Cosmetic surgeries and Casson–Walker–Lescop invariants

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1 Introduction

This note is based on the talk given by the second author at the conference *Intelligence of Low-dimensional Topology 2026*, held at RIMS in May 2026. The talk is based on results introduced in [9, 10].

Let K be a knot in a closed oriented 3-manifold M. The operation of forming a new 3-manifold $(M \setminus \mathrm{int}N(K)) \cup_f (D^2 \times S^1)$ from M is called *Dehn surgery* on K, where $N(K)$ denotes a regular neighborhood of K and f is a gluing map, that is, an orientation-reversing homeomorphism on boundary tori. The homeomorphism type of the resulting manifold is determined by the isotopy class of a loop on $\partial N(K)$ that is identified with the meridian of the attached solid torus by f. The isotopy class of a loop is called the *surgery slope*. When K is null-homologous in M, particularly in the case where $M = S^3$, surgery slopes are well known to be parametrized by $\mathbb{Q} \cup \{1/0\}$ using the standard meridian-longitude system, where $1/0$ corresponds to the meridional slope. Dehn surgery along the slope γ corresponding to $p/q \in \mathbb{Q} \cup \{1/0\}$ is called *p/q-surgery*. The resulting manifold is denoted by $M_K(\gamma)$ or $M_K(p/q)$. For p/q-surgery, we assume that $q > 0$ unless otherwise noted. Dehn surgery on a link is defined by performing Dehn surgery on each component of the link. For a link $L = K_1 \cup \cdots \cup K_n$ in S^3, *$(r_1, \ldots, r_n)$-surgery* means Dehn surgery on L along the slopes corresponding to the rational numbers $r_1, \ldots, r_n$, respectively.

A pair of Dehn surgeries on a knot is called *purely* (resp. *chirally*) *cosmetic* if the resulting 3-manifolds are orientation-preservingly (resp. orientation-reversingly) homeomorphic. It is conjectured that there are no “non-trivial” purely cosmetic surgeries.

Conjecture 1 (Cosmetic Surgery Conjecture, [1, Problem 1.12(a)], [5, Conjecture 6.1])**.** *Let K be a knot in a closed oriented 3-manifold M. Then two surgeries on K along inequivalent slopes are never purely cosmetic.*

In Conjecture 1, it is usually assumed that $M \setminus \mathrm{int}N(K)$ is irreducible and not homeomorphic to a solid torus.

On the other hand, there are examples of chirally cosmetic surgeries along inequivalent slopes. See [17, 2, 8] for such examples and [12, 11] for related review or questions.

A basic strategy to study cosmetic surgeries is to compare invariants of the resulting 3-manifolds. An important and reasonable invariant is (the order of) the first homology group as follows:

Proposition 1. *Let K be a null-homologous knot in a 3-manifold M. Then we have*

$$H_1(M_K(p/q);\mathbb{Z}) \cong H_1(M;\mathbb{Z}) \oplus \mathbb{Z}/|p|\mathbb{Z}.$$

By Proposition 1, if a null-homologous knot admits purely cosmetic surgeries, then the surgery slopes should be $\pm p/q$ and $\pm p/q'$. Here p, q, q' are positive integers and p is coprime to q and q'.

From the viewpoint of finite type invariants for 3-manifolds (for example, see [19]), the order of H_1 is regarded as the finite type invariant of degree zero. So, as a next step, it is natural to consider the finite type invariant of degree one, which essentially coincides with the Casson–Walker–Lescop invariant [14], which was originally introduced by Lescop [15] as a generalization of the Casson–Walker invariant for any closed oriented 3-manifolds. In this note, we review results on cosmetic surgeries obtained by using the Casson–Walker–Lescop invariant. We omit proofs which can be found in [10].

Remark 1. Here we make a remark about the coefficients of the Conway polynomial, specifically the difference between the usual ones and those defined à la Lescop [15]. The (usual) skein relation of the Conway polynomial $\nabla_L(z)$ of a link L in S^3 is given by

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z).$$

The k-th coefficient of $\nabla_L(z)$ is denoted by $a_k(L)$. In Lescop's book [15], the notation for the Conway polynomial and its coefficients is slightly different from the above. In this note, we denote by $\hat{\nabla}_L(z)$ the Conway polynomial used in [15], which satisfies the following skein relation:

$$\hat{\nabla}_{L_+}(z) - \hat{\nabla}_{L_-}(z) = -z\hat{\nabla}_{L_0}(z).$$

If L is a μ-component link, then $\hat{\nabla}_L(z)$ has the form

$$\hat{\nabla}_L(z) = z^{\mu-1}\left(\hat{a}_0(L) + \hat{a}_1(L)z^2 + \cdots\right).$$

Note that $\nabla_{L^*}(z) = \hat{\nabla}_L(z)$, where L^* denotes the mirror image of L. Also note that $\nabla_{L^*}(z) = (-1)^{\mu-1}\nabla_L(z)$. Therefore we have

$$\hat{a}_k(L) = (-1)^{\mu-1}a_{\mu+2k-1}(L).$$

In particular, $\hat{a}_k(K) = a_{2k}(K)$ holds for a knot K.

2 Results

We review results on cosmetic surgeries on null-homologous knots obtained via the Casson–Walker–Lescop invariant.

2.1 Cosmetic surgeries on a null-homologous knot and Conway polynomial

To our knowledge, the first result on purely cosmetic surgeries obtained by using the Casson–Walker invariant is due to Boyer and Lines [3]. This result was obtained as a consequence of extending the Casson invariant, defined for integral homology spheres, to homology lens spaces. Their results were obtained independently of Walker's extension [20] of the Casson invariant to rational homology spheres.

Proposition 2 ([3, (5.1) Proposition (iii)])**.** *Let K be a knot in an integral homology sphere. If $\Delta''_K(1) \neq 0$, then K admits no purely cosmetic surgeries.*

Here $\Delta_K(t)$ denotes the Alexander polynomial of K, normalized so that $\Delta_K(t^{-1}) = \Delta_K(t)$ and $\Delta_K(1) = 1$. It is known that for a knot K in S^3, $\Delta''_K(1)$ is equal to twice the second coefficient $a_2(K)$ of the Conway polynomial of K.

We extend Proposition 2 to a null-homologous knot in a 3-manifold with the first Betti number one or two. A link in S^3 is called *algebraically split* if the linking number of any pair of its components is zero.

Theorem 3 ([10, Theorem 1.5])**.** *Let K and K_1 be two knots in S^3 such that the link $K \cup K_1$ is algebraically split. Let M be the 3-manifold obtained by 0-surgery on K_1, i.e., surgery along the preferred longitudinal slope. Suppose that $\hat{a}_1(K \cup K_1) \neq 0$. Then, regarded as a knot in M, K admits no purely cosmetic surgeries. In particular, K is determined by the complement in M.*

Theorem 4 ([10, Theorem 1.6])**.** *Let K, K_1, and K_2 be knots in S^3 such that the link $K \cup K_1 \cup K_2$ is algebraically split. Let M_{p_1/q_1} be the 3-manifold obtained by $(p_1/q_1, 0)$-surgery on $K_1 \cup K_2$. Suppose that $p_1\hat{a}_1(K \cup K_2) + q_1\hat{a}_1(K \cup K_1 \cup K_2) \neq 0$. Then, regarded as a knot in M_{p_1/q_1}, K admits no purely cosmetic surgeries. In particular, K is determined by the complement in M_{p_1/q_1}.*

If the first Betti number of a 3-manifold M is odd, then the Casson–Walker–Lescop invariant of M, $\lambda(M)$, coincides with that of the orientation-reversed manifold $-M$, $\lambda(-M)$. Thus, Theorems 3, 4 also hold for chirally cosmetic surgeries.

Theorem 5 ([10, Theorem 1.7])**.** *Let K, K_1, and K_2 be knots in S^3 such that the link $K \cup K_1 \cup K_2$ is algebraically split. Let M_{00} be the 3-manifold obtained by $(0, 0)$-surgery on $K_1 \cup K_2$. Suppose that $\hat{a}_1(K \cup K_1 \cup K_2) \neq 0$. Then, regarded as a knot in M_{00}, K admits no purely cosmetic surgeries. In particular, K is determined by the complement in M_{00}.*

We note that the result above is, in a sense, the best possible, because $\hat{a}_1(L') = 0$ holds for $L' \subset L$ with $\sharp L' \geq 4$ when L is algebraically split [6, Lemma 1.5].

2.2 Integral purely cosmetic surgeries on a null-homologous knot

Next we focus on integral surgeries. Here, a Dehn surgery is said to be *integral* if the geometric intersection number of the meridian and a representative of the surgery slope is one. For a null-homologous knot, integral surgery is equivalent to $n/1$-surgery for $n \in \mathbb{Z}$. For a knot in an integral homology sphere, the following result holds as a corollary of a theorem by Boyer and Lines [3, Theorem 2.8].

Proposition 6. *A null-homologous knot in an integral homology sphere admits at most two pairs of integral purely cosmetic surgeries.*

The following result is an extension of Proposition 6 and [9, Theorem 2].

Theorem 7 ([10, Theorem 1.1])**.** *A null-homologous knot in a rational homology sphere admits at most two pairs of integral purely cosmetic surgeries.*

Remark 2. The finiteness of purely cosmetic surgeries on a knot in a closed oriented 3-manifold has been verified in [7].

2.3 Knot complement problem

The cosmetic surgery problem is closely related to the knot complement[1] problem (see Subsection 3.2). In fact, the following is conjectured:

Conjecture 2 (Oriented knot complement conjecture, [1, Problem 1.12 (b)], [5, Conjecture 6.2])**.** *If K_1 and K_2 are knots in a closed oriented 3-manifold N whose complements are homeomorphic via an orientation-preserving homeomorphism, then there exists a homeomorphism of N taking K_1 to K_2.*

As an application of our method for computing the Casson–Walker–Lescop invariant, we give some supporting evidence for the conjecture. In particular, the following corollaries can be regarded as a partial answer to this conjecture. For example, for a null-homologous knot in a 3-manifold obtained by Dehn surgery on a knot in S^3, we obtain the following.

Corollary 8 ([10, Corollary 1.2])**.** *Let M be a 3-manifold obtained by Dehn surgery on a knot in S^3. Then, for any null-homologous knot K in M, there are at most two knots in M that are inequivalent to K but whose exteriors are orientation-preservingly homeomorphic to the exterior of K. In particular, for any null-homologous knot in $S^2 \times S^1$ or a lens space, the same holds.*

Corollary 9 ([10, Corollary 1.3])**.** *Let M be a rational homology sphere obtained by Dehn surgery on an algebraically split link in S^3 with positive surgery slopes. Then, for any null-homologous knot K in M, there are at most two knots in M that are inequivalent to K but whose exteriors are orientation-preservingly homeomorphic to the exterior of K.*

Moreover, the next result gives a partial extension and an alternative proof of part of the result obtained in [4] using Heegaard Floer homology, which states that any null-homologous knot in an L-space is determined by the complement.

Corollary 10 ([10, Corollary 1.4])**.** *Let M be a connected sum of lens spaces $L(p_i, q_i)$ with coprime positive integers p_i and q_i, or a connected sum of $S^2 \times S^1$ and lens spaces $L(p_i, q_i)$ with coprime integers p_i and q_i. Then, for any null-homologous knot K in M, there are at most two knots in M that are inequivalent to K but whose exteriors are orientation-preservingly homeomorphic to the exterior of K.*

Remark 3. For a given knot, the finiteness of inequivalent knots whose exteriors are homeomorphic to that of the given knot is shown in [16].

[1]Here, the term "complement" is used to mean the "exterior" of a knot; that is, it refers to the compact 3-manifold obtained from the ambient manifold by removing an open tubular neighborhood of the knot.

3 Remarks

Here we review some facts that are well known to experts but lack readily accessible proofs in the literature.

3.1 Surgery description by algebraically split link

It is well-known that any integral homology sphere admits a surgery description on an algebraically split link along ± 1 slopes. On the other hand, there is a rational homology sphere never represented by Dehn surgery on an algebraically split link with rational coefficients. We give a proof of this folklore result below.

Proposition 11. *There exists a rational homology 3-sphere that cannot be represented by Dehn surgery on algebraically split links.*

Proof. Let M be a 3-manifold obtained by Dehn surgery on the torus link of type $(2,4)$ along slopes $(0,0)$, see Figure 1. The linking matrix of M for this surgery description is $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$. Let $\varphi\colon H_1(M) \times H_1(M) \to \mathbb{Q}/\mathbb{Z}$ be the torsion linking pairing on M, which is given by $\varphi(x,y) = xA^{-1}y \pmod 1$. Then for any $x \in H_1(M) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, we have $\varphi(x,x) = 0$. Suppose for a contradiction that M is obtained by Dehn surgery on an algebraically split link. Then the linking matrix of M for this surgery description is of the form

$$A' = \mathrm{diag}\,(\pm 2/q_1, \pm 2/q_2, \pm 1/q_3, \dots, \pm 1/q_n)$$

with positive odd integers q_1, q_2, and positive integers $q_3, \dots, q_n$. Then a presentation matrix of the torsion linking form φ' is $\mathrm{diag}\,(1/2, 1/2)$. Then for $\mu \in H_1(M)$ whose representative is the meridian of a component of the algebraically split link, we have $\varphi'(\mu,\mu) = 1/2 \pmod 1$. This contradicts $\varphi = \varphi'$. Therefore M is a rational homology sphere that cannot be represented by Dehn surgery on an algebraically split link. $\square$

$M =$

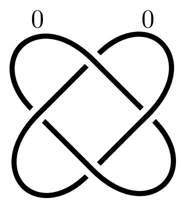

Figure 1: A rational homology sphere M in the proof of Proposition 11

Remark 4. The matrix $A^{-1} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ appears in [13] as the matrix E_0^1. Also see [21].

Remark 5. It is shown by Ohtsuki [18, Corollary 2.5] that for any closed oriented 3-manifold, taking the connected sum of lens spaces suitably, the resulting manifold can be represented by an algebraically split link along integral slopes.

3.2 Equivalence of two conjectures

Conjectures 1 and 2 are equivalent. While this fact is well known to experts, we verify it here. For a knot K in a 3-manifold M, we denote by M_K the exterior of K in M, that is, $M_K = M \setminus \mathrm{int}N(K)$, and by μ_K the meridian of K.

Assume that Conjecture 1 is true. Let K_1, K_2 be knots in a closed oriented 3-manifold N. Suppose that there exists an orientation-preserving homeomorphism $\psi\colon N_{K_1} \to N_{K_2}$. If $\psi(\mu_{K_1}) = \mu_{K_2}$, then ψ can be extended to a self-homeomorphism of N sending K_1 to K_2. If $\psi(\mu_{K_1}) \neq \mu_{K_2}$, then $\gamma := \psi(\mu_{K_1})$ is a slope on N_{K_2} such that $N_{K_2}(\gamma) \cong N_{K_1}(\mu_{K_1}) = N$. Since two Dehn surgeries on K_2 along slopes γ and μ_{K_2} yield N, by the assumption that Conjecture 1 is true, there exists a homeomorphism $f\colon N_{K_2} \to N_{K_2}$ such that $f(\gamma) = \mu_{K_2}$. Then $f \circ \psi$ can be extended to a self-homeomorphism of N sending K_1 to K_2.

Next assume that Conjecture 2 is true. Let K be a knot in a closed oriented 3-manifold M. Suppose that $M_K(\gamma_1)$ is orientation-preservingly homeomorphic to $M_K(\gamma_2)$. We identify $M_K(\gamma_1)$ and $M_K(\gamma_2)$ with a 3-manifold N. For $i = 1, 2$, let K_i be the dual knot of Dehn surgery along the slope γ_i in N, that is, K_i is the core of the attached solid torus of Dehn surgery along γ_i. Then N_{K_1} and N_{K_2} are orientation-preservingly homeomorphic. By the assumption that Conjecture 2 is true, there exists a homeomorphism $g\colon N \to N$ such that $g(K_1) = K_2$. Then $g(\mu_{K_1}) = \mu_{K_2}$, that is, $g(\gamma_1) = g(\gamma_2)$ since $\mu_{K_i} = \gamma_i$ in M_K.

References

[1] R. Inanc Baykur, Robion C. Kirby, and Daniel Ruberman, K3*: A new problem list in low-dimensional topology*, Mathematical surveys and monographs, vol. 295, American Mathematical Society, 2026.

[2] Steven A. Bleiler, Craig D. Hodgson, and Jeffrey R. Weeks, *Cosmetic surgery on knots*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), Geom. Topol. Monogr., vol. 2, Geom. Topol. Publ., Coventry, 1999, pp. 23–34.

[3] Steven Boyer and Daniel Lines, *Surgery formulae for Casson's invariant and extensions to homology lens spaces*, J. Reine Angew. Math. **405** (1990), 181–220.

[4] Fyodor Gainullin, *Heegaard Floer homology and knots determined by their complements*, Algebr. Geom. Topol. **18** (2018), no. 1, 69–109.

[5] Cameron McA. Gordon, *Dehn surgery on knots*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 631–642.

[6] Jim Hoste, *A formula for Casson's invariant*, Trans. Amer. Math. Soc. **297** (1986), no. 2, 547–562.

[7] Kazuhiro Ichihara, *Finiteness of purely cosmetic fillings*, Canadian Mathematical Bulletin (2026), 1–8.

[8] Kazuhiro Ichihara and In Dae Jong, *Cosmetic banding on knots and links*, Osaka J. Math. **55** (2018), no. 4, 731–745, With an appendix by Hidetoshi Masai.

[9] ______, *Cosmetic surgeries on knots in homology spheres and the Casson–Walker invariant*, Geom. Dedicata **219** (2025), no. 6, Paper No. 87.

[10] Kazuhiro Ichihara, In Dae Jong, and Yasuyoshi Tsutsumi, *Purely cosmetic surgeries and Casson–Walker–Lescop invariants*, arXiv:2603.11720 [math.GT] (2026).

[11] Tetsuya Ito, *Cosmetic surgery on knots*, Intelligence of Low-dimensional Topology 2022, in Tomotada Ohtsuki, editor, Problems in low-dimensional topology, RIMS Kôkyûroku, no. 2227, Res. Inst. Math. Sci. (RIMS), Kyoto, 2022, pp. 109–111.

[12] ______, *A short review on constraints of cosmetic surgery of knots in S^3*, Intelligence of Low-dimensional Topology 2022, RIMS Kôkyûroku, no. 2227, Res. Inst. Math. Sci. (RIMS), Kyoto, 2022, pp. 88–96.

[13] Akio Kawauchi and Sadayoshi Kojima, *Algebraic classification of linking pairings on 3-manifolds*, Math. Ann. **253** (1980), no. 1, 29–42.

[14] Thang Tu Quoc Le, Hitoshi Murakami, Jun Murakami, and Tomotada Ohtsuki, *A three-manifold invariant via the Kontsevich integral*, Osaka J. Math. **36** (1999), no. 2, 365–395.

[15] Christine Lescop, *Global surgery formula for the Casson-Walker invariant*, Annals of Mathematics Studies, vol. 140, Princeton University Press, Princeton, NJ, 1996.

[16] Bruno Martelli, *Links, two-handles, and four-manifolds*, Int. Math. Res. Not. (2005), no. 58, 3595–3623.

[17] Yves Mathieu, *Closed 3-manifolds unchanged by Dehn surgery*, J. Knot Theory Ramifications **1** (1992), no. 3, 279–296.

[18] Tomotada Ohtsuki, *A polynomial invariant of rational homology 3-spheres*, Invent. Math. **123** (1996), no. 2, 241–257.

[19] ______, *Quantum invariants*, Series on Knots and Everything, vol. 29, World Scientific Publishing Co., Inc., River Edge, NJ, 2002, A study of knots, 3-manifolds, and their sets.

[20] Kevin Walker, *An extension of Casson's invariant*, Annals of Mathematics Studies, vol. 126, Princeton University Press, Princeton, NJ, 1992.

[21] C. T. C. Wall, *Quadratic forms on finite groups, and related topics*, Topology **2** (1963), 281–298.

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