Constructing solutions of Polygon and Simplex equatio

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Abstract

This note is based on the author's talk "Constructing solutions of Polygon and Simplex equation" given in the conference "Intelligence of Low-dimensional Topology" held at the RIMS, Kyoto University, during May 20-May 22, 20126. The details appear in [18].

1 Introduction

The Yang–Baxter equation [20, 2],

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

originally introduced in the study of integrable models in statistical mechanics, is now fundamental to mathematical physics, topology, and many other areas of mathematics. A.B. Zamolodchikov [21, 22] considered a three-dimensional analog of the Yang–Baxter equation, known as the tetrahedron equation. Later, a higher-dimensional analog of these equations, called the simplex equations, was defined in [3], which includes the Yang–Baxter equation as the 2-simplex equation, Zamolodchikov's tetrahedron equation as the 3-simplex equation, and so on. See, for example, [9, 17, 4, 7, 16, 1] for more details.

Another important equation with a flavor similar to the Yang–Baxter equation is the pentagon (5-gon) equation,

$$T_{12}T_{13}T_{23} = T_{23}T_{12},$$

which again appears as a fundamental equation in mathematical physics and topology. Subsequently, A. Dimakis and F. Müller-Hoissen [7] defined the class of polygon equations. Just as simplex equations generalize the Yang–Baxter and tetrahedron equations, polygon equations generalize the pentagon equation. In [19], Müller-Hoissen explored (set-theoretic) polygon equations, investigating the relationship between neighboring polygon equations and how a solution of the k-gon equation lifts or descends to a solution of the $(k \pm 1)$-gon equation. Also, in topology, polygon equations play a key role in constructing state sum invariants of manifolds from their triangulations [15, 11, 14]. In particular, they guarantee invariance under a certain Pachner move.

As for the relationship between simplex equations and polygon equations, there have been several attempts to construct solutions of simplex equations from solutions of polygon equations. In [12], R.M. Kashaev and S.M. Sergeev showed that whenever a solution of the pentagon equation and a solution of its dual satisfy a certain compatibility condition, called the "ten-term relation", they give rise to solutions of the tetrahedron (3-simplex) equation and the 4-simplex equation. See also [13]. Later in [7], Dimakis and Müller-Hoissen associated simplex equations with higher Bruhat orders and polygon equations with higher Tamari orders. Then, by considering the "three color decomposition", which is a decomposition of the higher Bruhat order into the higher Tamari order, its dual, and what they called the mixed order, they constructed a solution of the $(n-1)$-simplex equation from a compatible pair of solutions of the n-gon equation and its dual. This generalizes Kashaev and Sergeev's construction of a solution of the 4-simplex equation from solutions of the pentagon equation and its dual. However, as pointed out in [7, Section 6], they were unable to fully explore the relationship between solutions of the 3-simplex equation and those of the pentagon equation. Also, the compatibility condition between solutions of the n-gon equation and its dual was not made explicit, by which we mean that although one can write the compatibility condition for each n individually, it is still unclear how to express it uniformly. A similar study, particularly for the class of odd-gon equations, also appears in Dimakis and I.G. Korepanov's work [5].

In this paper, we study polygon and simplex equations. We begin by exploring polygon equations within a general framework. We show that if two solutions of (dual) polygon equations are "commutative", their (partial) composition and tensor product yield a solution of a higher-order polygon equation. In the second part, we explore the relationship between simplex equations and polygon equations. We explicitly define the compatibility condition above for solutions of the n-gon and dual n-gon equations and observe how these two solutions give rise to a solution of the $(n-2)$-simplex equation and, for completeness, to a solution of the $(n-1)$-simplex equation. We provide direct proofs of these connections for both even- and odd-gon cases.

Throughout the paper, we fix a field $\Bbbk$, and vector spaces, linear maps, tensor products, duals, etc., are considered to be over $\Bbbk$ unless otherwise stated. Note that many of the arguments in this paper can be generalized to polygon and simplex equations in a symmetric monoidal category.

2 Polygon equations

Originally, polygon equations were introduced in [6, 7] as an algebraic realization of higher Stasheff–Tamari order. In this section, following [10], we define the n-gon equation in terms of an algebraic realization of Pachner moves on the standard simplex Δ^{n-1}.

2.1 Definition of polygon equation

Let $n \geq 3$ be an integer, and $\Delta^{n-1} = [0, 1, \ldots, n-1]$ be the standard $(n-1)$-simplex. Given a vector space V and a map

$$T\colon V^{\otimes \lfloor \frac{n-1}{2} \rfloor} \to V^{\otimes \lfloor \frac{n}{2} \rfloor},$$

we define the **n-gon equation** as follows:

- To each $(n-3)$-simplex $s \in \Delta^{n-1}$, we associate a copy of the vector space X, denoted by $V(s)$.
- To each $(n-2)$-simplex $p \in \Delta^{n-1}$, we associate a copy of the map T, denoted by $T(p)$:

$$T(p)\colon V(\partial_1 p) \times V(\partial_3 p) \times \cdots \times V(\partial_{2\lfloor \frac{n-1}{2} \rfloor - 1} p) \to V(\partial_0 p) \times V(\partial_2 p) \times \cdots \times V(\partial_{2\lfloor \frac{n}{2} \rfloor - 2} p).$$

As indicated, each input set and target set of $T(p)$ is labeled by the $(n-3)$-simplices in the boundary of the p.

Next, we consider the following splitting $\partial\Delta^{n-1} = D^+ \cup D^-$ of the boundary:

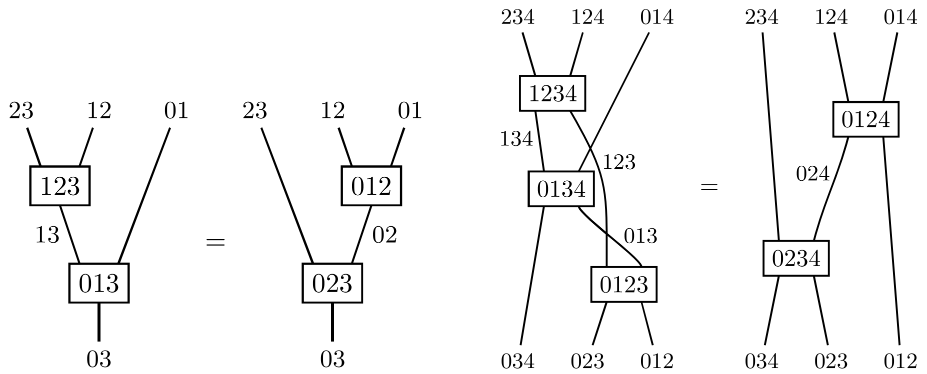
$$D^+ := \{\, \partial_i \Delta^{n-1} \,|\, i \text{ is even integer such that } 0 \leq i \leq n-1 \,\} \tag{1}$$
$$D^- := \{\, \partial_j \Delta^{n-1} \,|\, j \text{ is odd integer such that } 0 \leq j \leq n-1 \,\} \tag{2}$$

We compose the maps $T(\partial_i \Delta^{n-1})$ for $\partial_i \Delta^{n-1} \in D^+$ along the matching labeled outputs and inputs, and arrange the remaining free outputs and free inputs, each in reverse lexicographic order. We do the same for $T(\partial_j \Delta^{n-1})$ for $\partial_j \Delta^{n-1} \in D^-$. The resulting equality between these two compositions is called the **n-gon equation**. Figure 1 is a diagrammatic picture of polygon equations.

Similarly, for a map $S\colon X^{\times\lfloor\frac{n}{2}\rfloor} \to X^{\times\lfloor\frac{n-1}{2}\rfloor}$, if we associate

$$S(p) = S\colon X(\partial_0 p) \times X(\partial_1 p) \times \cdots \times X(\partial_{2\lfloor\frac{n}{2}\rfloor-2} p) \to X(\partial_1 p) \times X(\partial_3 p) \times \cdots \times X(\partial_{2\lfloor\frac{n-1}{2}\rfloor-1} p)$$

to each $(n-2)$-simplex $p \subset \Delta^{n-1}$, the same procedure yields another equation called the **dual n-gon equation**.



4-gon equation associated to Δ^3 5-gon (pentagon) equation associated to Δ^4

Figure 1: Diagrammatic pictures of 4- and 5-gon equations. The box labeled with the simplex p represent the map $T(p)$ and each edge labeled with simplex s represent the vector space $V(s)$. Read from bottom to top. Reading from top to bottom yields the dual 4- and 5-gon equations.

3 Diagrammatic definition of Polygon equation

3.1 Representing linear maps

Let V be a vector space, and $F\colon V^{\otimes k} \to V^{\otimes l}$ be a linear map. For integer $n \geq k$ and a pair (a, b) of multi-indices (i.e., row vectors)

$$a = [\,a_1, a_2, \ldots, a_k\,] \quad \text{and} \quad b = [\,b_1, b_2, \ldots, b_l\,]$$

with $0 < a_1 < a_2 < \cdots < a_k \leq n$ and $0 < b_1 < b_2 < \cdots < b_l \leq n - k + l$, we define a linear map

$$F_{(a,b)}\colon V^{\otimes n} \to V^{\otimes(n-k+l)}$$

by

$$\tau_{1,b_1}^{-1} \tau_{2,b_2}^{-1} \cdots \tau_{l,b_l}^{-1} (F \otimes \mathrm{id}_V^{\otimes(n-k)}) \tau_{k,a_k} \cdots \tau_{2,a_2} \tau_{1,a_1},$$

where $\tau_{i,j}$ is a map that sweeps the j-th factor to the i-th position if $i \neq j$, i.e.,

$$\tau_{i,j}(\cdots \otimes x_i \otimes \cdots \otimes x_{j-1} \otimes x_j \otimes x_{j+1} \otimes \cdots) := \cdots \otimes x_j \otimes x_i \otimes \cdots \otimes x_{j-1} \otimes x_{j+1} \otimes \cdots,$$

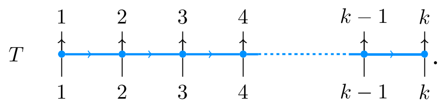
and identity if $i = j$. We mainly use the following notation:

- For a linear map $F\colon V^{\otimes k} \to V^{\otimes k}$ ($l = k$ case) and a multi-index a, F_a denotes the map $F_{(a,a)}$. This is just the standard notation specifying which factors of the tensor product the map F acts on.

- For a linear map $F\colon V^{\otimes k} \to V^{\otimes k+1}$ ($l = k+1$ case), if multi-indices a and b are of the forms
$$a = [\,a_1, a_2, \ldots, a_k\,] \quad \text{and} \quad b = [\,a_1, a_2, \ldots, a_k, a_k + 1\,]$$
then we write F_a for $F_{(a,b)}$.

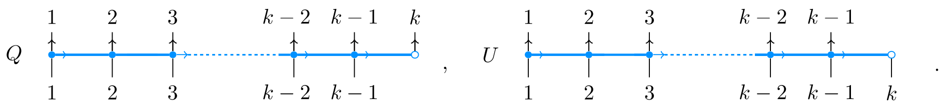
- For a linear map $F\colon V^{\otimes k} \to V^{\otimes k-1}$ ($l = k-1$ case), if multi-indices a and b are of the forms
$$a = [\,a_1, a_2, \ldots, a_{k-1}, a_{k-1} + 1\,] \quad \text{and} \quad b = [\,a_1, a_2, \ldots, a_{k-1}\,]$$
then we write F_b for $F_{(a,b)}$.

Next, we introduce graphical representations of linear maps. Given a linear map $T\colon V^{\otimes k} \to V^{\otimes k}$, we will represent T by an oriented line with k numbers of dots:

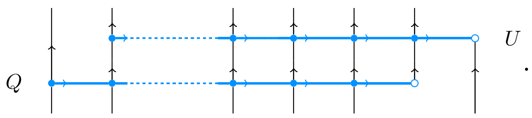


Starting from the initial dot with respect to the orientation of the line, the i-th perpendicular line represents the i-th input/output of the linear map T, where the incoming perpendicular line represents the input and the outgoing perpendicular line represents the output.

Similarly, given linear maps $Q\colon V^{\otimes k-1} \to V^{\otimes k}$ and $U\colon V^{\otimes k} \to V^{\otimes k-1}$, we will draw Q and U as



The composition of linear maps is represented by connecting the output leg to the input leg. For example, the linear map $(\mathrm{id}_V \otimes U) \circ (Q \otimes \mathrm{id}_V)$ is represented as



The left and right partial compositions $\circ_l, \circ_r$ of linear maps $F\colon V^{\otimes i} \to V^{\otimes j}$ and $G\colon V^{\otimes k} \to V^{\otimes l}$ are maps of the forms

$$F \circ_l G := (F \otimes \mathrm{id}_V^{\otimes l-1}) \circ (\mathrm{id}_V^{\otimes i-1} \otimes G)\colon V^{\otimes i+k-1} \to V^{\otimes j+l-1},$$
$$F \circ_r G := (\mathrm{id}_V^{\otimes j-1} \otimes G) \circ (F \otimes \mathrm{id}_V^{\otimes k-1})\colon V^{\otimes i+k-1} \to V^{\otimes j+l-1},$$

respectively.

Definition 3.1 (Odd-gon equation)**.** For an integer $k \geq 1$, the $(2k+1)$-gon equation for a linear map $T\colon V^{\otimes k} \to V^{\otimes k}$ is an equation in $\mathrm{End}(V^{\otimes \frac{k(k+1)}{2}})$ of the form:

$$T_{a_1} T_{a_2} \cdots T_{a_{k+1}} = T_{b_k} T_{b_{k-1}} \cdots T_{b_1},$$

where $a_i = [\,a_{i,1}, a_{i,2}, \ldots, a_{i,k}\,]$ and $b_j = [\,b_{j,1}, b_{j,2}, \ldots, b_{j,k}\,]$ are multi-indices indicating which factors of $V^{\otimes \frac{k(k+1)}{2}}$ the operator T acts on and are given by the following matrices A_{2k+1} and B_{2k+1}, respectively:

$$A_{2k+1} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k+1} \end{bmatrix} = \left[\begin{array}{c|ccc} 1 & 2 & \cdots & k \\ \hline 1 & & & \\ 2 & & & \\ \vdots & & A_{2k-1} + [\,k\,]_{k,k-1} & \\ k & & & \end{array}\right], \qquad B_{2k+1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} = \left[\begin{array}{c|ccc} 1 & 2 & \cdots & k \\ \hline 2 & & & \\ \vdots & & B_{2k-1} + [\,k\,]_{k-1,k-1} & \\ k & & & \end{array}\right],$$

with $A_3 = [1,1]^T$ and $B_3 = [1]$, where $[r]_{i,j}$ denotes the $i \times j$ matrix whose entries are all r.

Definition 3.2 (Dual odd-gon equation)**.** For an integer $k \geq 1$, the dual $(2k+1)$-gon equation for a linear map $T\colon V^{\otimes k} \to V^{\otimes k}$ is an equation in $\mathrm{End}(V^{\otimes \frac{k(k+1)}{2}})$ of the form:

$$T_{a_{k+1}} T_{a_k} \cdots T_{a_1} = T_{b_1} T_{b_2} \cdots T_{b_k},$$

where the multi-indices a_i and b_j are the same as in Definition 3.1.

Definition 3.3 (Even-gon equation)**.** For an integer $k \geq 2$, the $2k$-gon equation for a linear map $T\colon V^{\otimes k-1} \to V^{\otimes k}$ is an equation in $\mathrm{Hom}(V^{\otimes \frac{k(k-1)}{2}}, V^{\otimes \frac{k(k+1)}{2}})$ of the form:

$$T_{a_1} T_{a_2} \cdots T_{a_k} = T_{b_k} T_{b_{k-1}} \cdots T_{b_1},$$

where the multi-indices $a_i = [a_{i,1}, a_{i,2}, \ldots, a_{i,k-1}]$ and $b_j = [b_{j,1}, b_{j,2}, \ldots, b_{j,k-1}]$ are given by the following matrices A_{2k} and B_{2k}, respectively:

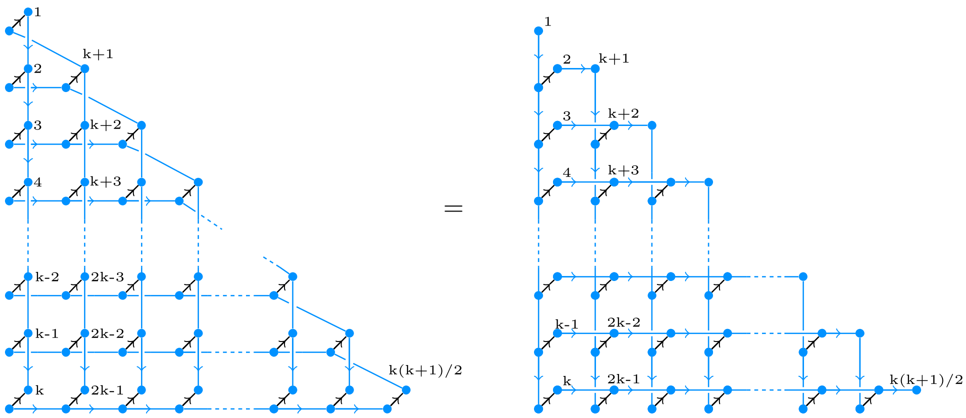
$$A_{2k} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} := A_{2k-1}, \quad B_{2k} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} := B_{2k+1} \text{ with the last column deleted.}$$

Definition 3.4 (Dual even-gon equation)**.** For an integer $k \geq 2$, the dual $2k$-gon equation for a linear map $T\colon V^{\otimes k} \to V^{\otimes k-1}$ is an equation in $\mathrm{Hom}(V^{\otimes \frac{k(k+1)}{2}}, V^{\otimes \frac{k(k-1)}{2}})$ of the form:

$$T_{a_k} T_{a_{k-1}} \cdots T_{a_1} = T_{b_1} T_{b_2} \cdots T_{b_k},$$

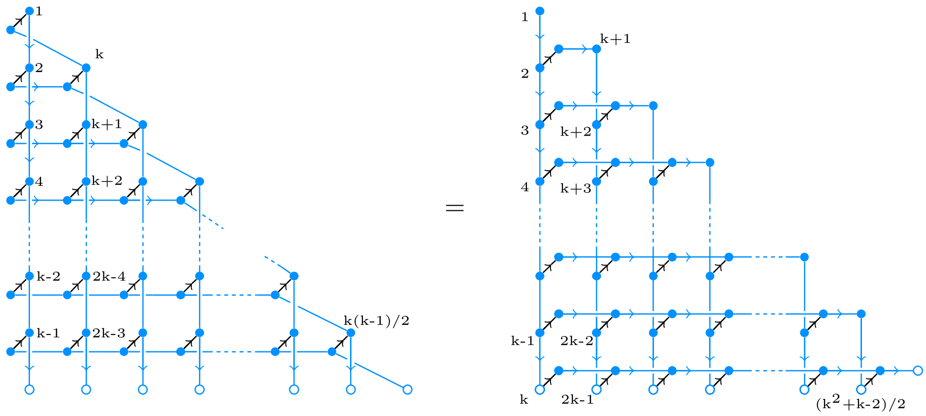
where the multi-indices a_i and b_j are the same as in Definition 3.3.

Alternatively, we can use a diagrammatic representation to define the polygon equation. The $(2k+1)$-gon equation can be defined graphically by



(it is read from bottom to top). Since the dual version is obtained by just reversing the order of compositions, the graphical representation of the dual $(2k+1)$-gon equation is given by reversing the composition arrows in the diagram of the $(2k+1)$-gon equation, or, in other words, reading the diagram from top to bottom.

The $2k$-gon equation can be defined graphically by



(it is read from bottom to top.) Again, since the dual version is obtained by reversing the order of compositions, the diagram of the dual $2k$-gon equation is obtained by reversing the composition arrows in the diagram of the $2k$-gon equations or by reading the diagram from top to bottom.

Example 3.5. The n-gon equations for lower n are given as follows:

3-gon: $T_1T_1 = T_1$, **4-gon:** $T_1T_1 = T_2T_1$,

5-gon: $T_{12}T_{13}T_{23} = T_{23}T_{12}$, **6-gon:** $T_{12}T_{13}T_{23} = T_{35}T_{24}T_{12}$,

7-gon: $T_{123}T_{145}T_{246}T_{356} = T_{356}T_{245}T_{123}$, **8-gon:** $T_{123}T_{145}T_{246}T_{356} = T_{479}T_{368}T_{256}T_{123}$,

9-gon: $T_{1234}T_{1567}T_{2589}T_{368,10}T_{479,10} = T_{479,10}T_{3689}T_{2567}T_{1234}$,

10-gon: $T_{1234}T_{1567}T_{2589}T_{368,10}T_{479,10} = T_{59,12,14}T_{48,11,13}T_{37,10,11}T_{2678}T_{1234}$.

We list some useful properties of solutions of polygon equations, many of which are similar to the properties of solutions of simplex equations [1].

Proposition 3.6. [18, Proposition 3.6]

1. The map T is a solution of the $(2k+1)$-gon equation if and only if T^{-1} (if exists) is a solution of the dual $(2k+1)$-gon equation
2. For a solution of the (dual) $(2k+1)$-gon equation T and any $\phi \in \mathrm{Aut}(V)$, the map $(\phi^{-1})^{\otimes k} \circ T \circ \phi^{\otimes k}$ is a solution of the (dual) $(2k+1)$-gon equation.
   For a solution of $2k$-gon (resp. dual $2k$-gon) equation T and any $\phi \in \mathrm{Aut}(V)$, the map $(\phi^{-1})^{\otimes k} \circ T \circ \phi^{\otimes (k-1)}$ (resp. $(\phi^{-1})^{\otimes (k-1)} \circ T \circ \phi^{\otimes k-1}$) is a solution of the $2k$-gon (resp. dual $2k$-gon) equation.
3. The map T is a solution of the $(2k+1)$-gon equation if and only if $\bar{\sigma}_k T \bar{\sigma}_k$ is a solution of the dual $(2k+1)$-gon equation, where $\bar{\sigma}_k$ is given in Remark **??**.
4. The map T is a solution of the $2k$-gon (resp. dual $2k$-gon) equation if and only if $\bar{\sigma}_{k-1} T \bar{\sigma}_k$ (resp. $\bar{\sigma}_k T \bar{\sigma}_{k-1}$) is a solution of the $2k$-gon (resp. dual $2k$-gon).
5. If T is an invertible solution of the (dual) $(2k+1)$-gon equation over a finite dimensional vector space V, then $\dim(V)^{-1}\,\mathrm{tr}_l(T)$ and $\dim(V)^{-1}\,\mathrm{tr}_r(T)$ are solutions of the (dual) $(2k-1)$-gon equation.

4 Stacking solutions

For linear map $F\colon V^{\otimes k} \to V^{\otimes l}$, let $F^{[n]}\colon (V^{\otimes n})^{\otimes k} \to (V^{\otimes n})^{\otimes l}$ denote the linear map defined by

$$(V^{\otimes n})^{\otimes k} \cong (V^{\otimes k})^{\otimes n} \to (V^{\otimes l})^{\otimes n} \cong (V^{\otimes n})^{\otimes l}.$$

Explicitly on tensors, first rearrange the input $(v_{1,1}\otimes v_{1,n})\otimes\cdots\otimes(v_{k,1}\otimes v_{k,n})$ to $(v_{1,1}\otimes v_{k,1})\otimes\cdots\otimes(v_{1,n}\otimes v_{k,n})$ then apply $F^{\otimes n}$. Then formally write the output as $\sum(w_{1,1}\otimes w_{l,1})\otimes\cdots\otimes(w_{1,n}\otimes w_{l,n})$, and rearrange back to the original position $\sum(w_{1,1}\otimes w_{1,n})\otimes\cdots\otimes(w_{l,1}\otimes w_{l,n})$.

A linear map $G\colon V^{\otimes i}\to V^{\otimes j}$ is said to commute with F, written as $F\leftrightarrow G$, if

$$G^{\otimes l}\circ F^{[i]}=F^{[j]}\circ G^{\otimes k}.$$

Note that this is equivalent to $F^{\otimes j}\circ G^{[k]}=G^{[l]}\circ F^{\otimes i}$, thus the notation $F\leftrightarrow G$ is independent of the order of F and G. For example when F is a product, i.e., $F\colon V^{\otimes 2}\to V$, then $F\leftrightarrow G$ is exactly the same condition for G to be an algebra map.

Let $T^{(n)}$ and $T'^{(n)}$ denote solutions of the n-gon equation, $S^{(n)}$ and $S'^{(n)}$ denote solutions of the dual n-gon equation.

Proposition 4.1. [18, Proposition 4.1] Let n and k be any positive integers. Then the following statements hold for the partial composition and tensor product:

1. If $T^{(2k+1)}\leftrightarrow T'^{(n)}$, then $\begin{cases} T^{(2k+1)}\circ_l T'^{(n)} \text{ is a solution of the } (n+2k-2)\text{-gon equation,}\\ T^{(2k+1)}\otimes T'^{(n)} \text{ is a solution of the } (n+2k)\text{-gon equation.}\end{cases}$
2. If $S^{(2k+1)}\leftrightarrow S'^{(n)}$, then $\begin{cases} S^{(2k+1)}\circ_r S'^{(n)} \text{ is a solution of the dual } (n+2k-2)\text{-gon equation,}\\ S^{(2k+1)}\otimes S'^{(n)} \text{ is a solution of the dual } (n+2k)\text{-gon equation.}\end{cases}$
3. If $S^{(2k)}\leftrightarrow T^{(n)}$, then $\begin{cases} S^{(2k)}\circ_l T^{(n)} \text{ is a solution of the dual } (n+2k-3)\text{-gon equation,}\\ S^{(2k)}\otimes T^{(n)} \text{ is a solution of the dual } (n+2k-1)\text{-gon equation.}\end{cases}$
4. If $T^{(2k)}\leftrightarrow S^{(n)}$, then $\begin{cases} T^{(2k)}\circ_r S^{(n)} \text{ is a solution of the } (n+2k-3)\text{-gon equation,}\\ T^{(2k)}\otimes S^{(n)} \text{ is a solution of the } (n+2k-1)\text{-gon equation.}\end{cases}$

As a corollary, we are now able to prove all the conjectures proposed in [19].

Corollary 4.2. [18, Corollary 4.2] Here we consider set-theoretic polygon equations over a set X. Let k be a positive integer.

1. [19, Conjecture 7.32] Let $S^{(2k)}$ be a solution of the dual $2k$-gon equation. Then
$$T^{(2k+1)}(a_1,\ldots,a_k):=(a_1,S^{(2k)}(a_1,\ldots,a_k))$$
is a solution of the $(2k+1)$-gon equation.
2. [19, Conjecture 7.33] Let $S^{(2k)}$ be a solution of the dual $2k$-gon equation, and $u\in X$. Then
$$T^{(2k+1)}(a_1,\ldots,a_k):=(u,S^{(2k)}(a_1,\ldots,a_k))$$
is a solution of the $(2k+1)$-gon equation if and only if $S^{(2k)}(u,\ldots,u)=(u,\ldots,u)$.
3. [19, Conjecture 7.34] Let $T^{(2k+1)}$ be a solution of the $(2k+1)$-gon equation. Then
$$T^{(2k+2)}(a_1,\ldots,a_k):=(T^{(2k+1)}(a_1,\ldots,a_k),a_k)$$
is a solution of the $(2k+2)$-gon equation.
4. [19, Conjecture 7.35] Let $T^{(2k+1)}$ be a solution of the $(2k+1)$-gon equation, and $u\in X$. Then
$$T^{(2k+2)}(a_1,\ldots,a_k):=(T^{(2k+1)}(a_1,\ldots,a_k),u)$$
is a solution of the $(2k+2)$-gon equation if and only if $T^{(2k+1)}(u,\ldots,u)=(u,\ldots,u)$.

5. [19, Conjecture 7.36] Let $S^{(2k+1)}$ be a solution of the dual $(2k+1)$-gon equation. Then

$$T^{(2k+2)}(a_1,\dots,a_k) \coloneqq (a_1, S^{(2k+1)}(a_1,\dots,a_k))$$

is a solution of the $(2k+2)$-gon equation.

6. [19, Conjecture 7.37] Let $S^{(2k+1)}$ be a solution of the dual $(2k+1)$-gon equation, and $u \in X$. Then

$$T^{(2k+2)}(a_1,\dots,a_k) \coloneqq (u, S^{(2k+1)}(a_1,\dots,a_k))$$

is a solution of the $(2k+2)$-gon equation if and only if $S^{(2k+1)}(u,\dots,u) = (u,\dots,u)$.

5 Examples of solutions of polygon equations

Let us give two examples which follows from the Proposition 4.1.

Example 5.1. Let H be a commutative and cocommutative bialgebra where M and Δ are product and coproduct, respectively. Then Δ is a solution of the 4-gon equation, and M is a solution of the dual 4-gon equation. For an integer $n \geq 3$, we define $T^{(n)}$ by

$$T^{(2k+1)} \coloneqq \underbrace{\Delta \circ_r M \circ_l \Delta \circ_r M \circ_l \cdots \circ_r M}_{2k-2}, \quad T^{(2k)} \coloneqq \underbrace{\Delta \circ_r M \circ_l \Delta \circ_r M \circ_l \cdots \circ_l \Delta}_{2k-3},$$

and $T^{(3)} \coloneqq \mathrm{id}_H$. Let $T = \Delta \circ_r M$. From the axiom of bialgebra and the (co)commutativity of M and Δ, one can easily check that $T \leftrightarrow T$ and $T \leftrightarrow \Delta$. Since $T^{(2k+1)} = T \circ_l \cdots \circ_l T$ and $T^{(2k)} = T \circ_l \cdots \circ_l T \circ_l \Delta$, $T^{(n)}$ is a solution of the n-gon equation.

Similarly, for an integer $n \geq 3$, if we define $S^{(n)}$ by

$$S^{(2k+1)} \coloneqq \underbrace{M \circ_l \Delta \circ_r M \circ_l \Delta \circ_r \cdots \circ_l \Delta}_{2k-2}, \quad S^{(2k)} \coloneqq \underbrace{M \circ_l \Delta \circ_r M \circ_l \Delta \circ_r \cdots \circ_r M}_{2k-3},$$

and $S^{(3)} \coloneqq \mathrm{id}_H$, then $S^{(n)}$ is a solution of the dual n-gon equation.

Example 5.2. A strict n-category $\mathcal{C}$ consists of sets $(\mathcal{C}_0, \mathcal{C}_1, \cdots, \mathcal{C}_n)$ and maps

$$\begin{aligned} s_k, t_k \colon \mathcal{C}_k \to \mathcal{C}_{k-1} &\quad \text{for } 1 \leq k \leq n, \\ 1_k \colon \mathcal{C}_k \to \mathcal{C}_{k+1} &\quad \text{for } 0 \leq k \leq n-1, \\ \circ_{p,q} \colon \{(f,g) \in \mathcal{C}_p \times \mathcal{C}_p \mid s_{pq}(f) = t_{pq}(g)\} \to \mathcal{C}_p &\quad \text{for } 0 \leq q < p, \end{aligned}$$

where $s_{pq} \coloneqq s_{q+1} \cdots s_{p-1} s_p$, $t_{pq} \coloneqq t_{q+1} \cdots t_{p-1} t_p$ and $1_{qp} = 1_{p-1} \cdots 1_{q+1} 1_q$. Element of $\mathcal{C}_p$ are called p-morphisms. We will write $\circ_q$ for $\circ_{p,q}$ when p is obvious from the context.

These maps should satisfy the following axioms:

- $s_{p-1} s_p = s_{p-1} t_p$, $t_{p-1} s_p = t_{p-1} t_p$ and $s_{p+1} 1_p = \mathrm{id}_{\mathcal{C}_p} = t_{p+1} 1_p$.
- For $f, g \in \mathcal{C}_p$ and $q < p$ with $s_{pq}(f) = t_{pq}(g)$,

$$s_p(f \circ_q g) = \begin{cases} s_p(f) \circ_q s_p(g) & \text{if } q < p-1, \\ s_p(g) & \text{if } q = p-1, \end{cases} \qquad t_p(f \circ_q g) = \begin{cases} t_p(f) \circ_q t_p(g) & \text{if } q < p-1, \\ t_p(f) & \text{if } q = p-1. \end{cases}$$

- For $f, g, h \in \mathcal{C}_p$, $(f \circ_q g) \circ_q h = f \circ_q (g \circ_q h)$.

- For $q < p < r$ and $f, g, h, k \in \mathcal{C}_r$ with $t_q(f) = s_q(g)$, $t_q(h) = s_q(k)$ and $t_p(f) = s_p(h)$,

$$(f \circ_q g) \circ_p (h \circ_q k) = (f \circ_p h) \circ_q (g \circ_p k).$$

- For $f \in \mathcal{C}_p$ and $q < p$, $f \circ_q 1_{qp}(s_{pq}(f)) = f = 1_{qp}(t_{pq}(f)) \circ_q f$.

A strict n-groupoid $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_n)$ is a strict n-category where each i-morphism is invertible with respect to $\circ_j$ for $0 \leq j < i$, i.e., for any $1 \leq i \leq n$, any $0 \leq j < i$ and any $a \in \mathcal{G}_i$, there exist $b \in \mathcal{G}_j$ such that $s_{ij}(a) = t_{ij}(b)$, $s_{ij}(b) = t_{ij}(a)$, $a \circ_j b = 1_{ji}(t_{ij}(a))$, and $b \circ_j a = 1_{ji}(s_{ij}(a))$.

Let H be the vector space spanned by all the n-morphisms of the strict n-groupoid $\mathcal{G}$. For each $0 \leq i \leq n-1$, set

$$\Delta_i \colon H \to H \otimes H;\ f \mapsto f \otimes f, \quad M_i \colon H \otimes H \to H;\ f \otimes g \mapsto \delta^{s_{ni}(f)}_{t_{ni}(g)} f \circ_i g,$$

where $\delta^{s_{ij}(f)}_{t_{ij}(g)} = 1$ if $s_{ij}(f) = t_{ij}(g)$ and 0 otherwise. Then for any i, the tuple (H, Δ_i, M_i) forms a (weak) bialgebra. For $2 \leq k \leq \frac{n-2}{2}$, if we set

$$T^{(2k+1)} := \Delta_0 \circ_r M_0 \circ_l \Delta_1 \circ_r \cdots \circ_r M_{k-2}, \quad T^{(2k)} := \Delta_0 \circ_r M_0 \circ_l \Delta_1 \circ_l \cdots \circ_l \Delta_{k-2},$$
$$S^{(2k+1)} := M_{n-1} \circ_l \Delta_{n-1} \circ_r \cdots \circ_l \Delta_{n-k+1}, \quad S^{(2k)} := M_{n-1} \circ_l \Delta_{n-1} \circ_r \cdots \circ_r M_{n-k+1},$$

then a similar argument to the ones above shows that these are solutions of the polygon and dual polygon equations.

6 Solutions of simplex equations from polygon equations

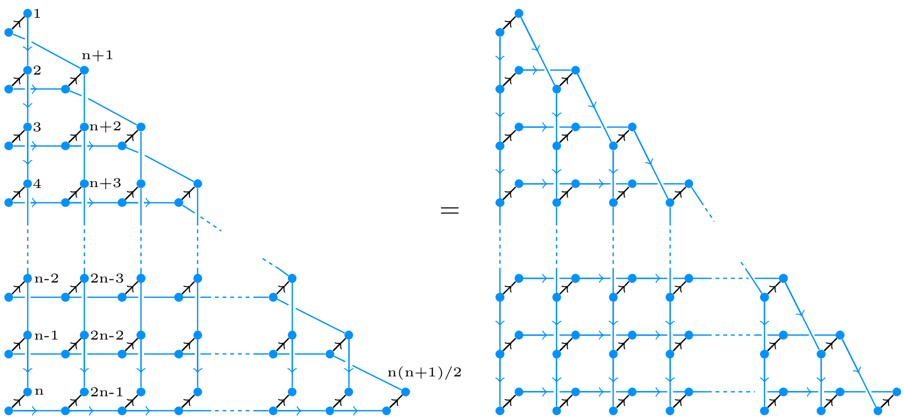
In this section, we explore the relationship between simplex equations and polygon equations. The main theorems (Theorem 6.4, 6.5) generalize the result from [12].

Definition 6.1 (Simplex equation)**.** For a positive integer n, the n-simplex equation for a linear map $R \colon V^{\otimes n} \to V^{\otimes n}$ is an equation in $\mathrm{End}(V^{\otimes \frac{n(n+1)}{2}})$ of the form:

$$R_{a_1} R_{a_2} \cdots R_{a_{n+1}} = R_{a_{n+1}} R_{a_n} \cdots R_{a_1},$$

where $[a_1, a_2, \ldots, a_n]^T$ is the matrix A_{2n+1} defined in Definition 3.1.

Alternatively, the n-simplex equation can be defined graphically by



(the left-hand side of the figure is read from bottom to top, and the right-hand side is read from top to bottom).

6.1 Mixed relation

Let us define the mixed relation explicitly, which corresponds to the equation associated with the mixed order in [7].

Definition 6.2 (Mixed relation for odd-gon)**.** Let T be a solution of $(2k+1)$-gon equation and S a solution of the dual $(2k+1)$-gon equation. Then the mixed relation is

$$T_{d_{k+1}} S_{e_k} T_{d_k} \cdots S_{e_1} T_{d_1} = S_{f_1} T_{g_1} S_{f_2} \cdots T_{g_k} S_{f_{k+1}},$$

where the multi-indices d_i, e_i, f_i and g_i are as follows:

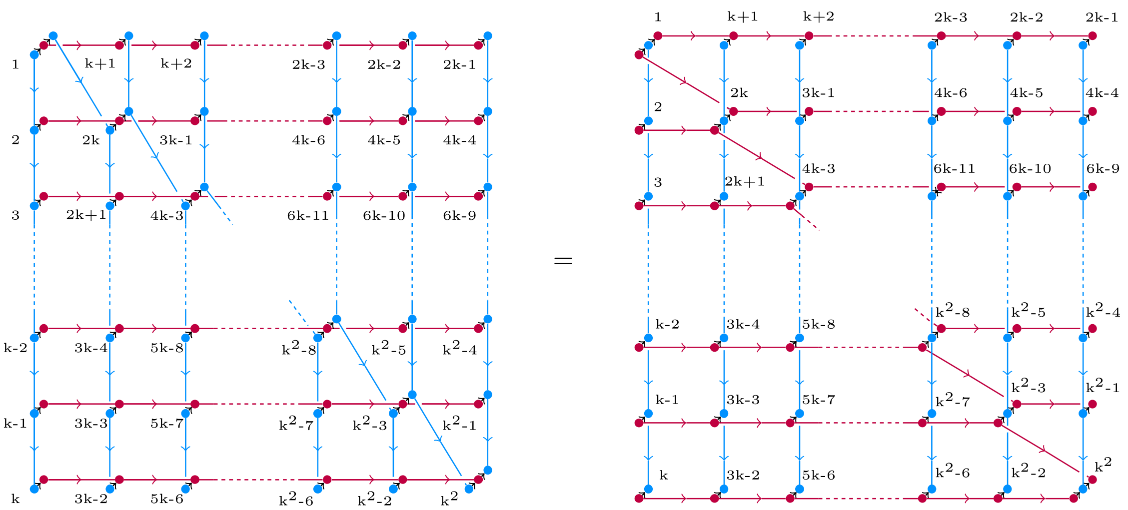
$$E_{2k+1} = \begin{bmatrix} e_1 \\ \vdots \\ e_k \end{bmatrix} = \left[\begin{array}{c|cccc} 1 & k+1 & k+2 & \cdots & 2k-1 \\ \hline 2 & & & & \\ 3 & & & & \\ \vdots & & E_{2k-1} + [2k-1]_{k-1,k-1} & & \\ k & & & & \end{array}\right], \quad E_3 = [\,1\,],$$

$$G_{2k+1} = \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix} = (E_{2k+1})^T,$$

$$D_{2k+1} = \begin{bmatrix} d_1 \\ \vdots \\ d_{k+1} \end{bmatrix} = \left[\begin{array}{c|cccc} 1 & 2 & 3 & \cdots & k \\ \hline 1 & & & & \\ k+1 & & & & \\ k+2 & & D_{2k-1} + [2k-1]_{k,k-1} & & \\ \vdots & & & & \\ 2k-1 & & & & \end{array}\right], \quad D_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$F_{2k+1} = \begin{bmatrix} f_1 \\ \vdots \\ f_{k+1} \end{bmatrix} = \left[\begin{array}{c|cccc} 1 & k+1 & k+2 & \cdots & 2k-1 \\ \hline 1 & & & & \\ 2 & & & & \\ \vdots & & F_{2k-1} + [2k-1]_{k,k-1} & & \\ k & & & & \end{array}\right], \quad F_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

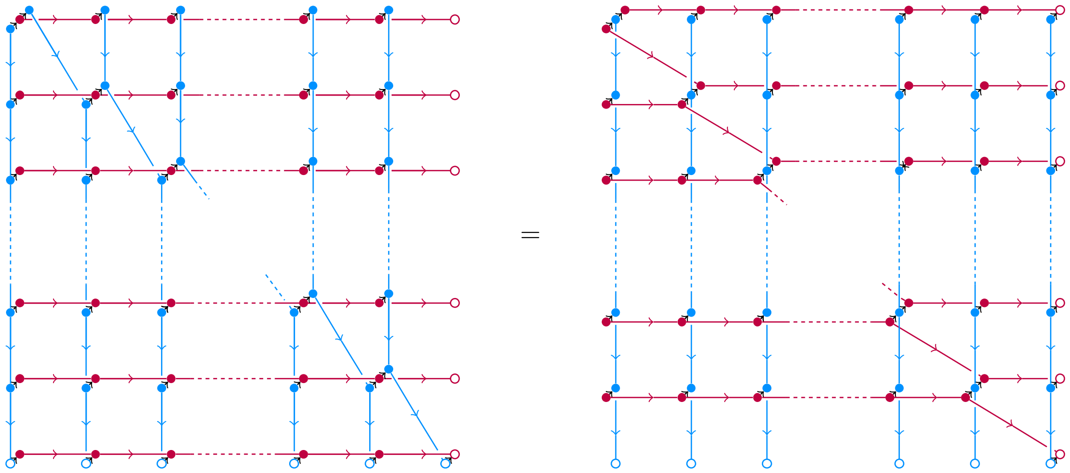
The graphical definition is



(the left-hand side is read from bottom to top, and the right-hand side is read from top to bottom).

For $k = 2$, this recovers the ten-term relation of the (dual) pentagon equations in [12].

Definition 6.3 (Mixed relation for even-gon)**.** Let T be a solution of $2k$-gon equation and S a solution of the dual $2k$-gon equation. Then the mixed relation is graphically defined by



(the left-hand side is read from bottom to top, and the right-hand side is read from top to bottom).[1]

6.2 From n-gon to $(n-1)$-simplex

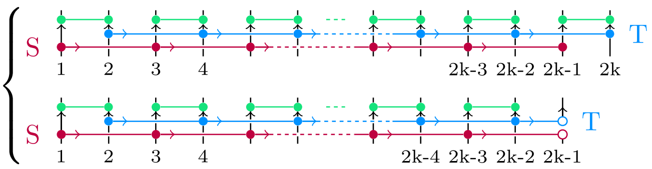
We construct a solution of the $(n-1)$-simplex equation from a pair consisting of solutions of the n-gon and dual n-gon equations, satisfying the mixed relation. Although this result was previously discussed in [7] and proved in [5] for the odd-gon equations, we provide a more direct proof using graphical calculus here.

Theorem 6.4. [18, Theorem 6.7] Let T be a solution of the n-gon equation, and S be a solution of its dual. Suppose that T and S satisfy the mixed relation. Then

$$R^{(n-1)} := \begin{cases} \sigma_{1,2}\sigma_{3,4}\cdots\sigma_{2k-1,2k}T_{2,4,\dots,2k}S_{1,3,\dots,2k-1} & \text{if } n = 2k+1, \\ \sigma_{1,2}\sigma_{3,4}\cdots\sigma_{2k-3,2k-2}T_{2,4,\dots,2k-2}S_{1,3,\dots,2k} & \text{if } n = 2k, \end{cases}$$

or graphically,

$R^{(n-1)} :=$



if $n = 2k+1$,

if $n = 2k$

is a solution of the $(n-1)$-simplex equation, where $\sigma_{i,j}$ flips the i-th and j-th factors if $i \neq j$ and identity if $i = j$, and is represented in green.

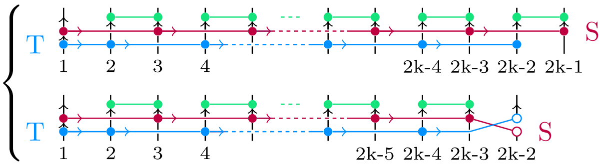
6.3 From n-gon to $(n-2)$-simplex

The same condition as in Theorem 6.4 also gives rise to a solution of the $(n-2)$-simplex equation. This generalizes the result from [12, Propositions 4, 5].

[1] Writing the mixed relation for even-gon in our tensor notation is complicated, and so we define it graphically.

Theorem 6.5. [18, Theorem 6.7] Let T be a solution of the n-gon equation and S be a solution of its dual. Suppose that T and S satisfy the mixed relation. Then

$R^{(n-2)} :=$



if $n = 2k+1$,

if $n = 2k$

is a solution of the $(n-2)$-simplex equation.[2]

Figure 2: Right partial trace version of $R^{(n-2)}$, from odd-gon (left), and even-gon (right)

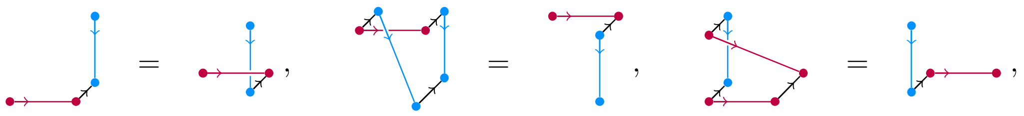
6.4 Examples

In [5], the authors constructed a solution of the (non-constant) $2n$-simplex equation from solutions of the (non-constant) $(2n+1)$-gon and dual $(2n+1)$-gon equations, which form a pair satisfying the mixed relation. Using Theorem 6.5, we know that their construction further descends to a solution of the $(2n-1)$-simplex equation. Another example of a pair of solutions of the 5-gon and dual 5-gon equations satisfying the mixed relation is given in [8].

Let T and S be solutions of the pentagon and dual pentagon equations, respectively. Kashaev and Sergeev [12, (4.1)] showed that if T and S satisfy

$$T_{13}S_{23} = S_{23}T_{13}, \tag{3}$$
$$T_{23}T_{13}S_{12} = S_{12}T_{23}, \tag{4}$$
$$T_{12}S_{13}S_{23} = S_{23}T_{12}, \tag{5}$$

then they satisfy the mixed relation. The graphical representations of these relations are



respectively (the blue line represents T, and the red line S).

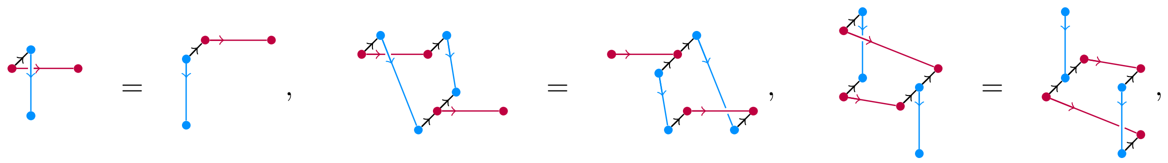
We now look for a similar condition for solutions of higher-order polygon equations. In particular, we consider solutions of the forms $T^{(2k+1)} := T \circ_l T \circ \cdots \circ_l T$ and $S^{(2k+1)} := S \circ_r S \circ \cdots \circ_r S$, where T and S are solution of pentagon and dual pentagon equation such that $T \leftrightarrow T$ and $S \leftrightarrow S$.

Lemma 6.6. Let $T^{(2k+1)}$ and $S^{(2k+1)}$ be as above. Suppose that T and S satisfy (3)–(5), and when $k > 1$ in addition, the following relations:

$$T_{12}S_{13} = S_{13}T_{12}, \tag{6}$$
$$T_{23}S_{34}T_{13}S_{12} = S_{34}T_{24}S_{12}T_{23}, \tag{7}$$
$$T_{12}S_{13}T_{34}S_{23} = S_{23}T_{12}S_{24}T_{34}. \tag{8}$$

[2] Writing the map $R^{(n-2)}$ in our tensor notation is complicated, and so we represent it graphically.

Graphically, they are



respectively. Then $T^{(2k+1)}$ and $S^{(2k+1)}$ satisfy the mixed relation.

Example 6.7. Let H be a commutative and cocommutative Hopf algebra with antipode s, and $T^{(2k+1)}$ be the solution of $(2k+1)$-gon equation in Example 5.1. Set $M_S := M \circ (s \otimes \mathrm{id}_H)$ and

$$S^{(2k+1)} := \underbrace{\Delta \circ_r M_S \circ_r \Delta \circ_r M_S \circ_r \cdots \circ_r M_S}_{2k-2} \quad (k > 1), \quad \text{and} \quad S^{(3)} := \mathrm{id}_H.$$

Then $S^{(2k+1)}$ is a solution of the dual $(2k+1)$-gon equation. One can also check that $T^{(3)}$ and $S^{(3)}$ satisfy (3)–(8) in Lemma 6.6, and therefore the pair $(T^{(2k+1)}, S^{(2k+1)})$ satisfies the mixed relation.

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