Height on virtual knots

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1 Introduction

The notion of virtual knots was introduced by Kauffman [8] as a generalization of classical knots. It is regarded as a knot theory in a neighborhood of a closed orientable surface, and as an equivalence class of oriented Gauss codes under abstractly defined Reidemeister moves, see §3 in [8]. Virtual knots are used to study ribbon torus-knots in $\mathbb{R}^4$ [13], and to analyze proteins [1].

A *virtual knot diagram* is a 4-valent graph on $\mathbb{R}^2$ such that a vertex represents either a classical crossing (consisting of one over-arc and two under-arcs) or a virtual crossing (marked by a circle). A *virtual knot* is an equivalence class of virtual knot diagrams modulo generalized Reidemeister moves [2], see Figure 1.

A bracket polynomial for knots was introduced by Kauffman [7], and it is equivalent to Jones polynomial [6]. It plays a fundamental role in the construction of Khovanov homology [9], [10], [11]. Kauffman bracket was also defined for virtual knots [8]. The calculation of Kauffman bracket for virtual knots follows four steps.
(1) We resolve classical crossings in a virtual knot diagram D, by A-resolutions and B-resolutions, to obtain a state, a finite union of circles with virtual crossings.
(2) We assign the polynomial, $q + q^{-1}$, to each circle with virtual crossings in a state.
(3) With the coefficient 1 (resp. $-q$) for every A-resolution (resp. B-resolution) in the first step, we sum, over all states, products of polynomials assigned to each circle in a

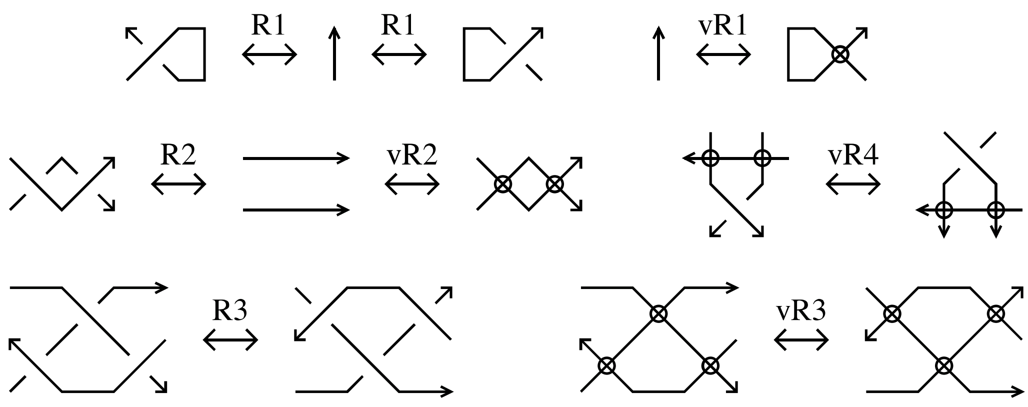

Figure 1: Reidemeister moves for oriented virtual knots

state.
(4) We multiply the polynomial, obtained in the third step, with powers of -1 and q using the numbers of classical crossings of positive and of negative sign in D.

Cheng [3] introduced a coloring on virtual knot diagrams, as in the upper row of Figure 2, to construct an invariant of virtual knots. In this manuscript, we introduce a height, as in the lower row of Figure 2, which is "dual" to Cheng coloring. We use this notion to modify the second and the fourth steps in the calculation of Kauffman bracket for virtual knots. In the second step, we use a notion of height to assign a polynomial, not necessarily equal to $q + q^{-1}$, to a circle with virtual crossings in a state. In the fourth step, we use a notion of height discrepancy at classical crossings to calculate an exponent of q.

2 Definition and a calculation

In this section, we introduce our definition. As a sample calculation, we use a diagram of the virtual knot 2_1 in Green's table [5], see also Appendix A in [4], illustrated in the center of Figure 3.

Let D denote a diagram of an oriented virtual knot, and let $\mathbf{v}_1, \cdots, \mathbf{v}_m$ denote virtual crossings of D. The diagram D outside $\mathbf{v}_1, \cdots, \mathbf{v}_m$ consists of edges $e_1, \cdots, e_m$. We assign a height $h(e_k) \in \mathbb{Z}$ to e_k ($k \in \{1, \cdots, m\}$) so that it changes at virtual crossings as illustrated in the lower row of Figure 2. We notice that the height stays the same along over-arcs and under-arcs at classical crossings. The height on edges $\{h(e_1), \cdots, h(e_m)\}$ is well-defined modulo a simultaneous addition of a constant.

Example 2.1. The diagram of 2_1 in Figure 3 has the virtual crossing $\mathbf{v}$ and classical crossings $\mathbf{x}_1, \mathbf{x}_2$. The thick edge e_1 (resp. thin edge e_2) has its height $b+1$ (resp. b).

In a calculation of Kauffman bracket, we resolve classical crossings, as in Figure 4. The diagram in the first (resp. second) term on the right-hand side of the equation is called *A-resolution* (resp. *B-resolution*) of the classical crossing $\mathbf{x}$ on the left-hand side. From the diagram D, we obtain states $S_1, \cdots, S_{2^n}$, where n is the number of classical crossings of D, and S_i ($i \in \{1, \cdots, 2^n\}$) consists of $i(\beta)$ circles $c_i(1), \cdots, c_i(i(\beta))$ with virtual crossings.

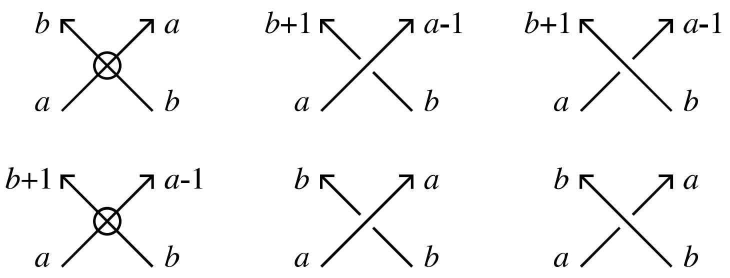

Figure 2: Cheng coloring (upper), and height (lower) near virtual crossing (left), and classical crossing of positive sign (center) and of negative sign (right)

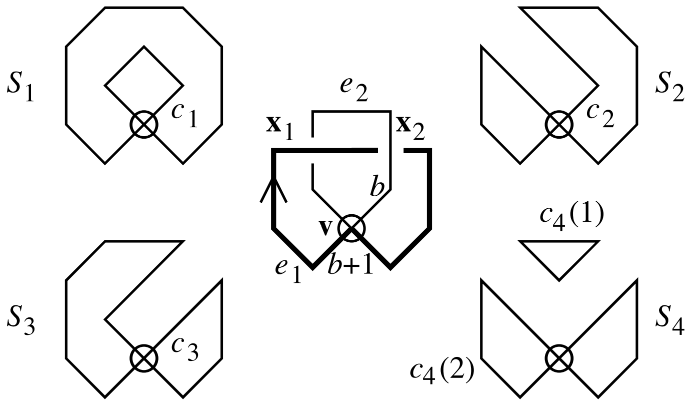

Figure 3: diagram of virtual knot 2_1 (center) and states S_1, S_2, S_3, S_4

Example 2.2. When we resolve classical crossings $\mathbf{x}_1$, $\mathbf{x}_2$ of the diagram of 2_1 in Figure 3, we obtain states S_1, S_2, S_3, S_4. The state S_γ ($\gamma = 1, 2, 3$) consists of one circle $c_\gamma(1)$, which we denote by c_γ in the figure, and S_4 consists of two circles $c_4(1)$ and $c_4(2)$.

Choose an orientation on $c_i(j)$ ($j \in \{1, \cdots, i(\beta)\}$). When we travel along $c_i(j)$ following its orientation, we count $+1$ (resp. -1) if we walk through a virtual crossing $\mathbf{v}_k$ ($k \in \{1, \cdots, m\}$) in the direction of increasing (resp. decreasing) heights, where heights on edges near $\mathbf{v}_k$ are inherited from those on D. We assign a number $va(c_i(j))$ on $c_i(j)$ which is equal to half the sum of the count while we travel along $c_i(j)$ once. We re-choose an orientation on $c_i(j)$ so that $va(c_i(j)) \geq 0$. We choose either orientation on $c_i(j)$ when $va(c_i(j)) = 0$.

If $va(c_i(j)) = 0$, then we assign the polynomial $kb(c_i(j)) = q + q^{-1}$ to $c_i(j)$. If $va(c_i(j)) \neq 0$, then we assign a function $kb(c_i(j)) = f(va(c_i(j)))$ to $c_i(j)$.

Example 2.3. In Figure 3, we observe that $va(c_1) = va(c_4(1)) = 0$ and $va(c_2) = va(c_3) = va(c_4(2)) = 1$. Therefore we obtain $kb(c_1) = kb(c_4(1)) = q + q^{-1}$, and $kb(c_2) = kb(c_3) = kb(c_4(2)) = f(1)$.

Let $o_{\mathbf{x}}$ be the over-arc in a neighborhood of a classical crossing $\mathbf{x}$ of D, and $r_{\mathbf{x}}$ (resp. $l_{\mathbf{x}}$) be the under-arc which we see on the right (resp. left) when we travel along $o_{\mathbf{x}}$ following the orientation on D. See Figure 4. We assign a number $h(\mathbf{x})$ at $\mathbf{x}$ which is equal to $h(o_{\mathbf{x}}) - h(r_{\mathbf{x}}) = h(o_{\mathbf{x}}) - h(l_{\mathbf{x}})$, where $h(s_{\mathbf{x}})$ ($s \in \{o, r, l\}$) is the height $h(e_k)$ such that e_k contains $s_{\mathbf{x}}$.

Example 2.4. In Figure 3, we observe $h(\mathbf{x}_1) = 1$ and $h(\mathbf{x}_2) = -1$ at classical crossings $\mathbf{x}_1$ and $\mathbf{x}_2$.

Let $kb(S_i) = \prod_j kb(c_i(j))$, and let $Kb(D) = \sum_i (-q)^{\#S_i} kb(S_i)$, where $\#S_i$ denotes the number of B-resolutions to obtain S_i from D. Let $n^-(D)$ denote the number of classical crossings of negative sign in D. Let $n_0^+(D)$ (resp. $n_0^-(D)$) denote the number of classical crossings $\mathbf{x}$ of positive (resp. negative) sign with $h(\mathbf{x}) = 0$ in D. We define $KB(D) = (-q)^{-n^-(D)} q^{n_0^+(D) - n_0^-(D)} Kb(D)$.

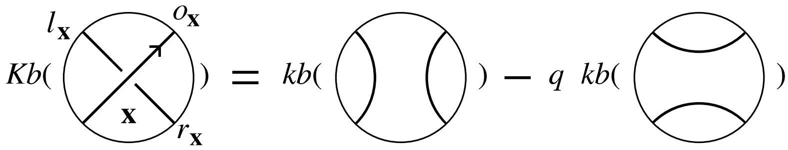

Figure 4: resolution at a classical crossing $\mathbf{x}$

Example 2.5. Let D denote the diagram of 2_1 in Figure 3. We obtain $Kb(D) = kb(S_1) + (-q)kb(S_2) + (-q)kb(S_3) + (-q)^2 kb(S_4) = (q + q^{-1}) - qf(1) - qf(1) + q^2(q + q^{-1})f(1)$, and $KB(D) = (-q)^{-0} q^{0-0} Kb(D) = (q + q^{-1}) + (q^3 - q)f(1)$.

The following theorem shows that $KB(D)$ is an invariant of an oriented virtual knot, where D is a diagram of the virtual knot.

Theorem 2.6. *Let D_1, D_2 denote diagrams of an oriented virtual knot such that D_2 is obtained from D_1 by applying one of generalized Reidemeister moves as in Figure 1. Then the equation $KB(D_1) = KB(D_2)$ holds.*

Remark 2.7. In [12], we use an information on $h(\mathbf{x})$ at a classical crossing $\mathbf{x}$ as in Figure 5.

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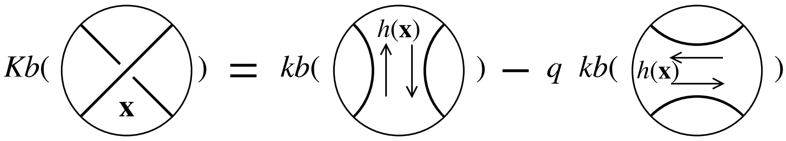

Figure 5: resolution at a classical crossing $\mathbf{x}$ with $h(\mathbf{x})$

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