Nielsen equivalence classes of mapping class groups

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1 Introduction

Nielsen equivalence is a natural equivalence relation on the set of n-tuples of elements in a group G. We say that two n-tuples $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ of elements in G are *Nielsen equivalent* if one can be transformed into the other via a finite sequence of the following three transformations, called *elementary Nielsen transformations*:

(1) replacing x_i with x_i^{-1} for some i;

(2) swapping x_i and x_j for some $i \neq j$; and

(3) replacing x_i with $x_i x_j$ for some $i \neq j$.

Elementary Nielsen transformations can be viewed as the group-theoretic version of elementary row operations in linear algebra. One reason why Nielsen equivalence arises naturally is that it provides a classification of certain extensions as follows. Let F_n be the free group of rank n generated by $f_1, \ldots, f_n$, and consider a surjective homomorphism $q_\chi \colon F_n \to G$ defined by $q_\chi(f_i) = x_i$, where $\chi = (x_1, \ldots, x_n)$ is a generating n-tuple of the group G. Based on Nielsen's result [23], which states that the elementary Nielsen transformations on $(f_1, \ldots, f_n)$ generate the automorphism group $\mathrm{Aut}(F_n)$ of F_n, two generating n-tuples χ and χ' are Nielsen equivalent if and only if $q_{\chi'} = q_\chi \circ \Phi$ for some $\Phi \in \mathrm{Aut}(F_n)$.

In this paper, we discuss the Nielsen equivalence classes of generating pairs of the mapping class group $\mathcal{M}_g$ of a closed oriented surface Σ_g of genus g. Recall that $\mathcal{M}_g$ is the group of isotopy classes of orientation-preserving homeomorphisms of Σ_g. A well-known result of Dehn [5] states that $\mathcal{M}_g$ is finitely generated (specifically, by Dehn twists). Subsequent work by Lickorish [18] and Humphries [11] aimed to reduce the number of Dehn twists needed to generate $\mathcal{M}_g$. Ultimately, Humphries determined the minimum number of Dehn twist generators. A minimal generating set of $\mathcal{M}_g$ that is not restricted to Dehn twists was given by Wajnryb [24]. His set consists of only two elements for $g \geq 1$. Since then, various minimal generating sets of $\mathcal{M}_g$ have been constructed (see, for example, [17, 25, 3, 12]). In particular, Hirose and the author [12] found infinitely many distinct generating pairs for $g \geq 9$. From this observation, Marco Linton and Makoto Sakuma posed the following question to the authors:

Question 1.1 (M. Linton and M. Sakuma)**.** *Are the generating pairs given in Theorem 1.2 of [12] Nielsen equivalent?*

Although the authors were unable to provide an answer to this question at the time, by constructing additional generating pairs, Hirose and the author subsequently obtained the following result:

Theorem 1.2 ([13])**.** *For each $g \geq 8$, there are infinitely many Nielsen equivalence classes of generating pairs of the mapping class group $\mathcal{M}_g$.*

The outline of this paper is as follows. In Section 2, we present some fundamental facts on Nielsen equivalence. Section 3 is devoted to outlining the proof of Theorem 1.2. In Section 4, we discuss the stabilization processes of generating pairs of $\mathcal{M}_g$. In the last section, we discuss related topics concerning the relationship between low-dimensional topology and Nielsen equivalence.

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2 Background on Nielsen equivalence

In this section, we present the background on Nielsen equivalence. The first result on Nielsen equivalence is, of course, that of Nielsen [23].

Theorem 2.1 ([23])**.** *Any two generating n-tuples of the free group F_n of rank n are Nielsen equivalent.*

The simplest example of Nielsen equivalence is found in finite cyclic groups.

Example 2.2. *Note that (t^ℓ) is a generating 1-tuple of a finite cyclic group $\mathbb{Z}_m = \langle t \mid t^m = 1 \rangle$ of order m if and only if ℓ satisfies $\gcd(\ell, m) = 1$ and $1 \leq \ell < m$. Therefore, $\mathbb{Z}_m$ has $\phi(m)$ generating 1-tuples, where ϕ is Euler's totient function. Since (1) is the unique Nielsen transformation for a generating 1-tuple, the only 1-tuple Nielsen equivalent to (t^ℓ) is $(t^{n-\ell})$. By the observation above, $\mathbb{Z}_m$ has $\phi(m)/2$ Nielsen equivalence classes of generating 1-tuples for $m > 2$. In particular, for $m = 2, 3, 4,$ and 6, $\mathbb{Z}_m$ has exactly one Nielsen equivalence class.*

Although the author may not be fully acquainted with its complete history, B. H. Neumann and Hanna Neumann were among the early researchers to develop the study of Nielsen equivalence. The following example was given by them:

Example 2.3 ([22])**.** *Let A_5 be the alternating group of degree five. Then, there are three Nielsen equivalence classes of generating pairs of A_5, which are represented by the pairs $((12345), (124)), ((12354), (125)),$ and $((12345), (12354))$.*

For example, we can show that $((12345), (12354))$ and $((123), (345))$ are Nielsen equivalent as follows: Note that $(x, y) \xrightarrow{(1)} (x, y^{-1}) \xrightarrow{(3)} (x, y^{-1}x) \xrightarrow{(1)} (x, x^{-1}y)$, and similarly, $(x, y) \to (xy^{-1}, y)$. Using these, we have

$$\begin{aligned}
&((12345), (12354)) \to ((12345), (12345)^{-1}(12354)) = ((12345), (345)) \\
&\to ((12345)(345)^{-1}, (345)) = ((123), (345)).
\end{aligned}$$

To prove that no two of $((12345), (124))$, $((12354), (125))$, and $((12345), (12354))$ are Nielsen equivalent, an invariant is needed. In fact, for any generating pair (x, y) of a group G, the conjugacy class of the commutator $[x, y] = xyx^{-1}y^{-1}$ is invariant under Nielsen equivalence, up to inversion. More precisely, the following lemma holds:

Lemma 2.4. *If two generating pairs (x_1, x_2) and (y_1, y_2) of a group G are Nielsen equivalent, then $[x_1, x_2]$ is conjugate to either $[y_1, y_2]$ or $[y_1, y_2]^{-1}$.*

The union of the conjugacy classes of $[x_1, x_2]^{\pm 1}$ is called the *Higman invariant.* By using this invariant, we complete the proof of Example 2.3.

The authors were unable to provide an answer to Question 1.1 since the commutators of the generating sets in Theorem 1.3 of [12] are all mutually conjugate. A negative answer to this question would imply the existence of Nielsen equivalence classes of $\mathcal{M}_g$ that cannot be distinguished by the Higman invariant. To approach the question, it is natural to consider the following question:

Question 2.5. *Does there exist an invariant of generating pairs of $\mathcal{M}_g$ under Nielsen equivalence that is finer than the Higman invariant?*

While an invariant word exists for generating pairs (i.e., $n = 2$), the situation for $n \geq 3$ is different:

Theorem 2.6 ([10])**.** *For $n \geq 3$, there exists no invariant word (such as the commutator) for generating n-tuples of a group G.*

The following theorem is a classical result on Nielsen equivalence.

Theorem 2.7 ([9], Grushko Theorem)**.** *Let $G = G_1 * G_2$ be the free product of two finitely generated groups G_1 and G_2. Any generating n-tuple $(x_1, \ldots, x_n)$ of G is Nielsen equivalent to an n-tuple $(y_1, \ldots, y_k, z_1, \ldots, z_{n-k})$ such that $G_1 = \langle y_1, \ldots, y_k \rangle$ and $G_2 = \langle z_1, \ldots, z_{n-k} \rangle$, where $\langle w_1, \ldots, w_m \rangle$ denotes the group generated by $w_1, \ldots, w_m$.*

The following is a standard example obtained from Theorem 2.7.

Example 2.8. *The groups $\mathrm{PSL}(2, \mathbb{Z})$ and $\mathrm{SL}(2, \mathbb{Z})$ each have exactly one Nielsen equivalence class of generating pairs.*

In fact, Theorem 2.7 implies that any generating pair (x_1, x_2) of $\mathrm{PSL}(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ is Nielsen equivalent to a pair (s, t) consisting of a generator s of $\mathbb{Z}_2$ and a generator t of $\mathbb{Z}_3$. This, together with Example 2.2, establishes the result for $\mathrm{PSL}(2, \mathbb{Z})$. Let $S, T \in \mathrm{SL}(2, \mathbb{Z})$ be matrices whose images under the natural projection $p\colon \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{PSL}(2, \mathbb{Z}) \cong \mathrm{SL}(2, \mathbb{Z})/\{\pm I\}$ are s and t, respectively. By applying the same sequence of Nielsen transformations used to transform $(p(A), p(B))$ into (s, t), it follows that any generating pair (A, B) of $\mathrm{SL}(2, \mathbb{Z})$ is Nielsen equivalent to one of (S, T), $(S, -T)$, $(-S, T)$, or $(-S, -T)$. These four generating pairs are Nielsen equivalent to each other in $\mathrm{SL}(2, \mathbb{Z})$ via the relations $S^2 = T^3 = -I$. For example, $(S, T) \xrightarrow{(3)} (S, TS) \xrightarrow{(3)} (S, TS^2) = (S, -T)$. This completes the proof for $\mathrm{SL}(2, \mathbb{Z})$. (In [13], Hirose and the author established the finiteness of the Nielsen equivalence classes for $\mathrm{SL}(2, \mathbb{Z})$, but the exact number of classes was not determined there.)

As far as the author knows, the first result that a certain group has infinitely many Nielsen equivalence classes is the following work of Dunwoody and Pietrowski [6]:

Example 2.9 ([6]). *The trefoil knot group $\langle x, y \mid x^2 = y^3 \rangle$ has infinitely many Nielsen equivalence classes of generating pairs.*

3 Outline of the proof of Theorem 1.2

This section provides an outline of the proof of Theorem 1.2. We review some results on mapping class groups. Throughout this paper, we will use the same symbol for a diffeomorphism and its isotopy class, or a simple closed curve and its isotopy class. The Dehn twist about a simple closed curve c on Σ_g is denoted by t_c. For f_1 and f_2 in $\mathcal{M}_g$, the notation $f_2 f_1$ means that we first apply f_1 and then f_2.

We assume that the surface of this paper is the yz-plane and that Σ_g is embedded in $\mathbb{R}^3$ as in Figure 1 such that it is invariant under the rotation r by $-\frac{2\pi}{g}$ about the x-axis. The symbols a_i, b_i, c_i will always denote the simple closed curves on Σ_g shown in Figure 1 for $i = 1, 2, \ldots, g$.

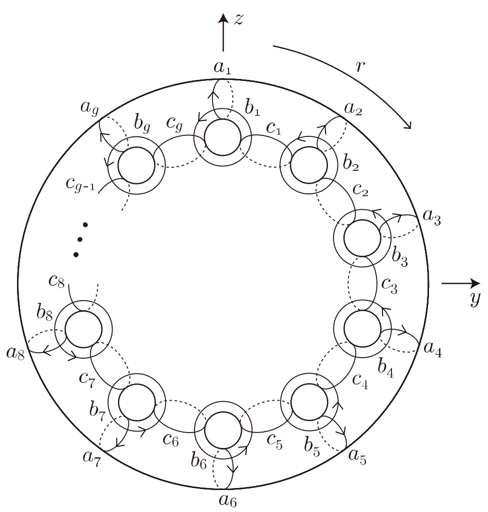

Figure 1: The rotation $r\colon \Sigma_g \to \Sigma_g$ by $-\frac{2\pi}{g}$ about the x-axis and the simple closed curves a_i, b_i, c_i on Σ_g for $i = 1, 2, \ldots, g$.

Theorem 1.2 follows from Theorem 3.1 and Proposition 3.2 below. For abbreviation, we write ρ_m and h_0 instead of $r t_{a_1}^m$ and $t_{b_1} t_{b_2} t_{a_3} t_{c_5}$, respectively.

Theorem 3.1 ([13]). *Let $g \geq 8$. For any integer m, the mapping class group $\mathcal{M}_g$ is generated by the two elements ρ_m and h_0.*

We omit the proof of Theorem 3.1, but the argument is familiar to mathematicians working on (minimal) generating sets of $\mathcal{M}_g$ (see, for example, [24, 12, 13]).

Proposition 3.2 ([13]). *The commutator $[h_0, \rho_\ell]$ is not conjugate to $[h_0, \rho_m]$ if $\ell \neq m$ and $\ell, m > 0$.*

Outline of the proof. For an element f in $\mathcal{M}_g$, we obtain the representation matrix A_ϕ of the induced action ϕ_* on the homology group $H_1(\Sigma_g; \mathbb{R})$ from a representative ϕ (i.e., an orientation-preserving homeomorphism of Σ_g) of f. This yields the homology representation $\mathcal{M}_g \to \mathrm{GL}(2g, \mathbb{R})$. Therefore, if two elements of $\mathcal{M}_g$ are conjugate, then the sets of eigenvalues of their representation matrices coincide. After a short but tedious calculation, we see that the sets of eigenvalues for the representation matrices of $[h_0, \rho_\ell]$ and $[h_0, \rho_m]^{\pm 1}$ are distinct when $\ell \neq m$ and $\ell, m > 0$. Therefore, if $\ell \neq m$ and $\ell, m > 0$, then $[h_0, \rho_\ell]$ is not conjugate to $[h_0, \rho_m]^{\pm 1}$, which completes the proof. $\square$

Proof of Theorem 1.2. We see that Theorem 1.2 immediately follows from Theorem 3.1, Proposition 3.2 and Lemma 2.4. $\square$

While there exist infinitely many Nielsen equivalence classes for $g \geq 8$, the situation is different for small genus. For $g = 1$, the mapping class group $\mathcal{M}_1 \cong SL(2, \mathbb{Z})$ possesses exactly one Nielsen equivalence class of generating pairs (see Example 2.8). The situation for the intermediate cases $2 \leq g \leq 7$ remains unclear, leading to the following question:

Question 3.3. *For $2 \leq g \leq 7$, is the number of Nielsen equivalence classes of generating pairs of $\mathcal{M}_g$ finite or infinite?*

4 Stabilization process of tuples

Since any element y in G generated by $x_1, \ldots, x_n$ can be expressed as a product of elements in the set $\{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$, the $(n+1)$-tuples $(x_1, \ldots, x_n, 1)$ and $(x_1, \ldots, x_n, y)$ are Nielsen equivalent. We call the process of adding 1 to a generating tuple the *stabilization* of the tuple. By an inductive argument, for any $k \geq 1$ and any elements $y_1, \ldots, y_k \in G$, the $(n+k)$-tuples

$$(x_1, \ldots, x_n, \underbrace{1, \ldots, 1}_{k}) \quad \text{and} \quad (x_1, \ldots, x_n, y_1, \ldots, y_k)$$

are Nielsen equivalent. If $(y_1, \ldots, y_n)$ is another generating n-tuple of G, then we obtain the following chain of Nielsen equivalences:

$$(x_1, \ldots, x_n, \underbrace{1, \ldots, 1}_{n}) \sim (x_1, \ldots, x_n, y_1, \ldots, y_n) \sim (y_1, \ldots, y_n, \underbrace{1, \ldots, 1}_{n}),$$

where $\sim$ stands for Nielsen equivalence. For this reason, it is natural to ask whether $(x_1, \ldots, x_n, \underbrace{1, \ldots, 1}_{k})$ and $(y_1, \ldots, y_n, \underbrace{1, \ldots, 1}_{k})$ are Nielsen equivalent for $1 \leq k \leq n-1$. For the generating pairs (h_0, ρ_m) of $\mathcal{M}_g$ mentioned above, it is easy to check that $(h_0, \rho_\ell, 1) \sim (h_0, \rho_m, 1)$ for any $\ell \neq m$. (This observation was communicated to the authors by Seiichi Kamada.) Indeed, for any two generating pairs (x, y) and (x, z) of a group G sharing the same first element, we have

$$(x, y, 1) \sim (x, y, z) \sim (x, z, y) \sim (x, z, 1).$$

This suggests that the infinitely many Nielsen equivalence classes of generating pairs might collapse into a few (or even one) classes after a single stabilization. This observation raises the following question:

Question 4.1. *For $g \geq 2$, is the number of Nielsen equivalence classes of generating 3-tuples of $\mathcal{M}_g$ finite or infinite?*

Remark 4.2. *We note, however, that the stabilization process does not always lead to a single equivalence class. Kapovich and Weidmann [16] showed that for every $n \geq 2$, there exists a group G with $d(G) = n$ admitting two generating n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ such that the $(2n-1)$-tuples*

$$(x_1, \ldots, x_n, \underbrace{1, \ldots, 1}_{n-1}) \quad \textit{and} \quad (y_1, \ldots, y_n, \underbrace{1, \ldots, 1}_{n-1})$$

are not Nielsen equivalent in G.

5 Low-dimensional topology and Nielsen equivalence

In this section, we describe the relationship between low-dimensional topology and Nielsen equivalence. Handle decompositions of connected manifolds are closely related to Nielsen equivalence. Let us consider a handle decomposition of a connected manifold M with a single 0-handle, n 1-handles, and m 2-handles. Suppose that the fundamental group $\pi_1(M)$ of M is presented with n generators and m relators. Then, the 1-handles $h_1^{(1)}, \ldots, h_n^{(1)}$ correspond to the generators, and the 2-handles $h_1^{(2)}, \ldots, h_m^{(2)}$ correspond to the relators. The Nielsen transformations (1), (2), and (3) on generating tuples can be translated into terms of operations on the 1-handles, which do not change the diffeomorphism type of M, as follows:

(1) reversing the orientation of the 1-handle $h_i^{(1)}$;

(2) permuting the indices of $h_i^{(1)}$ and $h_j^{(1)}$; and

(3) sliding the 1-handle $h_i^{(1)}$ over the 1-handle $h_j^{(1)}$.

Note that if we consider the submanifold H of M consisting of a 0-handle and n 1-handles, and the embedding $i \colon H \hookrightarrow M$ inducing the surjection i_* from the fundamental group $\pi_1(H) \cong F_n$ to $\pi_1(M)$, then i_* corresponds to the surjection q_χ described in the Introduction.

In 3-dimensional topology, Heegaard splittings and Nielsen equivalence are deeply connected. A *Heegaard splitting* of genus n for a closed connected 3-manifold M is a decomposition

$$M = H_1 \cup H_2, \quad H_1 \cap H_2 = \partial H_1 = \partial H_2,$$

where each H_i $(i = 1, 2)$ is a handlebody of genus n. We say that two Heegaard splittings $M = H_1^1 \cup H_2^1 = H_1^2 \cup H_2^2$ are *homeomorphic* (resp. *isotopic*) if there is a homeomorphism $f \colon M \to M$ (resp. an isotopy $f \colon M \times I \to M$) such that $f(H_1^1) = H_1^2$ or $f(H_1^1) = H_2^2$ (resp. $f_1(H_1^1) = H_1^2$ or $f_1(H_1^1) = H_2^2$). The following lemma has been used extensively to distinguish Heegaard splittings (see, for example, [4, 19] for applications):

Lemma 5.1. *Let $M = H_1^1 \cup H_2^1 = H_1^2 \cup H_2^2$ be two Heegaard splittings of genus n for a 3-manifold M.*

- *If the splittings are isotopic, then the image of the standard generating n-tuple of $\pi_1(H_i^1)$ is Nielsen equivalent to that of $\pi_1(H_i^2)$ in $\pi_1(M)$ for each $i = 1, 2$.*
- *If the splittings are homeomorphic by a homeomorphism $f\colon M \to M$ such that $f(H_i^1) = H_i^2$, then the f_*-image of the standard generators of $\pi_1(H_i^1)$ is Nielsen equivalent to the standard generators of $\pi_1(H_i^2)$ in $\pi_1(M)$.*

A similar statement to Lemma 5.1 holds for trisections of 4-manifolds whose definition we omit here. Specifically, for a 4-manifold M, the Nielsen equivalence class of the generating sets of $\pi_1(M)$ induced by the trisection parts is invariant under isotopy and diffeomorphisms preserving the trisection (see [14]; see also [14, 15] for applications).

Next, we present the (stable) *Andrews-Curtis conjecture* posed by J. J. Andrews and M. L. Curtis [1] in 1965, which is connected to the 4-dimensional smooth Poincaré conjecture via Nielsen transformations. To state these conjectures, we consider the following five elementary transformations for a presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ of a group G:

(AC1) replacing r_i with r_i^{-1} for some i;

(AC2) swapping r_i and r_j for some $i \neq j$;

(AC3) replacing r_i with $r_i r_j$ for some $i \neq j$;

(AC4) replacing r_i with $w^{-1} r_i w$ for some word w in the generators; and

(AC5) adding or deleting a new generator x_{n+1} and a trivial relator $r_{n+1} = x_{n+1}$.

Transformations (AC1)–(AC3) correspond to elementary Nielsen transformations (1)–(3). We say that two presentations P and P' of G are *AC-equivalent* (resp. *stably AC-equivalent*) if one can be transformed into the other via a finite sequence of transformations (AC1)–(AC4) (resp. (AC1)–(AC5)). We can now state the Andrews-Curtis conjecture and the stable Andrews-Curtis conjecture.

Conjecture 5.2 (Andrews-Curtis [1])**.** *Given a presentation $P = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ of the trivial group, P is AC-equivalent to $\langle x_1, \ldots, x_n \mid x_1, \ldots, x_n \rangle$.*

Conjecture 5.3. *Given a presentation $P = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ of the trivial group, P is stably AC-equivalent to the empty presentation.*

Let us consider a handle decomposition of a connected manifold with one 0-handle, n 1-handles, and n 2-handles. The transformations (AC1)–(AC5) can be translated into operations on the 2-handles that do not change the diffeomorphism type of the manifold, as follows:

(AC1) reversing the orientation of the 2-handle $h_i^{(2)}$;

(AC2) permuting the indices of $h_i^{(2)}$ and $h_j^{(2)}$;

(AC3) sliding the 2-handle $h_i^{(2)}$ over the 2-handle $h_j^{(2)}$;

(AC4) sliding the attaching curve of $h_i^{(2)}$ over the 1-handles; and

(AC5) adding or deleting a cancelling 1-handle/2-handle pair.

A topological consequence of Conjecture 5.3 is the following: Let X^k be a contractible smooth k-manifold that admits a handle decomposition with one 0-handle, n 1-handles, and n 2-handles (and no handles of index ≥ 3) realizing a presentation $P = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ of the trivial group $\pi_1(X^k)$.

Proposition 5.4. *If $k \geq 5$ and P is stably AC-equivalent to the empty presentation, then X^k is diffeomorphic to the standard k-disk $\mathbb{D}^k$.*

When $k = 4$, this proposition does not hold due to the peculiarities of four dimensions. However, it remains true that $X^4 \times [0, 1]$ is diffeomorphic to $\mathbb{D}^5$. Moreover, we have the following:

Proposition 5.5 (Exercise 5.1.10(d) in [8])**.** *If $k = 4$ and P is stably AC-equivalent to the empty presentation, then the double $DX^4 = X^4 \cup_{\mathrm{id}_{\partial X^4}} \overline{X^4}$ of X^4 is diffeomorphic to the standard 4-sphere $\mathbb{S}^4$, where $\overline{X^4}$ is the manifold X^4 with the opposite orientation.*

Against this background, researchers have been attempting to construct potential counterexamples to the 4-dimensional smooth Poincaré conjecture—specifically, homotopy 4-spheres whose fundamental group presentations would serve as candidates to disprove the stable Andrews-Curtis conjecture. The homotopy 4-sphere $DX_{\mathrm{AK}(m)}$ constructed in [2] is one such famous example. Its fundamental group admits the presentation $\mathrm{AK}(m) = \langle x, y \mid x^m = y^{m+1}, xyx = yxy \rangle$ for $m \geq 2$. In [20], it was shown that AK(2) is AC-equivalent to the trivial presentation $\langle x, y \mid x, y \rangle$, implying that $DX_{\mathrm{AK}(2)}$ is diffeomorphic to $\mathbb{S}^4$. While it remains open whether $\mathrm{AK}(m)$ is (stably) AC-equivalent to $\langle x, y \mid x, y \rangle$ for $m \geq 3$, Gompf [7] showed that $DX_{\mathrm{AK}(m)}$ is diffeomorphic to $\mathbb{S}^4$.

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