Graph complexes and diffeomorphism groups

Tadayuki Watanabe
Department of Mathematics, Kyoto University

1 Introduction

Let X be a compact topological manifold. The following general fact is well-known.

Fact 1.1 (Moduli space of smooth manifolds)**.** *There is a space $\mathcal{M}(X)$ such that for any smooth manifold B there is a natural bijection*

$$\mathrm{Bun}(B;X) \cong [B, \mathcal{M}(X)],$$

where

- $\mathrm{Bun}(B;X)$ *is the set of isomorphism classes of smooth fiber bundles over B with fiber homeomorphic to X,*
- $[B, \mathcal{M}(X)]$ *is the set of homotopy classes of continuous maps $B \to \mathcal{M}(X)$.*

Remark 1.2. There is a relative version $\mathcal{M}_\partial(X)$ for smooth manifolds with a fixed smooth structure on ∂X and fixed behavior near ∂X. The components of $\mathcal{M}_\partial(X)$ are the classifying spaces of the diffeomorphism groups:

$$\mathcal{M}_\partial(X) \cong \coprod_{[\Sigma]} B\mathrm{Diff}_\partial(X_\Sigma),$$

where the disjoint union is taken over all diffeomorphism classes of smooth structures Σ on X that extend the fixed structure on ∂X, and $X_\Sigma = (X, \Sigma)$. Considering such a disjoint union is natural from the viewpoint of smoothing theory.

The following problem is fundamental for the moduli spaces.

Problem 1.3. *Determine the group $\pi_i(\mathcal{M}(X))$ (or $\pi_i(\mathcal{M}_\partial(X))$).*

This problem can be considered as a natural analogue of the diffeomorphism classification of smooth manifolds, which studies $\pi_0(\mathcal{M}(X))$ (or $\pi_0(\mathcal{M}_\partial(X))$). Higher analogues of some of the major problems in differential topology are summarized in Table 1.

manifold (π_0)	family of manifolds (π_i)
• Classification up to diffeo. ($\pi_0(\mathcal{M}_\partial(X))$)	• $\pi_i(\mathcal{M}_\partial(M))$ e.g. $\pi_1(\mathcal{M}_\partial(X)) = \pi_0(\mathrm{Diff}_\partial(X))$.
• Smooth Poincaré conjecture ($\pi_0(\mathcal{M}_\partial(D^d)) = 0$?)	• Smale conjecture ($\mathcal{M}_\partial(D^d) \simeq *$?)
• h-cobordism theorem	• Pseudo-isotopy implies isotopy, i.e. $\pi_1(\mathcal{M}_\sqcup(X \times I)) \to \pi_1(\mathcal{M}_\partial(X))$ is zero (Cerf).
• Schoenflies Problem (Is any $S^3 \hookrightarrow S^4$ isotopic to standard S^3?)	• $\pi_i(\mathrm{Emb}_\partial(D^3, D^4)) = 0$? ($\pi_i(\mathrm{Emb}_\partial(D^3, D^4)) \cong \pi_{i-1}(\mathcal{M}_\partial(D^4))$ for $i \geq 1$.)
• Classification of 2-knots ($\pi_0(\mathrm{Emb}_\partial(D^2, D^4))$)	• $\pi_i(\mathrm{Emb}_\partial(D^2, D^4))$ ($\exists \pi_i(\mathrm{Emb}^{\mathrm{fr}}_\partial(D^2, D^4))_{\mathrm{std}} \hookrightarrow \pi_i(\mathcal{M}_\partial(D^3 \times S^1))$ for $i \geq 1$)

Table 1: Higher analogues of problems in differential topology.

1.1 Related problems

Observation 1.4 (Budney–Gabai [BG]). *If we understand $\pi_1(\mathcal{M}_\partial(D^3 \times S^1)) = \pi_0(\mathrm{Diff}_\partial(D^3 \times S^1))$ well enough, the 4D Schoenflies problem will be solved.*

Theorem 1.5 (Morlet [BL]). *We have*

$$\mathcal{M}_\partial(D^d) \simeq \Omega^d \mathrm{Top}(d)/O(d) \qquad (\textit{if } d \neq 4),$$

where

- $\mathrm{Top}(d) = \mathrm{Homeo}(\mathbb{R}^d, 0)$ *is the "structure group" for topological* $\mathbb{R}^d$*-bundles,*
- $O(d)$ *is the orthogonal group, which is the structure group for linear* $\mathbb{R}^d$*-bundles.*

Hence determining $\pi_i(\mathcal{M}_\partial(D^d))$ is closely related to the classification of topological $\mathbb{R}^d$-bundles, at least for $d \neq 4$.

1.2 Recent results for $X = D^d$

It is known that there is a homotopy equivalence

$$\mathcal{M}_\partial(D^d) \simeq \Theta_d \times B\mathrm{Diff}_\partial(D^d)$$

for $d \geq 5$, where Θ_d is the group of h-cobordism classes of homotopy d-spheres ([GRW]). Several strong results on $\mathcal{M}_\partial(D^d)$ were obtained after a pioneering work by M. Weiss [We].

Theorem 1.6 (Kupers [Kup]). *$B\mathrm{Diff}_\partial(D^d)$ is of finite type (i.e. π_1 finite and π_i ($i \geq 2$) finitely generated) for $d \neq 4, 5, 7$.*

Theorem 1.7 (Kupers–Randal-Williams [KuRW]). *For $2n \geq 6$,*

$$\pi_i(\mathcal{M}_\partial(D^{2n})) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & (i \geq 2n-1 \textit{ and } i \equiv 2n-1 \textit{ mod } 4), \\ 0 & (\textit{otherwise}) \end{cases}$$

outside the "bands" $\bigcup_{k \geq 2} [2kn - 4k - 1, 2kn - 1]$ of degrees.

Theorem 1.8 (Krannich–Randal-Williams [KrRW]). *For $2n+1 \geq 5$ and $i \leq 3n-8$,*

$$\pi_i(\mathcal{M}_\partial(D^{2n+1})) \otimes \mathbb{Q} \cong K_{i+1}(\mathbb{Z}) \otimes \mathbb{Q} \oplus \begin{cases} \mathbb{Q} & (i \equiv 2n-2 \ \mathit{mod}\ 4,\ j \geq 2n-2), \\ 0 & (\mathit{otherwise}), \end{cases}$$

where $K_{i+1}(\mathbb{Z})$ is the algebraic K-theory of $\mathbb{Z}$.

2 Kontsevich graph complex $(\mathcal{G}, \partial)$

Before stating the main result, we recall the definition of Kontsevich's graph complex ([Ko]). Let $\mathcal{G}$ be the vector space over $\mathbb{Q}$ spanned by pairs (Γ, o), where

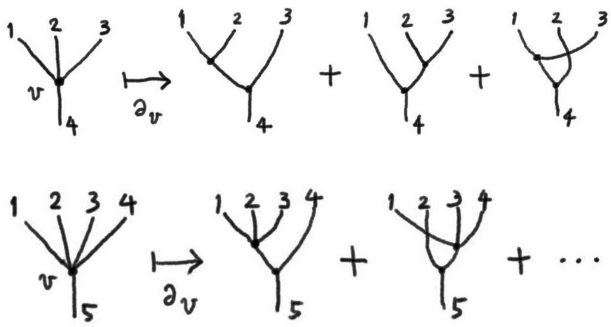
- Γ is a finite connected graph with only ≥ 3-valent vertices,
- o is a choice of orientation of $\mathbb{R}^{\{\text{edges of } \Gamma\}}$,

modulo the relation

$$(\Gamma, -o) \sim -(\Gamma, o).$$

The boundary operator $\partial \colon \mathcal{G} \to \mathcal{G}$ is defined by

$$\partial\Gamma = \sum_{\substack{v:\\ \text{vertex of } \Gamma}} \partial_v \Gamma,$$



$$\vdots$$

The terms in the right hand sides are equipped with naturally induced orientations. Thanks to the orientation, we have $\partial^2 = 0$. The homology $H(\mathcal{G})$ is called the *graph homology*.

2.1 Bigrading

Let $E(\Gamma)$ (resp. $V(\Gamma)$) be the set of all edges (resp. vertices) of a graph Γ. We equip $\mathcal{G}$ with the following bidegree:

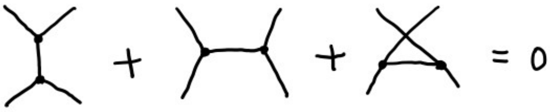
$$\mathcal{G} = \bigoplus_{k,\ell \geq 0} \mathcal{G}^{(k,\ell)},$$

where $k = |E(\Gamma)| - |V(\Gamma)| = b_1(\Gamma) - 1$, $\ell = 2|E(\Gamma)| - 3|V(\Gamma)|$ (excess from 3-valent graphs). Then the restriction of ∂ to $\mathcal{G}^{(k,\ell)}$ is $\mathcal{G}^{(k,\ell)} \to \mathcal{G}^{(k,\ell+1)}$. The graph homology is decomposed accordingly.

$$H(\mathcal{G}) = \bigoplus_{k,\ell \geq 0} H_{k,\ell}(\mathcal{G}).$$

Example 2.1.

$$H_{k,0}(\mathcal{G}) = \mathcal{G}^{(k,0)}/\mathrm{IHX}.$$

IHX: 

The space $\mathcal{G}^{(k,0)}$ is spanned by 3-valent graphs with $2k$ vertices. The dimensions of $H_{k,0}(\mathcal{G})$ for lower k were computed as follows ([BNM]).

$k = b_1 - 1$	1	2	3	4	5	6	7	8	9
$\dim H_{k,0}(\mathcal{G})$	0	1	0	0	1	0	0	0	1

Example 2.2 (Euler characteristic)**.** Since the subcomplex $\mathcal{G}^{(k,*)} = \bigoplus_{\ell \geq 0} \mathcal{G}^{(k,\ell)}$ of $\mathcal{G}$ is of finite length, its Euler characteristic is defined. The values of $\chi(\mathcal{G}^{(k,*)}$ for lower k are as follows ([WZ]).

$k = b_1 - 1$	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\chi(\mathcal{G}^{(k,*)})$	0	1	0	2	-1	3	-1	4	-4	6	-5	$\cdots$

3 3-valent graphs

Kontsevich's characteristic class is defined by integrals over the configuration spaces of points on the fibers of a smooth $\mathbb{R}^d$-bundles ([Ko]). It is an element of

$$H^{k(2n-3)+\ell}(\mathcal{M}_\partial^{\mathrm{fr}}(D^{2n}); H_{k,\ell}(\mathcal{G}) \otimes \mathbb{R}),$$

where $\mathcal{M}_\partial^{\mathrm{fr}}(D^{2n})$ differs from $\mathcal{M}_\partial(D^{2n})$ by the fiber sequence

$$\Omega_0^{2n} SO(2n) \to \mathcal{M}_\partial^{\mathrm{fr}}(D^{2n}) \to \mathcal{M}_\partial(D^{2n}).$$

By utilizing the Kontsevich characteristic classes, we obtain the following result.

Theorem 3.1 ([Wa18, Wa21])**.** *For $d = 2n \geq 4$, $k \geq 1$, the Kontsevich characteristic classes give epimorphisms*

$$\pi_{k(2n-3)}(\mathcal{M}_\partial(D^{2n})) \otimes \mathbb{R} \to H_{k,0}(\mathcal{G}) \otimes \mathbb{R}.$$

Remark 3.2. The degree $k(2n-3) \in [2kn - 4n - 1, 2kn - 1]$ (a "band" of Kupers–Randal-Williams). We have similar results for $d = 2n + 1 \geq 5$ ([Wa09]).

To prove Theorem 3.1, we construct D^{2n}-bundle over $D^{k(2n-3)}$ by a higher dimensional analogue of Goussarov–Habiro's surgery ([Gu, Hab]) and detect it by a higher dimensional analogue of Kuperberg–Thurston's computation of configuration space integrals ([KT], see also [Les]).

Corollary 3.3 (Disproof of the Smale conjecture in 4D)**.**

$$\mathrm{Diff}_\partial(D^4) \not\simeq *.$$

Remark 3.4.

- $\mathrm{Homeo}_\partial(D^d) \simeq *$ for $d \geq 1$ (by Alexander trick).
- $\mathrm{Diff}_\partial(D^d) \simeq *$ for $d = 1, 2, 3$ ($d = 2$: Smale [Sm], $d = 3$: Hatcher [Hat]).
- $\mathrm{Diff}_\partial(D^d) \not\simeq *$ for $d \geq 5$ (Cerf [Ce], Milnor [Mil], Kervaire–Milnor [KM],...).
- $d = 4$: Kirby, Problem 4.34 [Ki].

Theorem 3.5 (J. Lin, Y. Xie [LX])**.** *Let* $\mathrm{Top}(4) = \mathrm{Homeo}(\mathbb{R}^4, 0)$*. We have*

$$\pi_{k+4}(\mathrm{Top}(4)) \otimes \mathbb{Q} \neq 0$$

whenever $H_{k,0}(\mathcal{G}) \neq 0$*.*

Remark 3.6. It is unknown whether the Morlet equivalence

$$\mathrm{Diff}_\partial(D^d) \simeq \Omega^{d+1}\mathrm{Top}(d)/O(d)$$

is true for $d = 4$. (This is known to be true for $d \neq 4$.)

Remark 3.7. Some properties of our graph surgery construction about concordance are described in [BW].

4 Theta graph in 4-manifolds

Observation 4.1 (Budney–Gabai [BG])**.** *If we understand* $\pi_1(\mathcal{M}_\partial(D^3 \times S^1)) = \pi_0(\mathrm{Diff}_\partial(D^3 \times S^1))$ *well enough, the 4D Schoenflies problem will be solved.*

Theorem 4.2 (Budney–Gabai [BG], W [Wa20])**.** *The group* $\pi_1(\mathcal{M}_\partial(D^3 \times S^1))$ *is of infinite rank.*

Our proof uses a twisted analogue of the Kontsevich characteristic class for 2-loop graphs. It turned out that the infinite rank subgroup of $\pi_1(\mathcal{M}_\partial(D^3 \times S^1))$ does not give a counterexample to the 4D Schoenflies conjecture.

The same method works for the cylinder over a spherical 3-manifold.

Theorem 4.3 (W [Wa20], Ohta–W [OW23])**.** *Let* Σ^3 *be a spherical 3-manifold,* $\Sigma^3 \not\cong S^3$*. Let* $\mathcal{A}_\Theta^{\mathrm{even}}(\mathbb{C}\pi)$ *be the space of 2-loop 3-valent graphs with decorations by* $\mathbb{C}\pi$ *modulo the IHX relation and the "holonomy relation". Then we have*

$$\dim \pi_1(\mathcal{M}_\partial(\Sigma^3 \times I)) \otimes \mathbb{Q} \geq \dim \mathcal{A}_\Theta^{\mathrm{even}}(\mathbb{C}\pi).$$

Remark 4.4. In a joint work with H. Kodani ([KW]), we found formulas computing $\dim \mathcal{A}_\Theta^{\mathrm{even}}(\mathbb{C}\pi)$. For example, when $p \equiv 0 \pmod 6$, we have

$$\dim \mathcal{A}_\Theta^{\mathrm{even}}(\mathbb{C}[D_{4p}^*]) = \frac{p^2}{3} - \frac{p}{2} + 1,$$

where D_{4p}^* is the binary dihedral group of order $4p$.

5 General graphs

In a joint work with Boris Botvinnik, we obtained the following result.

Theorem 5.1 (Botvinnik–W [BWI]). *Let $\mu \geq 1$. Let $\mathcal{G}^{\leq \mu}$ be the truncation of $\mathcal{G}$ to the excess $\leq \mu$ part. For $2n \gg 0$, we have a natural chain map*

$$\overline{\phi}\colon \mathcal{G}^{\leq \mu} \to S_*(\mathcal{M}_\partial^{\mathrm{fr}}(D^{2n}); \mathbb{Q})$$

that induces a monomorphism

$$H_{k,m}(\mathcal{G}^{\leq \mu}) \hookrightarrow H_{k(2n-3)+m}(\mathcal{M}_\partial^{\mathrm{fr}}(D^{2n}); \mathbb{Q})$$

for $m \leq \mu - 1$.

Thus the rational homology of the moduli space $\mathcal{M}_\partial^{\mathrm{fr}}(D^{2n})$ may be very big, as suggested by the known computations of the graph homology.

To prove this theorem, we construct higher excess analogues of Goussarov–Habiro graph claspers, topologically realizing the boundary operator of the graph complex. In particular, the realization of an ℓ-valent vertex is given by a certain family of string links $I^{a_1} \cup \cdots \cup I^{a_\ell} \to I^{2n}$.

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Department of Mathematics
Kyoto University
Kyoto 606-8502
JAPAN
E-mail address: `tadayuki.watanabe@math.kyoto-u.ac.jp`

京都大学大学院理学研究科・数学教室　渡邉忠之