the double branched cover of a knot in S^3 is important to construct exotic P^2 knots. He proved that the degree invariant is calculated by Lefshetz fixed point formula with respect to the Real involution on $Pin(2)$-monopole Floer homology in [9].

In [1], Baraglia comprehensively studied Real Seiberg–Witten theory. He gave more general definition of the degree invariant and proved fundamental properties of them. As an application, he gave exotic embeddings of non-orientable surfaces in admissible 4-manifolds, which is a large class of 4-manifolds.

In this article, we only focused on invariants for 4-manifolds with involutions. In these years, invariants for 3-manifolds with involutions have also developed. For 3-manifolds with involutions, Li defined the Real monopole Floer homology in [11]. There is a refinement of this Floer homology to $\mathbb{Z}/4$-equivariant spectrum valued invariant, called The Real Seiberg–Witten Floer homotopy type. Xiao [16] announced that $\mathbb{Z}/2$-equivariant homology is isomorphic to Li's Real monopole Floer homology. The Real Seiberg–Witten Floer homotopy types for the double branched covers along knots in S^3 are important because the absolute values of the Euler numbers of them are the degree invariant of the knot, which is used in lemma 3.6. In Kang–Park–Taniguchi [5, 6] compute Real Seiberg–Witten Floer homotopy type using lattice homotopy type. As an application, they proved that non-trivial cables of the figure eight knot are not slice.

Though some people find ways to compute the Real Seiberg–Witten Floer homotopy type, it is still hard to compute it for random knots. One way to give more examples of calculation of this invariant is to consider the satellite operation for knots. In [13], we proved a formula to compute Real Seiberg–Witten Floer homotopy type of $P(K)$ from that of K and $P(\text{unknot})$, where $P \subset S^1 \times D^2$ is an odd satellite pattern.

References

[1] Baraglia, David, Exotic embedded surfaces and involutions from Real Seiberg–Witten theory, International Journal of Mathematics, Vol. 37, No. 04, p. 2650028, 2026.

[2] Finashin, S. M. and Kreck, M. and Viro, O. Ya., Nondiffeomorphic but homeomorphic knottings of surfaces in the 4-sphere, Topology and geometry—Rohlin Seminar, Lecture Notes in Math., Vol. 1346, pp.157–198, 1988.

[3] Michael H. Freedman and Frank Quinn, Topology of 4-manifolds, *Princeton Math. Series*, Vol. 39, Princeton University Press,1990.

[4] Hughes, Mark and Kim, Seungwon and Miller, Maggie, Branched covers of twist-roll spun knots and turned twisted tori, arXiv preprint arXiv:2402.11706,2024.

[5] Kang, Sungkyung and Park, JungHwan and Taniguchi, Masaki, Cables of the figure-eight knot via real Frøyshov invariants, arXiv preprint arXiv:2405.09295, 2024.

[6] Kang, Sungkyung and Park, JungHwan and Taniguchi, Masaki, Smooth concordance of cables of the figure-eight knot, arXiv preprint arXiv:2505.03720, 2025.

[7] Hokuto Konno, Jin Miyazawa, and Masaki Taniguchi. Involutions, links, and Floer cohomologies. *Journal of Topology*, Vol. 17, No. 2, p. e12340, 2024.

[8] Kuhrman, Judson More Exotic $\mathbb{R}P^2$-knots and Homotopy Spheres, arXiv preprint arXiv:2507.03798,2025.

[9] Kuhrman, Judson Miyazawa's Invariant, Lefschetz Numbers, and Seifert Solids, arXiv preprint arXiv:2605.14996,2026.

[10] Lawson, Terry Normal bundles for an embedded RP2 in a positive definite 4-manifold, Journal of Differential Geometry, Vol. 22, No. 2, pp.215–231, 1985.

[11] Li, Jiakai, Monopole Floer homology and real structures, arXiv preprint arXiv:2211.10768, 2022.

[12] Jin Miyazawa. A gauge theoretic invariant of embedded surfaces in 4-manifolds and exotic P^2-knots. *arXiv preprint arXiv–2312.02041*, 2023.

[13] Jin Miyazawa, JungHwan Park, and Masaki Taniguchi. A satellite formula for real Seiberg-Witten Floer homotopy types. *arXiv preprint arXiv:2504.03270*, 2025.

[14] Nakamura, Nobuhiro, $\mathrm{Pin}^-(2)$-monopole invariants, *J. Differential Geom.*,Vol. 101, No. 3, pp.507–549,2015.

[15] Plotnick, Steven P, Fibered knots in S4—twisting, spinning, rolling, surgery, and branching, *Contemp. Math*, Vol. 5, pp.437–459,1984.

[16] Xiao, Yonghan, The equivalence between two real Seiberg-Witten homologies, arXiv preprint arXiv:2510.03709, 2025.

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502
JAPAN
E-mail address: miyazawa.jin.5a@kyoto-u.ac.jp

京都大学・数理解析研究所　宮澤仁