Combinatorics of ordered ideal triangulations and Andersen-Kashaev volume conjecture

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1 Introduction

In the early 2010s, Andersen and Kashaev constructed a TQFT based on quantum Teichmüller theory. In particular, they defined a partition function associated with every ordered ideal triangulation of a hyperbolic knot complement in $\mathbb{S}^3$ equipped with an angle structure. The Andersen–Kashaev volume conjecture predicts that this partition function can be expressed in terms of a Jones function of the knot and that, in the semiclassical limit, it decays exponentially at a rate determined by the hyperbolic volume of the knot complement.

In [5], Ben Aribi and the author introduced a new combinatorial notion called FAMED triangulations, which plays an important role in the study of the Andersen–Kashaev volume conjecture for Teichmüller TQFT. The main result of [5] states that, if an ordered ideal triangulation is both FAMED and geometric, then the Andersen–Kashaev volume conjecture and its generalizations hold.

The main goal of this survey is to introduce the Teichmüller TQFT partition function, the associated volume conjecture, and the FAMED conditions introduced in [5]. Throughout this paper, we restrict our attention to hyperbolic knot complements in $\mathbb{S}^3$, although the same definition applies more generally to one-cusped 3-manifolds with trivial second homology.

2 Ordered ideal triangulation

The partition function in Teichmüller TQFT is defined using triangulations equipped with an additional structure called an *ordering*. More precisely, an *ordered tetrahedron T* is a tetrahedron whose four vertices are labeled by (0,1,2,3). Each edge of T is oriented from the vertex with the smaller label to the vertex with the larger label. An ordered ideal tetrahedron is called *positive* or *negative* according to whether it is depicted on the left- or right-hand side of Figure 1, respectively. We denote the sign of T by $\epsilon(T) \in \pm 1$.

Definition 2.1. *An ordered triangulation is a triangulation consisting of ordered tetrahedra such that every face-pairing map preserves the induced orientations of the edges.*

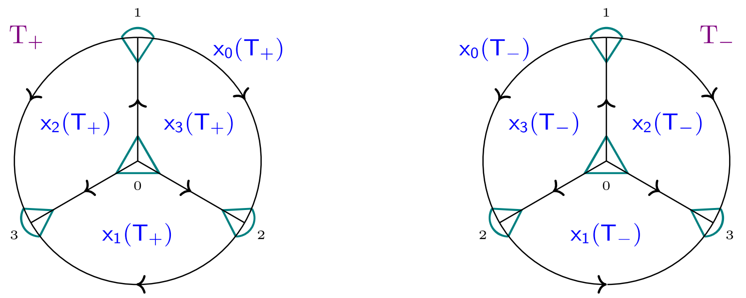


Figure 1: A positive tetrahedron T_+ is shown on the left, and a negative tetrahedron T_- is shown on the right. For each $T = T_\pm$ and $k = 0, 1, 2, 3$, the face opposite to the k-th vertex is denoted by $x_k(T)$ and is indicated in the figure.

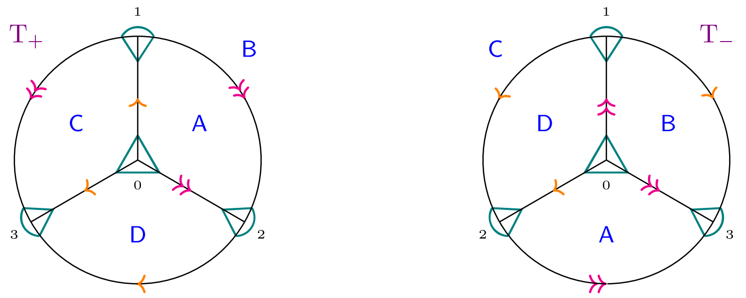


Figure 2: Thurston's ideal triangulation of the figure eight knot complement with an ordering.

For example, as shown in Figure 2, Thurston's well-known ideal triangulation of the figure eight knot complement can be endowed with an ordering that makes it an ordered triangulation.

This additional combinatorial structure of triangulation provides an algebraic interpretation of the triangulation as follows. We regard each ordered tetrahedron as a suitable linear map and the gluing of tetrahedra as a tensor contraction of such linear maps. Note that when a tetrahedron is viewed from outside, two of the three oriented edges travel in the same clockwise or counterclockwise direction. Using the right hand rule, this convention induces two incoming faces and two outgoing faces for each tetrahedron. We assign a finite dimensional vector space V to each face, so that the two incoming faces and the two outgoing faces each naturally correspond to a copies of $V \otimes V$. An ordered tetrahedron can therefore be interpreted as a linear map $S^\pm$ from the tensor product $V \otimes V$ to itself. In particular, once we fix a basis $\{e_1, \ldots, e_N\}$ of V, we can write $S^\pm : V \otimes V \to V \otimes V$ as

$$S^\pm(e_i \otimes e_j) = \sum_{k,l=1}^{N} (S^\pm)_{i,j}^{k,l} e_k \otimes e_l,$$

where $(S^\pm)_{i,j}^{k,l}$'s are the matrix coefficients of $S^\pm$.

To obtain a scalar from an ordered triangulation, whenever we glue faces together, we identified the corresponding indices and summing over all possible values. For example, applying this construction to the triangulation in Figure 2 yields an expression of the form

$$\sum_{A,B,C,D=1}^{N} (S^{+})_{A,D}^{B,C}(S^{-})_{B,C}^{A,D}.$$

To ensure that the resulting scalar is independent of the chosen triangulation, the operators $S^{\pm}$ cannot be chosen arbitrarily. In particular, they must satisfy certain algebraic identities corresponding to combinatorial moves that connect different triangulations. One such relation is the pentagon identity, which algebraically realizes the 2–3 Pachner move illustrated in Figure 3.

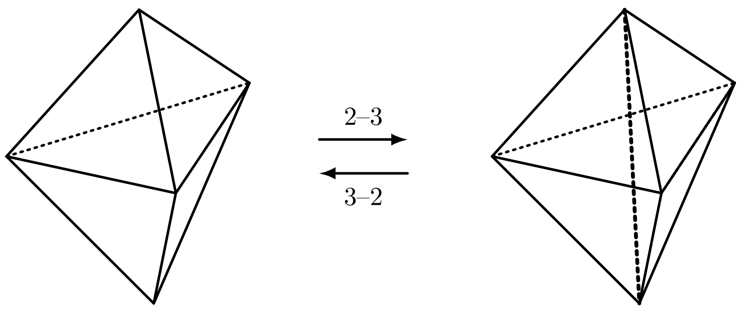


Figure 3: The 2–3 and 3–2 Pachner moves. On the left, the bipyramid is decomposed into two tetrahedra, one above and one below, glued together along a common face. On the right, a new interior edge is introduced and the bipyramid is decomposed into three tetrahedra.

It turns out that such kind of suitable linear map $S^{\pm}$ can be constructed using representation theory of quantum groups. In particular, when $S^{\pm}$ arise from cyclic representations of the Borel subalgebra of $\mathrm{U}_q(\mathfrak{sl}_2(\mathbb{C}))$, the resulting state sums are closely related to Kashaev's invariant [7] and to the quantum hyperbolic invariants of Baseilhac–Benedetti [6].

3 Partition function in Teichmüller TQFT

In [2], Andersen and Kashaev applied an analogous construction using infinite-dimensional representations. In this setting, the finite sums are replaced by improper integrals, giving rise to questions of absolute convergence. Remarkably, these convergence issues are closely connected to the theory of *angle structures* introduced by Casson and Rivin.

Definition 3.1. *An angle structure of a triangulation X is an assignment of dihedral angles in $(0, \pi)$ to edges of the tetrahedra such that*

1. *opposite edges of each tetrahedron share the same dihedral angle,*
2. *the angle sum of each truncated triangle is equal to π, and*
3. *the angle sum around each edge of the triangulation is equal to 2π.*

The set of all angle structure is denoted by $\mathscr{A}_X$.

For each T, the angles at the edges $01, 02, 03$ are denoted by $\alpha_1(T), \alpha_2(T), \alpha_3(T)$ respectively. Recall that for any ideal triangulation of a knot complement, the truncated triangles around the ideal vertices are glued together to form a triangulation of the boundary a small tubular neighborhood of the knot. Given a peripheral curve γ on the boundary torus, up to isotopy, we can assume that γ cuts off a corner of each triangle its intersect, as illustrated in Figure 4. Given an angle structure $\alpha \in \mathscr{A}_X$, the *angular holonomy* of γ is defined to be the signed sum of the angles at the corners cut off by γ, where the sign is positive/negative if the corner is on the left/right of the curve γ. See Figure 4 for a concrete example.

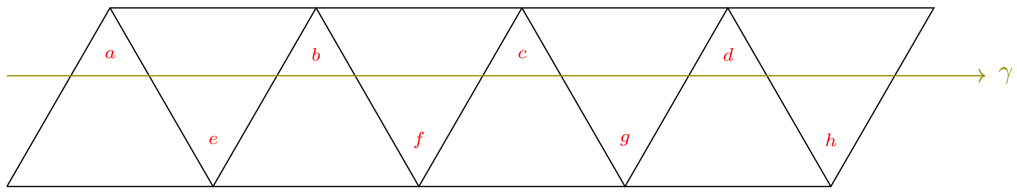


Figure 4: The angular holonomy of the peripheral curve γ is given by $a+b+c+d-e-f-g-h$.

Now, let $\mathrm{b} > 0$ and $\hbar = (\mathrm{b}+\mathrm{b}^{-1})^{-2} > 0$ We will incorporate the dihedral angles into *Faddeev's quantum dilogarithm function*, a special function that plays an important role in quantum topology.

Definition 3.2. *For $z \in \mathbb{R} + i(-1/2\sqrt{\hbar}, 1/2\sqrt{\hbar})$, the Faddeev's quantum dilogarithm is the holomorphic function defined by*

$$\Phi_{\mathrm{b}}(z) = \exp\left(\frac{1}{4}\int_{w\in\mathbb{R}+i0^+} \frac{e^{-2izw}}{\sinh(\mathrm{b}w)\sinh(\mathrm{b}^{-1}w)w} dw\right).$$

The quantum dilogarithm function is the fundamental building block assigned to the tetrahedron. It satisfies several important properties and asymptotics properties. In particular, when $\mathrm{b} \to 0$,

$$\Phi_{\mathrm{b}}\left(\frac{z}{2\pi\mathrm{b}}\right) = \exp\left(\frac{-i}{2\pi\mathrm{b}^2}\mathrm{Li}_2(-e^z)\right)\left(1 + O_{\mathrm{b}\to0^+}(\mathrm{b}^2)\right),$$

where Li_2 is the dilogarithm function, which can be used to compute the hyperbolic volume of an ideal tetrahedron.

Using the ingredients introduced above, we can now state the definition of the Teichmüller TQFT partition function. For the remainder of this survey, we restrict our attention to ideal triangulations of hyperbolic knot complements in $\mathbb{S}^3$.

Definition 3.3. *Let X be an ordered ideal triangulation of $\mathbb{S}^3 \setminus K$ with an angle structure $\alpha \in \mathscr{A}_X$. The partition function of (X, α) is defined by*

$$\mathscr{Z}_\hbar(X,\alpha) = \int_{\mathbf{x}\in\mathbb{R}^{2N}} \prod_{tetra.T} \frac{\delta(x_0(T)-x_1(T)+x_2(T))e^{(2\pi i x_0(T)+\epsilon(T)\alpha_3(T)/\sqrt{\hbar})(x_3(T)-x_2(T)))}}{\Phi_{\mathrm{b}}\Big(\epsilon(T)((x_3(T)-x_2(T)) - \frac{i}{2\pi\sqrt{\hbar}}(\alpha_2(T)+\alpha_3(T)))\Big)^{\epsilon(T)}} d\mathbf{x},$$

where the product in the integrand is over all tetrahedra in X,

- Φ_{b} *is the Faddeev's quantum dilogarithm function;*
- $x_k(T)$ *is the face opposite to the k-th vertex of T;*
- $\alpha_2(T), \alpha_3(T)$ *are angles on* $02, 03$ *edges of T;*
- $\epsilon(T) \in \{\pm 1\}$ *is the sign of T; and*
- δ *is the Dirac delta distribution.*

Intuitively, the presence of the Dirace delta distribution means that many "entires" of the linear maps $S^{\pm}$ vanish. More precisely, the product of the Dirac delta distribution restricts the integral to the linear subspace defined by the system of linear equations $x_0(T) - x_1(T) + x_2(T)$ for all tetrahedron T.

In [2], Andersen and Kashaev proved that the resulting integral converges absolutely. Moreover, they prove that the modulus $|\mathscr{Z}_\hbar(X, \alpha)|$ of the partition function is invariant under the angled 3-2 Pachner move, which is a 3-2 Pachner move between two ideal triangulations with compatible angle structures.

Furthermore, in [2], Andersen and Kashaev propose the following volume conjecture for the partition function, which relates the partition function with volume of the underlying hyperbolic 3-manifolds. We use the following version of Andersen-Kashaev volume conjecture as stated in [3].

Conjecture 3.4. *Let K be a hyperbolic knot in $\mathbb{S}^3$. There exists an ordered ideal triangulation X of $\mathbb{S}^3 \setminus K$ and a Jones function J_X such that :*

1. *for any angle structures $\alpha \in \mathscr{A}_X$ and for all $\hbar > 0$, we have:*

$$|\mathscr{Z}_\hbar(X, \alpha)| = \left| \int_{\mathbb{R} + i \frac{\mu_X(\alpha)}{2\pi\sqrt{\hbar}}} J_X(\hbar, x) e^{\frac{1}{2\sqrt{\hbar}} x \lambda_X(\alpha)} dx \right|,$$

   where $\lambda_X(\alpha)$ is the angular holonomies of the preferred longitude of K.

2. *In the semi-classical limit $\hbar \to 0^+$, we retrieve the hyperbolic volume of K as:*

$$\lim_{\hbar \to 0^+} 2\pi\hbar \log |J_X(\hbar, 0)| = -\mathrm{Vol}(\mathbb{S}^3 \setminus K).$$

The conjecture is proved for the hyperbolic knots 4_1 and 5_2 by Andersen–Kashaev [2], 6_1 by Andersen–Nissen [1], 7_3 by Uemura [8], and all hyperbolic twist knots by Ben Aribi–Guéritaud–Piguet-Nakazawa [3].

4 Andersen-Kashaev volume conjecture for FAMED geometric triangulations

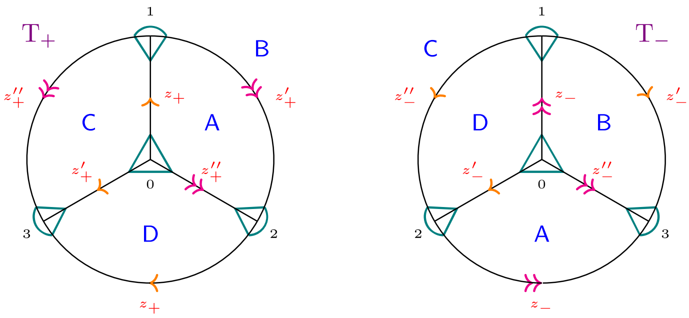
Note that two distinguished linear combinations of face variables appear in Definition 3.3, namely, $x_0 - x_1 + x_2$ and $x_2 - x_3$. We now introduce the *face adjacency matrices*, which encode this combinatorial information associated with an ordered ideal triangulation. First, we let $\mathscr{E}$ be the diagonal matrix whose diagonal entries encode the signs ± 1 of the tetrahedra. For each tetrahedron T and each $k = 0, 1, 2, 3$, let $x_k(T)$ denote the face opposite the k-th vertex of T, and let $\mathscr{X}_k$ be the coefficient matrix with entries

$$(\mathscr{X}_k)_{i,j} := \delta_{j\text{-th face},\, x_k(T_i)}.$$

Define

$$\mathscr{B} := \begin{pmatrix} 0_N \\ \mathscr{E} \end{pmatrix} \quad \text{and} \quad \mathscr{A} := \begin{pmatrix} \mathscr{X}_0 - \mathscr{X}_1 + \mathscr{X}_2 \\ \mathscr{X}_2 - \mathscr{X}_3 \end{pmatrix}.$$

See Figure 5 for the face adjacency matrices of Thurston's ideal triangulation.



$$\mathscr{X}_0 = \begin{array}{c} \\ T_+ \\ T_- \end{array}\!\!\begin{array}{c} \begin{matrix} \mathsf{A} & \mathsf{B} & \mathsf{C} & \mathsf{D} \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$\mathscr{X}_1 = \begin{array}{c} \\ T_+ \\ T_- \end{array}\!\!\begin{array}{c} \begin{matrix} \mathsf{A} & \mathsf{B} & \mathsf{C} & \mathsf{D} \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\mathscr{X}_2 = \begin{array}{c} \\ T_+ \\ T_- \end{array}\!\!\begin{array}{c} \begin{matrix} \mathsf{A} & \mathsf{B} & \mathsf{C} & \mathsf{D} \end{matrix} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$\mathscr{X}_3 = \begin{array}{c} \\ T_+ \\ T_- \end{array}\!\!\begin{array}{c} \begin{matrix} \mathsf{A} & \mathsf{B} & \mathsf{C} & \mathsf{D} \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\mathscr{A} = \begin{pmatrix} \mathscr{X}_0 - \mathscr{X}_1 + \mathscr{X}_2 \\ \mathscr{X}_2 - \mathscr{X}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

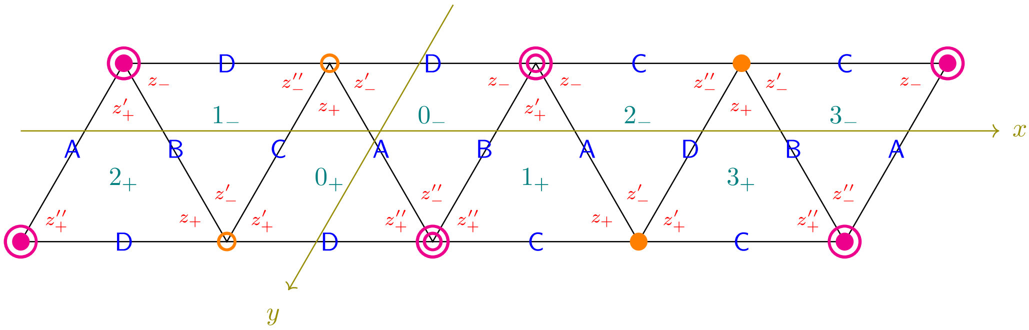
$$\mathscr{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathscr{E} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ (matrix of tetrahedra signs)}$$

Figure 5: This figure from [5, Figure 1] shows an ordered Thurston's triangulation of $M = \mathbb{S}^3 \setminus 4_1$ and its face adjacency matrices

We are going to relate the face adjanceny matrices with the *Neumann–Zagier matrices*, which plays an important role in studying hyperbolic geometry of the manifold. Figure 6 shows the cusp triangulation of the torus obtained from the triangulation in Figure 5 by truncating the vertices. The gluing equations corresponding to the orange single arrow and the pink double arrow can be read off from the cusp triangulation by going around a vertex, which represents a truncated edge. Similarly, the holonomy equations associated with the meridian y and the preferred longitude $l = x + 2y$ can be read from the corresponding peripheral curves. The associated Neumann–Zagier matrices $\mathbf{G}, \mathbf{G}', \mathbf{G}''$ with respect to the preferred longitude l are detailed in Figure 6. We define

$$\mathbf{A} := \mathbf{G} - \mathbf{G}' \quad \text{and} \quad \mathbf{B} := \mathbf{G}'' - \mathbf{G}'.$$



(→) starts at ○ and ends at ● : $2\mathrm{Log}(z_+) + \mathrm{Log}(z'_+) + 2\mathrm{Log}(z'_-) + \mathrm{Log}(z''_-) = 2i\pi$

(↠) starts at ◎ and ends at ◉ : $\mathrm{Log}(z'_+) + 2\mathrm{Log}(z''_+) + 2\mathrm{Log}(z_-) + \mathrm{Log}(z''_-) = 2i\pi$

Meridian y : $\mathrm{H}^{\mathbb{C}}_{X,y}(\mathbf{z}) = -\mathrm{Log}(z'_-) + \mathrm{Log}(z''_+)$

Preferred longitude $x + 2y$: $\mathrm{H}^{\mathbb{C}}_{X,x+2y}(\mathbf{z}) = 2i\pi - 4\mathrm{Log}(z'_-) - 2\mathrm{Log}(z''_-)$

$$\mathbf{G} = \begin{array}{c} \\ (\rightarrow) \\ x+2y \end{array} \begin{array}{c} \begin{array}{cc} z_+ & z_- \end{array} \\ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \quad \mathbf{G}' = \begin{array}{c} \\ (\rightarrow) \\ x+2y \end{array} \begin{array}{c} \begin{array}{cc} z'_+ & z'_- \end{array} \\ \begin{pmatrix} 1 & 2 \\ 0 & -4 \end{pmatrix} \end{array}, \quad \mathbf{G}'' = \begin{array}{c} \\ (\rightarrow) \\ x+2y \end{array} \begin{array}{c} \begin{array}{cc} z''_+ & z''_- \end{array} \\ \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \end{array}$$

Figure 6: This figure from [5, Figure 2] shows the cusp triangulation of $\mathbb{S}^3 \setminus 4_1$ and the Neumann-Zagier matrices

Definition 4.1. *Let K be a hyperbolic knot in $\mathbb{S}^3$. Let X be an ordered ideal triangulation of M with N tetrahedra, and let $\mathscr{A}$ be the face adjacency matrix associated with X as above. Let $l \in \pi_1(\partial M)$ be the preferred longitude of K. Let $\mathbf{A}$ and $\mathbf{B}$ be the Neumann–Zagier matrices with respect to l as above. We say that X is FAMED, for "Face Adjacency Matrices with Edge Duality", with respect to l if:*

1. *the space of angle structures is nonempty, so that the partition function is well-defined;*
2. $\det \mathscr{A} \neq 0$*;*
3. $\det \mathbf{B} \neq 0$*;*
4. *we have*

$$\mathbf{B}^{-1}\mathbf{A} = \mathscr{X}_0\mathscr{A}^{-1}\mathscr{B} + \left(\mathscr{X}_0\mathscr{A}^{-1}\mathscr{B}\right)^{\top} + \frac{\mathscr{E} + \mathrm{Id}_N}{2}.$$

In [5], Ben Aribi and the author proved that the Andersen–Kashaev volume conjecture holds whenever the triangulation is both FAMED and geometric. Thus, the FAMED and geometric conditions provide a purely combinatorial and geometric sufficient criterion for Conjecture 3.4.

Theorem 4.2. *Conjecture 3.4 holds if X is FAMED and geometric.*

5 Further discussion

It is important to note that not every ordered ideal triangulation is FAMED. Nevertheless, in [5], Ben Aribi and the author proved that all hyperbolic twist knot complements admit FAMED geometric triangulations. Moerover, in [4], by using computer verification, Ben Aribi, Guilloux and the author proved that each of 42,000 hyperbolic knots in Snappy census admits a FAMED geometric triangulation. Besides, it is observed in [4] that whenever condition (2) in Definition 4.1 is satisfied, conditions (3) and (4) are also satisfied. In all those examples, one has $\det \mathscr{A} = \pm 1$ and $\det \mathbf{B} = \pm 2$.

Later, in [9], the author generalizes Theorem 4.2 by proposing a generalization of Definition 4.1 and by considering triangulations consisting of both geometric and flat tetrahedra. At the time of writing, computer verification has found such triangulations for more than one million hyperbolic knots, in particular implying Conjecture 3.4 for these cases.

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