Simplicity of the kernel of the flux homomorphisms of non-orientable open surfaces

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1 Introduction

In this paper, we discuss the latter half of the preprint [KM25] by Kim and the author.

A group G is called *simple* if every normal subgroup is trivial or G itself. In other words, the group G is simple if and only if any homomorphism from G to any group is either trivial or injective.

For a C^∞-manifold M, let $\mathrm{Diff}^r_c(M)_0$ denote the group of compactly-supported C^r-diffeomorphisms that are isotopic to the identity. Thurston [Thu74] proved that $\mathrm{Diff}^\infty_c(M)_0$ is simple, and Mather [Mat74, Mat75] proved that $\mathrm{Diff}^r_c(M)_0$ is simple unless $r = \dim(M)+1$.

From here we concentrate on the case where M is an open surface. Let S be an orientable open surface and ω an area form. Let $\mathrm{Diff}_c(S;\omega)_0$ be the group of compactly-supported ω-preserving C^∞-diffeomorphisms of S that are isotopic to the identity. In contrast to the case of $\mathrm{Diff}^\infty_c(M)_0$, the group $\mathrm{Diff}_c(S;\omega)_0$ is not simple, and in fact there exists a surjective homomorphism

$$\mathrm{Flux}_S \colon \mathrm{Diff}_c(S;\omega)_0 \to \mathrm{H}^1_c(S;\mathbb{R})$$

called the flux homomorphism (see Section 2 for the definition). The kernel of the flux homomorphism is known to coincide with the group of Hamiltonian diffeomorphisms. Moreover, the kernel $\ker \mathrm{Flux}_S$ of the flux homomorphism is not simple, either. In fact, there is a surjective homomorphism

$$\mathrm{Cal}_S \colon \ker \mathrm{Flux}_S \to \mathbb{R}$$

called the Calabi homomorphism (see Section 2 for the definition). Banyaga [Ban78] proved that the kernel $\ker \mathrm{Cal}_S$ is simple.

In this paper, we will consider the case where M is a non-orientable open surface N. For a non-orientable surface, the notion of area density plays a similar role to area forms for orientable surfaces. Let ω be an area density of N, and $\mathrm{Diff}_c(N;\omega)_0$ the group of compactly-supported ω-preserving C^∞-diffeomorphisms of N that are isotopic to the identity. Similarly to the orientable surface cases, the group $\mathrm{Diff}_c(N;\omega)_0$ admits a surjective homomorphism

$$\mathrm{Flux}_N \colon \mathrm{Diff}_c(N;\omega)_0 \to \mathrm{H}^1_c(N;L_N)$$

called the flux homomorphism. Here $\mathrm{H}^1_c(N; L_N)$ is the compactly-supported first cohomology of N with coefficients in the orientation line bundle $L_N \to N$ of N.

In contrast to the fact that $\ker \mathrm{Flux}_S$ is not simple for an orientable open surface S, we show the following:

Theorem 1.1 ([KM25])**.** For a non-orientable open surface N, the kernel $\ker \mathrm{Flux}_N$ is simple.

In this paper, we will explain the outline of the proof of the above theorem for the open Möbius band.

2 Flux homomorphism and Calabi homomorphism

2.1 Case of orientable open surfaces

In this section, we will explain the definitions of the flux homomorphism and the Calabi homomorphism (see [Ban97] and [MS98] for details).

Let S be an orientable open surface and $\omega \in \Omega^2(S)$ an area form. Since S is an open surface, the second cohomology $\mathrm{H}^2(S;\mathbb{R})$ is trivial, and hence ω is exact. Let η be a 1-form on S satisfying $d\eta = \omega$. For an element g of $\mathrm{Diff}_c(S;\omega)_0$, the difference $\eta - g^*\eta$ is closed since $d(\eta - g^*\eta) = \omega - g^*\omega = \omega - \omega = 0$. Moreover, $\eta - g^*\eta$ is compactly-supported since g is compactly-supported. Hence $\eta - g^*\eta$ represents a compactly-supported first cohomology class of S. It is verified that the compactly-supported cohomology class $[\eta - g^*\eta]$ does not depend on the choice of η. This gives rise to a well-defined map

$$\mathrm{Flux}_S \colon \mathrm{Diff}_c(S;\omega)_0 \to \mathrm{H}^1_c(S;\mathbb{R});\ g \mapsto [\eta - g^*\eta],$$

and it turns out to be a surjective homomorphism, called the *flux homomorphism*.

For an element g of $\ker \mathrm{Flux}_S$, we can take a compactly-supported 0-form φ_g on S satisfying $d\varphi_g = \eta - g^*\eta$. By the compactly-supportedness, such φ_g is unique. Hence we obtain a map

$$\mathrm{Cal}_S \colon \ker \mathrm{Flux}_S \to \mathbb{R};\ g \mapsto \int_S \varphi_g \cdot \omega,$$

and it turns out to be a surjective homomorphism, called the *Calabi homomorphism*.

2.2 Case of non-orientable open surfaces

Let N be a non-orientable open surface and $L_N \to N$ the orientation line bundle. Let $\Omega^p(N; L_N)$ (resp. $\Omega^p_c(N; L_N)$) be the set of all sections (resp. compactly-supported sections) of the bundle $\wedge^p T^*N \otimes L_N \to N$. An element of $\Omega^p(N; L_N)$ (resp. $\Omega^p_c(N; L_N)$) is called a *twisted p-form* of N (resp. *compactly-supported twisted p-form*) (see [BT82] and [dR84] for details on twisted differential forms). An *area density* on N is an element of $\Omega^2(N; L_N)$ which is nowhere zero.

We can define the exterior derivative $d \colon \Omega^p(N; L_N) \to \Omega^{p+1}(N; L_N)$, and hence $(\Omega^\bullet(N; L_N), d)$ forms a cochain complex. Let $\mathrm{H}^*(N; L_N)$ denote the cohomology of $(\Omega^*(N; L_N), d)$. Moreover, the compactly-supported twisted differential forms $\Omega^\bullet_c(N; L_N)$ turns out to be a subcomplex of $(\Omega^\bullet(N; L_N), d)$, and hence it gives rise to a cohomology group $\mathrm{H}^*_c(N; L_N)$. It is easily verified that the second cohomology group $\mathrm{H}^2(N; L_N)$ is trivial, and hence the area density is exact on a non-orientable open surface.

Let $\mathrm{Diff}_c(N; \omega)_0$ be the group of compactly-supported ω-preserving diffeomorphisms of N that are isotopic to the identity. Take a twisted 1-form $\eta \in \Omega^1(N; L_N)$ such that $d\eta = \omega$. As in the case of an orientable surface, the difference $\eta - g^*\eta \in \Omega^1_c(N; L_N)$ gives rise to a surjective homomorphism

$$\mathrm{Flux}_N \colon \mathrm{Diff}_c(N; \omega)_0 \to \mathrm{H}^1_c(N; L_N); \;\; g \mapsto [\eta - g^*\eta],$$

which is called the *flux homomorphism*.

3 Simplicity trick

In this section, we will explain a well-known criterion for a given subgroup of the group of diffeomorphisms to be simple.

Let M be a C^∞-manifold and G a subgroup of $\mathrm{Diff}_c(M)$. The simplicity trick has two ingredients: the displacing property and the strong fragmentation property.

Definition 3.1 (Displacing property)**.** *G is said to have the* displacing property *if for any distinct points x and y in M, there exist $z \in M \setminus \{x, y\}$ and $g \in G$ such that $g(y) = z$ and $x \notin \mathrm{supp}(g)$.*

For an open set $U \subset M$, set

$$G_U = \{g \in G \mid \mathrm{supp}(g) \subset U\}_0.$$

Definition 3.2 (Fragmentation property)**.** *G is said to have the* fragmentation property *if for every open cover $\mathcal{U}$ of M by open balls and for every $g \in G$, there exist $U_1, \dots, U_n \in \mathcal{U}$ and $g_1, \dots, g_n \in G$ such that*

(1) $g = g_1 \cdots g_n$,

(2) $g_i \in G_{U_i}$ *for every i.*

Definition 3.3 (Strong fragmentation property)**.** *G is said to have the* strong fragmentation property *if for every open cover $\mathcal{U}$ of M by open balls and for every $g \in G$, there exist $U_1, \dots, U_n \in \mathcal{U}$ and $g_1, \dots, g_n \in G$ such that*

(1) $g = g_1 \cdots g_n$,

(2) $g_i \in [G_{U_i}, G_{U_i}]$ *for every i.*

Here $[G_{U_i}, G_{U_i}]$ denotes the commutator subgroup of G_{U_i}.

Note that if G_U is perfect for every open ball $U \subset M$, then the strong fragmentation property is equivalent to the fragmentation property.

Theorem 3.4 (Simplicity trick; see [Ban97] for example)**.** *If G acts on M transitively and has the displacing property and the strong fragmentation property, then G is simple.*

For the simplicity of $\mathrm{Diff}_c(M)_0$ (Thurston) and of $\ker \mathrm{Cal}_M$ (Banyaga), the simplicity trick was used in the following way. Let G be either $\mathrm{Diff}_c(M)_0$ or $\ker \mathrm{Cal}_M$. It is not hard to prove that the group G acts on the manifold transitively and has the displacing property and the fragmentation property. Thurston and Banyaga proved that for every open ball U, the group G_U is perfect by using Mather–Thurston theorem. Hence the fragmentation property is equivalent to the strong fragmentation property, and thus the simplicity trick works for G.

Our goal is to prove the simplicity of the group $\ker \mathrm{Flux}_N$ for a non-orientable surface N. For this group $\ker \mathrm{Flux}_N$, the transitivity and the displacing property can be easily verified. However, the above strategy does not work well. Indeed, the group $(\ker \mathrm{Flux}_N)_U$ is not perfect for an open disk $U \subset N$, since $(\ker \mathrm{Flux}_N)_U$ is isomorphic to $\mathrm{Diff}_c(U;\omega)_0$ and this group admits the Calabi homomorphism $\mathrm{Cal}_U\colon \mathrm{Diff}_c(U;\omega)_0 \to \mathbb{R}$. Hence we need to establish the strong fragmentation property without perfectness.

4 From fragmentation property to strong fragmentation property by using non-orientability

In this section, we will explain how we improve the fragmentation property to the strong fragmentation property from the non-orientability. This is called the *cell division trick* in [KM25]. Here we concentrate on the toy model on the open Möbius band, which is the most elementary example containing the essence of the trick.

Let $\widetilde{M}$ be the infinite strip $\mathbb{R} \times (-1,1)$ and $\tau\colon \widetilde{M} \to \widetilde{M}$ the map defined by $\tau(x,y) = (x+1,-y)$. Let M be the quotient $M = \widetilde{M}/\langle\tau\rangle$ and $p\colon \widetilde{M} \to M$ the projection. We set $\epsilon = 1/100$, $U = p((-1/2+\epsilon, 1/2-\epsilon)\times(-1,1))$ and $V = p((\epsilon, 1-\epsilon)\times(-1,1))$. Obviously $\{U,V\}$ is an open cover of M by open disks. Then $W = U \cap V$ has two connected components, say $W_1 = p((-1/2+\epsilon,-\epsilon)\times(-1,1))$ and $W_2 = p((\epsilon, 1/2-\epsilon)\times(-1,1))$.

Assume that we have already proved the fragmentation property for the open cover $\{U,V\}$. Then, for a given $g \in \ker \mathrm{Flux}_M$, we have the decomposition

$$g = g_1 \cdots g_n$$

such that the support of each g_i is contained in U or V.

Assume that the support of g_i is contained in U, that is, $g_i \in (\ker \mathrm{Flux}_M)_U$. Since $(\ker \mathrm{Flux}_M)_U$ is isomorphic to $\mathrm{Diff}_c(U;\omega)_0$, we can consider the value of g_i under the Calabi homomorphism Cal_U on $\mathrm{Diff}_c(U;\omega)_0 \cong (\ker \mathrm{Flux}_M)_U$. By the theorem of Banyaga, the kernel of the Calabi homomorphism is simple, and hence if $\mathrm{Cal}_U(g_i) = 0$, then g_i is contained in $[G_U, G_U]$.

Even if $\mathrm{Cal}_U(g_i) \neq 0$, we are still able to decompose g_i into commutators as follows.

Proposition 4.1. For an element g of G_U, there exist $g_1 \in (\ker \mathrm{Flux}_M)_U$ and $g_2 \in (\ker \mathrm{Flux}_M)_V$ such that $g = g_1 g_2$ and $\mathrm{Cal}_U(g_1) = \mathrm{Cal}_V(g_2) = 0$.

Proof. Set $a = \mathrm{Cal}_U(g)$. Take elements $h_1 \in (\ker \mathrm{Flux}_M)_{W_1}$ and $h_2 \in (\ker \mathrm{Flux}_M)_{W_2}$ such that $\mathrm{Cal}_U(h_1) = \mathrm{Cal}_U(h_2) = \frac{a}{2}$. Such h_i's exist since Cal_{W_i}'s are surjective. Then, by taking a natural coordinate system for V, we can see that $\mathrm{Cal}_V(h_1) = -\frac{a}{2}$. Then $g_1 = g h_2^{-1} h_1^{-1}$ and $g_2 = h_1 h_2$ are the desired elements. □

By applying Proposition 4.1 to each g_i, we can improve the fragmentation property for $\{U, V\}$ to the strong fragmentation property. A similar argument works for an arbitrary open cover by open disks of an arbitrary non-orientable surface, and hence we obtain the strong fragmentation property from the fragmentation property. See [KM25] for details.

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