On the Goldman bracket and separability of loops

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Abstract

We give a brief overview of algebraic criteria for the separability of loops on oriented surfaces in terms of the Goldman bracket. The purpose of this note is not to present the full proof, but rather to summarize the background, the main results, and the geometric ideas behind the argument. We also explain an application to the center of the Goldman Lie algebra of a pair of pants.

1 Introduction

Let Σ be a connected oriented surface, possibly with boundary and punctures. We assume that the Euler characteristic of Σ is negative, so that Σ admits a complete hyperbolic metric. Denote by $\hat{\pi}$ the set of free homotopy classes of loops on Σ. Let K be a unital commutative ring of characteristic zero. For any set X, we denote by KX the free K-module generated by X.

Goldman [3] introduced a Lie bracket on $K\hat{\pi}$, now called the Goldman bracket, by using the intersection points of generic representatives of two free homotopy classes. This bracket is one of the fundamental algebraic structures associated with loops on surfaces. It gives a bridge between the topology of curves on surfaces and the algebraic structure of the free module generated by free homotopy classes.

For $x, y \in \hat{\pi}$, let $i(x, y)$ denote the geometric intersection number of x and y. Thus $i(x, y)$ is the minimal number of transversal intersection points among all pairs of generic representatives of x and y. In particular, the condition $i(x, y) = 0$ means that x and y admit disjoint representatives. In this note, we call this property the separability of x and y.

A classical theorem of Goldman [3] states that, if x is represented by a simple closed curve, then the vanishing of the Goldman bracket $[x, y]$ is equivalent to the vanishing of $i(x, y)$. Thus, for simple closed curves, the Goldman bracket gives an algebraic criterion for separability. However, the simplicity assumption is essential. If x is not simple, the implication

$$[x, y] = 0 \quad \Longrightarrow \quad i(x, y) = 0$$

does not hold in general. For example, the bracket $[x, x]$ always vanishes by antisymmetry, while a non-simple class x may satisfy $i(x, x) > 0$. Moreover, in [1], Chas found infinitely many nontrivial examples of distinct classes x and y such that $[x, y] = 0$ but $i(x, y) > 0$.

The purpose of this note is to explain a way to recover separability information from the Goldman bracket even when the loops are not assumed to be simple. The point is that, instead of considering only $[x, y]$, one considers brackets involving powers of one of the loops, such as $[x^m, y]$. The use of two different powers removes the exceptional case and turns this into a criterion for separability.

This note is based on the author's talk at the RIMS workshop. We give an overview of some results from [5], focusing on the main statements relevant to our discussion and the idea of the proof.

2 The Goldman bracket

We recall the definition of the Goldman bracket. Let $x, y \in \hat{\pi}$, and let a and b be generic representatives of x and y, respectively. For each intersection point $P \in a \cap b$, one obtains a loop product $a_P b_P$ by first going around a from P and then going around b from P. Let $|a_P b_P|$ denote its free homotopy class. The Goldman bracket is defined by

$$[x, y] = \sum_{P \in a \cap b} \varepsilon(P; a, b)\, |a_P b_P|,$$

and $\varepsilon(P; a, b) \in \{\pm 1\}$ is the local intersection sign at P. Goldman [3] proved that this bracket is well-defined on free homotopy classes and gives $K\hat{\pi}$ the structure of a Lie algebra.

The definition shows that the Goldman bracket is defined in terms of the intersection points of two loops. Therefore it is natural to ask how much geometric information is contained in the bracket $[x, y]$. If x and y have disjoint representatives, then $[x, y] = 0$. The converse, however, is not true in general, because different intersection points may produce the same free homotopy class with opposite signs and therefore cancel each other.

When one of the two classes is simple, Goldman's theorem gives a clean answer.

Theorem 1 (Goldman [3]). *Let $x, y \in \hat{\pi}$, and suppose that x is represented by a simple closed curve. Then*

$$[x, y] = 0$$

if and only if

$$i(x, y) = 0.$$

This theorem can be viewed as an algebraic characterization of separability from a simple closed curve. The main question considered in this note is whether one can obtain a similar criterion without assuming that x is simple.

There are two immediate difficulties. First, if x is non-simple, then $[x, x] = 0$ holds by antisymmetry although $i(x, x)$ may be positive. Second, there exist more complicated cancellations in the Goldman bracket which are not explained only by antisymmetry. Thus the vanishing of $[x, y]$ itself is too weak to characterize the condition $i(x, y) = 0$.

3 Separability criteria

For $m \in \mathbb{Z} \setminus \{0\}$ and $x \in \hat{\pi}$, we denote by x^m the free homotopy class obtained by going around a representative of x exactly m times. More precisely, if $\gamma : S^1 \to \Sigma$ is a representative of x and if $f_m : S^1 \to S^1$ is the map $f_m(z) = z^m$, then x^m is the free homotopy class represented by $\gamma \circ f_m$.

Our first main theorem states that the bracket $[x^m, y]$ detects separability up to one obvious exceptional case.

Theorem 2. *Let $x, y \in \hat{\pi}$ and let $m \geq 2$. Then*

$$[x^m, y] = 0$$

in $K\hat{\pi}$ if and only if

$$i(x, y) = 0 \quad or \quad y = x^m.$$

The implication from right to left is easy. If $i(x, y) = 0$, then x and y admit disjoint representatives, so the Goldman bracket vanishes. If $y = x^m$, then $[x^m, y] = [x^m, x^m] = 0$ by antisymmetry. The content of the theorem is the converse direction.

This theorem shows that the bracket $[x^m, y]$ gives an algebraic test for whether x and y have disjoint representatives, except for the unavoidable antisymmetric case.

As a corollary of this theorem, we obtain the following criteria.

Corollary 3. *Let m_1 and m_2 be distinct positive integers. Then, for any $x, y \in \hat{\pi}$, the following conditions are equivalent.*

1. $i(x, y) = 0$.
2. $[x^{m_1}, y] = [x^{m_2}, y] = 0$.
3. $[x^{m_1}, y] = [x, y^{m_2}] = 0$.

This gives a purely algebraic criterion for separability. The condition $i(x, y) = 0$ is a geometric condition, while the other conditions are expressed only in terms of the Goldman bracket. The point is that the exceptional equality $y = x^m$ cannot hold for two distinct values of m. Thus the simultaneous vanishing conditions imply that x and y have disjoint representatives.

4 Idea of the proof

We now explain the main geometric idea of the proof. The purpose of this section is not to give all details, but to indicate why hyperbolic geometry is useful in the study of cancellations in the Goldman bracket.

Fix a complete hyperbolic metric on Σ. For a nontrivial free homotopy class, we take its geodesic representative. Geodesic representatives are particularly useful because their intersections are in minimal position. In the simple case this is classical. For non-simple loops, one has to use an appropriate interpretation of intersection points, but the same hyperbolic viewpoint remains effective.

Assume that $[x^m, y] = 0$ for some $m \geq 2$ and that $i(x, y) \neq 0$. We will show that this implies $y = x^m$. Take geodesic representatives of x^m and y, and suppose that they intersect. The Goldman bracket is a sum over the intersection points. Since the bracket is assumed to vanish, each term in the sum must be cancelled by another term. Thus, for an intersection point P, there exists another intersection point Q such that the loop product associated with P represents the same free homotopy class as the loop product associated with Q, with the opposite sign.

Among all intersection points of the geodesic representatives, choose one point P whose forward angle is minimal. The forward angle is the angle less than π between the two tangent vectors determined by the directions of the geodesics at the intersection point. The minimality of this angle is the key choice in the argument.

The cancellation of the term corresponding to P gives another intersection point Q. We lift the situation to the hyperbolic plane. Starting from a lift of P, we alternately lift the geodesic representatives of x^m and y. In this way, we obtain a bi-infinite zigzag curve in the hyperbolic plane. Similarly, starting from a lift of Q, we obtain another bi-infinite zigzag curve.

The equality of the corresponding loop products implies that these two zigzag curves have related endpoints at infinity. Therefore their relative positions are strongly restricted. If $y \neq x^m$, then by analyzing these relative positions, we find another intersection point R of the geodesic representatives of x^m and y. The forward angle at R is strictly smaller than that at P.

This contradicts the choice of P as an intersection point with minimal forward angle. Hence the original assumption that the geodesic representatives intersect must be false, unless the exceptional case $y = x^m$ occurs. This proves the desired implication.

The role of the condition $m \geq 2$ appears when one has to deal with self-intersections of the geodesic representative of x^m. In the analysis of the two zigzag curves, it can happen that a lift of x^m in one zigzag curve intersects a lift of x^m in the other. This phenomenon does not occur in the simple case. The higher power condition provides enough room to find the new intersection point with a smaller forward angle.

This is the main geometric point of the proof. The argument is inspired by earlier work of Kabiraj [4] and by related work of Chas and Kabiraj [2], but the new point here is that the self-intersections arising from non-simple loops can be controlled by using the assumption $m \geq 2$.

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References

[1] Moira Chas. Combinatorial Lie bialgebras of curves on surfaces. *Topology*, 43(3):543–568, 2004.

[2] Moira Chas and Arpan Kabiraj. The Lie bracket of undirected closed curves on a surface. *Trans. Amer. Math. Soc.*, 375(4):2365–2386, 2022.

[3] William M. Goldman. Invariant functions on Lie groups and Hamiltonian flows of surface group representations. *Invent. Math.*, 85(2):263–302, 1986.

[4] Arpan Kabiraj. Center of the Goldman Lie algebra. *Algebr. Geom. Topol.*, 16(5):2839–2849, 2016.

[5] Aoi Wakuda. Separability criteria for loops via the Goldman bracket. arXiv preprint arXiv:2511.18503, 2025.

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