On quasi-strongly keen Heegaard splittings

Kanako Oie
Graduate School of Humanities and Sciences, Nara Women's University

1 Introduction

The curve complex, introduced by Harvey [1], is one of the fundamental tools in the study of Heegaard splittings of 3-manifolds. Hempel [2] used it to define the distance of a Heegaard splitting, namely the distance between the disk sets of the two handlebodies in the curve complex of the Heegaard surface. Since then, the Hempel distance has been used as an effective measure of the complexity of Heegaard splittings.

In order to refine the notion of distance, Ido, Jang and Kobayashi [3] introduced the notions of keen and strongly keen Heegaard splittings. A Heegaard splitting is called keen if there exists a unique pair of vertices in the two disk sets realizing the Hempel distance. It is called strongly keen if, moreover, the geodesic connecting this unique pair is also unique. Thus the strongly keen condition records not only the distance between the disk sets, but also the uniqueness of the way in which this distance is realized in the curve complex.

These uniqueness conditions are related to the symmetry of Heegaard splittings. A particularly relevant result is due to Iguchi and Koda [5]. They proved that if a Heegaard splitting is strongly keen and has distance two, then its Goeritz group is either a finite cyclic group or a finite dihedral group. They also proved that if a Heegaard splitting is keen and has distance at least three, then the same conclusion holds. These results suggest that uniqueness properties of distance-realizing vertices and geodesics impose strong restrictions on the automorphisms preserving the Heegaard splitting.

From this point of view, the distance-two case is especially natural to consider. For distance at least three, the keen condition already gives a strong restriction on the Goeritz group. In contrast, for distance two, the result of Iguchi and Koda uses the stronger assumption of being strongly keen. This leads to the following question: if the geodesic realizing the distance is not assumed to be unique, but the number of such geodesics is finite, does this give a genuinely broader class of Heegaard splittings?

Motivated by this question, the author introduced in [8] the notion of a quasi-strongly keen Heegaard splitting. Roughly speaking, a quasi-strongly keen Heegaard splitting is a keen Heegaard splitting such that the set of geodesics connecting the unique distance-realizing pair of vertices is finite.

The purpose of this article is to give a brief overview of quasi-strongly keen Heegaard splittings and to explain why, in distance two, this notion does not give a new class between keen and strongly keen Heegaard splittings. The precise statement will be given in Section 3.

2 Preliminaries and motivation

In this section, we recall the basic notions used in this article and explain the motivation for considering quasi-strongly keen Heegaard splittings. Throughout this section, S denotes a closed orientable surface of genus at least two.

2.1 Curve complexes and geodesics

A simple closed curve in S is called essential if it does not bound a disk in S. We consider essential simple closed curves up to isotopy.

The curve complex $\mathcal{C}(S)$ is the simplicial complex whose vertices are isotopy classes of essential simple closed curves on S. A collection of distinct vertices spans a simplex if the corresponding curves can be realized pairwise disjointly on S. We denote by $\mathcal{C}^0(S)$ the set of vertices of $\mathcal{C}(S)$.

For two vertices $\alpha, \beta \in \mathcal{C}^0(S)$, the distance $d_S(\alpha, \beta)$ is the minimal length of an edge path in $\mathcal{C}(S)$ connecting them. A path

$$[\ell_0, \ell_1, \dots, \ell_n]$$

is called a geodesic if $n = d_S(\ell_0, \ell_n)$.

2.2 Type F pairs and distance-two geodesics

Let $\alpha, \beta \in \mathcal{C}^0(S)$. We say that the pair (α, β) is of type F if the set of geodesics connecting α and β is finite. We say that (α, β) is of strongly type F if there exists exactly one geodesic connecting them.

Thus, the type F condition is weaker than the strongly type F condition. The converse does not hold in general. In [7], examples of pairs of vertices in the curve complex which are of type F but not strongly type F were constructed. Hence the condition that the set of geodesics is finite is genuinely different from the condition that the geodesic is unique.

The paper [6] studies this question from the viewpoint of the spectrum of the number of geodesics. For an integer $d \geq 2$, let $\mathrm{Sp}_d(S)$ denote the set of positive integers k such that there exist vertices $\alpha, \beta \in \mathcal{C}^0(S)$ satisfying

$$d_S(\alpha, \beta) = d$$

and such that the number of geodesics connecting α and β is equal to k.

For a closed surface S of genus $g \geq 3$, the distance-two spectrum is given by

$$\mathrm{Sp}_2(S) = \left\{1, 2, \dots, \left\lfloor \frac{3g}{2} \right\rfloor \right\}.$$

This shows that, in the curve complex, there are many pairs of vertices which are connected by finitely many geodesics but not by a unique geodesic.

For distance two, the structure of geodesics can be described by their middle vertices. Suppose that $\ell_0, \ell_2 \in \mathcal{C}^0(S)$ satisfy

$$d_S(\ell_0, \ell_2) = 2,$$

and suppose that (ℓ_0, ℓ_2) is of type F but not strongly type F. Then there exist only finitely many geodesics connecting ℓ_0 and ℓ_2, and the number of such geodesics is at least two. We denote all of them by

$$[\ell_0, \ell_1^{(i)}, \ell_2] \quad (i = 1, \dots, n).$$

We will use the following lemma in the proof of the main theorem.

Lemma 1 *Under the above notation, for any $1 \leq i < j \leq n$, the curves $\ell_1^{(i)}$ and $\ell_1^{(j)}$ are disjoint.*

This lemma can be proved by an argument similar to the one used in [6]: if two middle vertices intersect, then one can find infinitely many curves giving distance-two geodesics. In the present article, Lemma 1 is used to analyze quasi-strongly keen Heegaard splittings of distance two.

2.3 Heegaard splittings and keen conditions

Let M be a closed orientable 3-manifold. A Heegaard splitting of M is a decomposition

$$M = V_1 \cup_S V_2,$$

where V_1 and V_2 are handlebodies and

$$V_1 \cap V_2 = \partial V_1 = \partial V_2 = S.$$

The surface S is called the Heegaard surface.

For a handlebody V, the disk complex $\mathcal{D}(V)$ is the full subcomplex of $\mathcal{C}(\partial V)$ spanned by vertices represented by essential simple closed curves on ∂V which bound properly embedded disks in V.

The Hempel distance of the Heegaard splitting $V_1 \cup_S V_2$ is defined by

$$d(V_1 \cup_S V_2) = d_S(\mathcal{D}(V_1), \mathcal{D}(V_2)) = \min\{d_S(\alpha, \beta) \mid \alpha \in \mathcal{D}(V_1),\ \beta \in \mathcal{D}(V_2)\}.$$

This distance measures how far the two disk complexes are from each other in the curve complex of the Heegaard surface.

Ido, Jang and Kobayashi [3] introduced refinements of this notion. A Heegaard splitting $V_1 \cup_S V_2$ is called keen if there exists a unique pair

$$(\alpha, \beta) \in \mathcal{D}(V_1) \times \mathcal{D}(V_2)$$

such that

$$d_S(\alpha, \beta) = d(V_1 \cup_S V_2).$$

Thus, in a keen Heegaard splitting, the pair of disk-bounding curves realizing the Hempel distance is unique.

Even if the distance-realizing pair is unique, the geodesic connecting the pair in the curve complex may not be unique. A keen Heegaard splitting is called strongly keen if the unique distance-realizing pair is connected by exactly one geodesic in $\mathcal{C}(S)$. Equivalently, a keen Heegaard splitting is strongly keen if its unique distance-realizing pair is of strongly type F.

Motivated by the distinction between type F and strongly type F, the author introduced in [8] the notion of a quasi-strongly keen Heegaard splitting. Let $V_1 \cup_S V_2$ be a keen Heegaard splitting, and let

$$(\alpha, \beta) \in \mathcal{D}(V_1) \times \mathcal{D}(V_2)$$

be the unique pair realizing the Hempel distance. We say that $V_1 \cup_S V_2$ is quasi-strongly keen if the pair (α, β) is of type F in $\mathcal{C}(S)$.

In other words, a quasi-strongly keen Heegaard splitting is a keen Heegaard splitting for which the unique distance-realizing pair is connected by only finitely many geodesics. Thus, quasi-strongly keen Heegaard splittings form a natural intermediate class between strongly keen Heegaard splittings and keen Heegaard splittings.

2.4 Motivation from Goeritz groups

We also recall a motivation coming from Goeritz groups. Let $M = V_1 \cup_S V_2$ be a Heegaard splitting. The Goeritz group of this splitting is the group of isotopy classes of orientation-preserving self-homeomorphisms of M which preserve each of the handlebodies V_1 and V_2 setwise.

The structure of Goeritz groups is closely related to the Hempel distance of Heegaard splittings. For example, questions about when the Goeritz group is finite, finitely generated, or finitely presented have been studied in connection with the distance of the splitting; see [5] and the references therein.

In particular, Iguchi and Koda [5] proved that if a Heegaard splitting of genus at least two is strongly keen and has distance two, then its Goeritz group is either a finite cyclic group or a finite dihedral group. They also proved that if a Heegaard splitting is keen and has distance at least three, then the same conclusion holds.

This provides one motivation for considering conditions between keen and strongly keen Heegaard splittings.

3 Main result

We now state the main theorem reviewed in this article.

Theorem 1 *Every quasi-strongly keen Heegaard splitting of genus at least three and distance two is strongly keen.*

The proof of Theorem 1 uses Lemma 1 to control the possible middle vertices of distance-two geodesics. We do not give the proof here; for the complete proof, we refer the reader to [8].

Remark. Theorem 1 does not hold in low genus. In genus one, for example, the genus-one Heegaard splitting of the lens space $L(5, 2)$ has a distance-realizing pair which is connected by exactly two geodesics, since the curve complex of the torus is the Farey graph. For details, see Appendix B in [4]. In genus two, it is known that any Heegaard splitting is not keen; see Remark 3.2 in [3]. Thus, the assumption that the genus is at least three is essential.

Acknowledgement. The author is grateful to Professor Yeonhee Jang for helpful communications. She also thanks the anonymous referee for valuable comments and suggestions which improved the manuscript. This work was supported by JST SPRING, Grant Number JPMJSP2139.

References

[1] W. J. Harvey, *Boundary structure of the modular group*, In: Riemann Surfaces and Related Topics, Ann. of Math. Stud. **97**, Princeton Univ. Press (1981), 245–251.

[2] J. Hempel, *3-manifolds as viewed from the curve complex*, Topology **40** (2001), no. 3, 631–657.

[3] A. Ido, Y. Jang and T. Kobayashi, *On keen Heegaard splittings*, Advanced Studies in Pure Mathematics 78, Singularities in Generic Geometry (2018), 293—311.

[4] A. Ido, Y. Jang and T. Kobayashi, *On keen bridge splittings of links*, Topology and its Applications 371 (2025), 109355.

[5] D. Iguchi and Y. Koda, *Twisted book decompositions and the Goeritz groups*, Topology Appl. **272** (2020), 107064.

[6] R. Matsuda, K. Oie and H. Shiga, *On the spectrum of the number of geodesics and tight geodesics in the curve complex*, Contemporary Mathematics, Proceedings of the AMS Special Session on Topics in Complex Analysis and Dynamics, to appear.

[7] K. Oie, *On finiteness of the geodesics joining a pair of points in curve complexes*, JP J. Geom. Topol. **30** (2024), 69–82.

[8] K. Oie, *On quasi-strongly keen Heegaard splittings*, Osaka J. Math., to appear.

Graduate School of Humanities and Sciences
Nara Women's University
Nara 630-8506
JAPAN
E-mail address: `yak_oie@cc.nara-wu.ac.jp`

奈良女子大学大学院人間文化総合科学研究科自然科学専攻　大家佳奈子