

PARTITION RELATIONS FOR TREES I: INCOMPARABLE TREES

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ABSTRACT. Todorćević proved that Martin's axiom $\text{MA}_{\aleph_1}$ implies that every two coherent $\aleph_1$ -Aronszajn trees are comparable. Here, from cardinal arithmetic assumptions, we obtain the failure of the analogous statement for higher trees. In particular, for every $n < \omega$, assuming $2^{\aleph_n} = \aleph_{n+1}$, there is a family of $\aleph_{n+2}$ -many pairwise incomparable $\aleph_{n+1}$ -coherent $\aleph_{n+1}$ -complete $\aleph_{n+2}$ -Aronszajn trees. The proof uses an anti-Ramsey colourings for trees recently introduced by the authors.

1. INTRODUCTION

Given two trees $\mathbf{S} = (S, <_S)$ and $\mathbf{T} = (T, <_T)$, a map $g : S \rightarrow T$ is a *weak embedding from $\mathbf{S}$ to $\mathbf{T}$* iff for all $s, s' \in S$, $s <_S s'$ implies $g(s) <_T g(s')$.¹ A map g from a subset of S to T is *level-preserving* iff for all $s \in \text{dom}(g)$, $\text{ht}_{\mathbf{S}}(s) = \text{ht}_{\mathbf{T}}(g(s))$. A map g from a subset of S to T is *Lipschitz* iff it is level-preserving and for all $s, s' \in \text{dom}(g)$, $\text{ht}_{\mathbf{S}}(s \wedge_{\mathbf{S}} s') \leq \text{ht}_{\mathbf{T}}(g(s) \wedge_{\mathbf{T}} g(s'))$.

Two trees are *comparable* iff there exists a weak embedding from one of the trees to the other one. This note is motivated by the following theorem of Todorćević (see [Tod07a, Theorem 5.4] or [Tod07b, Theorem 4.3.10]).

Theorem (Todorćević). *Martin's axiom $\text{MA}_{\aleph_1}$ implies that every two coherent $\aleph_1$ -Aronszajn trees are comparable.*

Our goal here is to obtain the failure of higher analogs of Todorćević's theorem from hypotheses that are conceivably compatible with potential higher forcing axioms.

Theorem A. *For every $n < \omega$, if $2^{\aleph_n} = \aleph_{n+1}$, then there is a family of $\aleph_{n+2}$ -many pairwise incomparable $\aleph_{n+1}$ -coherent $\aleph_{n+1}$ -complete $\aleph_{n+2}$ -Aronszajn trees. Furthermore, no two trees from the family admit a Lipschitz map from an $\aleph_{n+2}$ -sized subset of one to the other.*

The point is that the existence of a weak embedding between trees gives rise to one that is also level-preserving, and that level-preserving weak embeddings are Lipschitz.

For updates on this paper, visit <https://p.assafrinot.com/73>.

¹Unlike usual embeddings that are required to be injective, weak embeddings may be constant on antichains.

Theorem A follows from the following more general result, since any successor cardinal (in fact, any regular uncountable cardinal that is not greatly Mahlo) admits a *nontrivial C -sequence*.

Theorem B. *Suppose that $\lambda = \lambda^{<\lambda}$ is an uncountable cardinal, and any of the following:*

- $\square(\lambda^+, <\lambda)$ holds and $\theta = \lambda^+$;
- λ admits a nontrivial C -sequence and $\theta = \lambda$.

Then there is a family of 2^θ many streamlined λ -coherent λ -complete λ^+ -Aronszajn trees such that no two of them admit a Lipschitz map from a λ^+ -sized subset of one to the other.

The proofs go through obtaining instances of the negative partition relations for trees introduced in [IR25], and reproduced in Definition 2.9 below.

2. PRELIMINARIES

2.1. Conventions. Throughout this paper, κ stands for a regular uncountable cardinal, and $\theta, \lambda, \chi, \mu, \nu$ denote arbitrary cardinals. E_θ^κ stands for $\{\alpha < \kappa \mid \text{cf}(\alpha) = \theta\}$, $\text{acc}(\kappa)$ stands for $\{\alpha < \kappa \mid \sup(\alpha) = \alpha > 0\}$, and H_κ stands for the collection of all sets of hereditary cardinality less than κ . Two functions f, g from the ordinals to the universe are said to be *comparable* iff $f \subseteq g$ or $g \subseteq f$; we also let

$$\Delta(f, g) := \min\{\text{dom}(f), \text{dom}(g), \delta \mid \delta \in \text{dom}(f) \cap \text{dom}(g) \text{ \& } f(\delta) \neq g(\delta)\},$$

so that f and g are incomparable iff $\Delta(f, g) < \min\{\text{dom}(f), \text{dom}(g)\}$.

2.2. Trees. For preliminaries on trees, see the introduction of [BR17]. Given two distinct nodes x, y of a Hausdorff tree $\mathbf{T} = (T, <_T)$, we let $x \wedge y$ denote the highest node z of $\mathbf{T}$ to satisfy $z \leq_T x$ and $z \leq_T y$. In particular, $x \wedge y \in \{x, y\}$ iff x and y are comparable nodes of $\mathbf{T}$.

Definition 2.1. A *streamlined tree* is a collection T of functions from ordinals to the universe that is closed under taking initial segments.

Convention 2.2. We identify a streamlined tree T with the Hausdorff tree $(T, \subsetneq)$; in this case, $\text{ht}(t) = \text{dom}(t)$ for every $t \in T$.

Note that in the context of streamlined trees, a partial function $g : S \rightarrow T$ is Lipschitz iff it is dom-preserving and for all $s, s' \in \text{dom}(g)$, $\Delta(s, s') \leq \Delta(g(s), g(s'))$.

Definition 2.3 (Default lexicographic ordering). For a subset $T \subseteq {}^{<\kappa}H_\kappa$, fix $\prec$ such that $(H_\kappa, \prec)$ is a well-order extending $(\kappa, \in)$, and then define a linear ordering $<_{\text{lex}}$ of T , as follows:

- If f and g are comparable, then $f <_{\text{lex}} g$ iff $g \subseteq f$;
- Otherwise, let $\delta := \Delta(f, g)$, and then $f <_{\text{lex}} g$ iff $f(\delta) \prec g(\delta)$.

Remark 2.4. If $T \subseteq {}^{<\kappa}H_\kappa$ is a streamlined κ -Aronszajn tree then $(T, <_{\text{lex}})$ is a κ -Aronszajn line, i.e., a κ -sized linear order that contains no κ -sized linear order of density less than κ , no copy of $(\kappa, \in)$ nor a copy of $(\kappa, \ni)$.

Definition 2.5. A streamlined tree T is λ -coherent iff $|\{\beta \in \text{dom}(t_0) \cap \text{dom}(t_1) \mid t_0(\beta) \neq t_1(\beta)\}| < \lambda$ for all $t_0, t_1 \in T$.

Fact 2.6 (essentially Specker, [Spe49]). *For every cardinal $\lambda = \lambda^{<\lambda}$, there exists a streamlined λ -coherent, λ -complete λ^+ -Aronszajn tree $T \subseteq {}^{<\lambda^+}\lambda$.*

Fact 2.7 (König, [Kön03, Theorem 3.9]). *If $\square(\kappa)$ holds, then there exists a streamlined coherent (that is, $\aleph_0$ -coherent) κ -Aronszajn tree $T \subseteq {}^{<\kappa}2$.*

2.3. Partition relation for trees.

Definition 2.8 (Rapid families). For a family S of sequences, and a map $h : \bigcup_{x \in S} \text{Im}(x) \rightarrow \text{Ord}$, we say that S is h -rapid iff for all $x \neq y$ in S , either $\sup(h[\text{Im}(x)]) < \min(h[\text{Im}(y)])$ or $\sup(h[\text{Im}(y)]) < \min(h[\text{Im}(x)])$.

The next definition was introduced in [IR25], where sufficient conditions for instances of it to hold were obtained, and applications were presented.

Definition 2.9. For a structure $\mathbf{T} = (T, <, \triangleleft, h)$ such that $(T, <)$ is a Hausdorff tree, $(T, \triangleleft)$ is a partially ordered set, and $h : T \rightarrow \text{Ord}$ is a map, we define three partition relations, as follows.

- (I) $\text{weak } \mathbf{T} \xrightarrow{\wedge} [\kappa]_\theta^{<\nu, <\mu}$ asserts the existence of a colouring $c : T \rightarrow \theta$ satisfying that for every nonzero $\sigma < \nu$, for every h -rapid $S \subseteq {}^\sigma T$ of size κ , for every $\tau < \theta$, there are $x \neq y$ in S such that the following three hold:
 - (a) for every $i < \sigma$, $c(x(i) \wedge y(i)) = \tau$;
 - (b) for every $i < \sigma$, if $x(i) \triangleleft y(i)$ then $h(x(i)) < h(y(i))$;
 - (c) $|\{h(x(i) \wedge y(i)) \mid i < \sigma\}| < \mu$.
- (II) $\mathbf{T} \xrightarrow{\wedge} [\kappa]_\theta^{<\nu, <\mu}$ asserts the existence of a colouring $c : T \rightarrow \theta$ witnessing weak $\mathbf{T} \xrightarrow{\wedge} [\kappa]_\theta^{<\nu, <\mu}$ in addition to satisfying the following. For every $T' \in [T]^\kappa$, for every nonzero $\sigma < \nu$, there exists an h -rapid $S' \subseteq {}^\sigma T'$ of size κ such that for every $S \subseteq S'$ of size κ , and every $\tau \in {}^\sigma \theta$, there are $x \neq y$ in S such that the following three hold:
 - (a) for every $i < \sigma$, $c(x(i) \wedge y(i)) = \tau(i)$;
 - (b) same as Clause (b) above;
 - (c) same as Clause (c) above.
- (III) $\mathbf{T} \xrightarrow{\Delta} [\kappa]_\theta^{<\nu, <\mu}$ asserts the existence of a colouring $d : \text{Im}(h) \rightarrow \theta$ satisfying following. For every nonzero $\sigma < \nu$, for every h -rapid $S \subseteq {}^\sigma T$ of size κ , for every $\tau < \theta$, there are $x \neq y$ in S such that the following three hold:
 - (a) for every $i < \sigma$, $d(h(x(i) \wedge y(i))) = \tau$;
 - (b) same as Clause (b) above;
 - (c) same as Clause (c) above.

Convention 2.10. We may replace the quadruple $\mathbf{T}$ by a triple $(T, <, \triangleleft)$, and then we mean $\mathbf{T} := (T, <, \triangleleft, \text{ht})$. We may replace $\mathbf{T}$ by a streamlined tree T , and then we mean $\mathbf{T} := (T, \subseteq, <_{\text{lex}}, \text{dom})$. We may replace the superscript $<^\nu$ by ν , and then we mean that σ is equal to ν . We may replace the superscript $<^\mu$ by μ , in which case Clause (c) becomes “ $|\{h(x(i) \wedge y(i)) \mid i < \sigma\}| \leq \mu$ ”. We may omit μ , in which case requirement (c) is waived.

The utility of the parameter μ in the partition relations of Definition 2.9 will be demonstrated in Section 4 below. It is clear that the partition relation of Clause (III) implies that of Clause (I). We now give a sufficient condition for Clause (I) to be no weaker than Clause (II).

Lemma 2.11. *Suppose that weak $\mathbf{T} \xrightarrow{\wedge} [\kappa]_\theta^{<^\nu, <^\mu}$ holds, where:*

- κ is a regular uncountable cardinal;
- $\nu \in [2, \kappa)$ is a cardinal such that $\lambda^{<^\nu} < \kappa$ for every $\lambda < \kappa$;
- $\mathbf{T} = (T, <, \triangleleft, h)$ and h is $(<^\kappa)$ -to-one.

All of the following hold:

- (1) *If $\theta > 1$, then $(T, <)$ has no chains of size κ ;*
- (2) *If $\theta = \kappa$ and $(T, <)$ is a κ -tree, then $\mathbf{T} \xrightarrow{\wedge} [\kappa]_\theta^{<^\nu, <^\mu}$ holds.*

Proof. (1) Assuming $\theta > 1$, by monotonicity, we may fix a colouring $c : T \rightarrow 2$ witnessing weak $\mathbf{T} \xrightarrow{\wedge} [\kappa]_2^1$. Now, if $(T, <)$ were to have a chain B of size κ , then we could find some $C \in [B]^\kappa$ on which C is constant with value, say, j . As h is $(<^\kappa)$ -to-one, we may find an $S \subseteq C$ of size κ on which h is injective. So, S is h -rapid, and then, by the choice of c , there are $x \neq y$ in S such that $c(x \wedge y) = 1 - j$. However, $x \wedge y \in \{x, y\} \subseteq C$ and hence $c(x \wedge y) = j$. This is a contradiction.

(2) Suppose that $(T, <)$ is a κ -tree and that $c : T \rightarrow \kappa$ is a colouring witnessing weak $\mathbf{T} \xrightarrow{\wedge} [\kappa]_\kappa^{<^\nu, <^\mu}$. By Clause (1), $(T, <)$ is a κ -Aronszajn tree. We shall need the following direct consequence of this fact.

Claim 2.11.1. *For every $T' \in [T]^\kappa$, for every $\sigma < \nu$, there exists an $\alpha < \kappa$ for which $Z_\alpha := \{z \in T_\alpha \mid |\{t \in T' \mid z < t\}| = \kappa\}$ satisfies $|Z_\alpha| \geq |\sigma|$.*

Proof. Suppose not, and let T' and σ be a counterexample. For every $z \in T$ and $\alpha < \text{ht}(z)$, let $z \restriction \alpha$ denote the unique $y \in T_\alpha$ to satisfy $y < z$. By transitivity, for all $\alpha < \beta < \kappa$ and $z \in Z_\beta$, we have $z \restriction \alpha \in Z_\alpha$.

Fix a regular cardinal $\chi \in [\sigma, \nu]$. As $(T, <)$ is a Hausdorff tree, for each $\alpha \in E_\chi^\kappa$, we may fix a large enough $\delta_\alpha < \alpha$ such that $z \mapsto z \restriction \delta_\alpha$ is injective over Z_α . As $(T, <)$ is a κ -tree and $\lambda^{<^\nu} < \kappa$ for every $\lambda < \kappa$, we may fix some $\delta < \kappa$ and a subset $Y \in [T_\delta]^{<^\sigma}$ for which the following set is stationary:

$$A := \{\alpha \in E_\chi^\kappa \mid \{z \restriction \delta_\alpha \mid z \in Z_\alpha\} = Y\}.$$

Fix $y \in Y$. Then for every $\alpha \in A$, there is a unique $z_\alpha \in Z_\alpha$ such that $y < z_\alpha$. Let $\alpha < \beta$ be a pair of ordinals from A . As $z_\beta \in Z_\beta$, we have that $z_\beta \restriction \alpha \in Z_\alpha$ and it extends y , so that $z_\beta \restriction \alpha = z_\alpha$ necessarily. So $\{z_\alpha \mid \alpha \in A\}$ is a κ -sized chain in T . This is a contradiction. $\square$

Fix a bijection $\pi : \kappa \leftrightarrow \bigcup \{Z^\kappa \mid Z \in [T]^{<\nu}, Z \neq \emptyset\}$, and define a colouring $c' : T \rightarrow \kappa$ as follows. Given $t \in T$, first denote $\pi(c(t))$ by $g : Z \rightarrow \kappa$. Then, if there exists a unique $z \in Z$ such that $z < t$, then let $c'(t) := g(z)$ for this z . Otherwise, let $c'(t) := \min(\text{Im}(g))$. We verify that c' witnesses $\mathbf{T} \xrightarrow{\Delta} [\kappa]_\kappa^{<\nu, <\mu}$.

- (I) Given $\sigma < \nu$, an h -rapid $S \subseteq {}^\sigma T$ of size κ , and a prescribed colour $\tau < \kappa$, fix an arbitrary $z \in T$, and consider the constant function $g : \{z\} \rightarrow \{\tau\}$. Next, by the choice of c , fix $x \neq y$ in S such that the following three hold:

- for every $i < \sigma$, $c(x(i) \wedge y(i)) = \pi^{-1}(g)$;
- for every $i < \sigma$, if $x(i) \triangleleft y(i)$ then $h(x(i)) < h(y(i))$;
- $|\{h(x(i) \wedge y(i)) \mid i < \sigma\}| < \mu$.

It is clear that for every $i < \sigma$, $c'(x(i) \wedge y(i)) = \tau$.

- (II) Given $\sigma < \nu$ and $T' \in [T]^\kappa$, since T is a κ -tree, by appealing to Claim 2.11.1, we find some $\alpha < \kappa$ and an injective $z \in {}^\sigma(T_\alpha)$ such that $|\{t \in T' \mid z(i) \subseteq t\}| = \kappa$ for every $i < \sigma$. As h is $(<\kappa)$ -to-one, it follows that we may find an h -rapid $S' \subseteq {}^n T'$ of size κ satisfying that for every $x \in S$, it is the case that $z(i) < x(i)$ for every $i < \sigma$. Finally, given $S \subseteq S'$ of size κ , and $\tau \in {}^\sigma \kappa$, define $g : \{z(i) \mid i < \sigma\} \rightarrow \kappa$ via $g(z(i)) := \tau(i)$. By the choice of c , fix $x \neq y$ in S such that the following three hold:

- for every $i < \sigma$, $c(x(i) \wedge y(i)) = \pi^{-1}(g)$;
- for every $i < \sigma$, if $x(i) \triangleleft y(i)$ then $h(x(i)) < h(y(i))$;
- $|\{h(x(i) \wedge y(i)) \mid i < \sigma\}| < \mu$.

It is clear that for every $i < \sigma$, $c'(x(i) \wedge y(i)) = \tau(i)$. □

3. INSTANCES OF THE PARTITION RELATION

We start with a fact announced without a proof in [IR25]:

Proposition 3.1. *Suppose that T is a streamlined ν -free κ -Souslin tree.*

Then $T \xrightarrow{\Delta} [\kappa]_\kappa^{<\nu, 1}$ holds.

Proof. First, to avoid superficial complications, and as there are less than κ many nodes in T with fewer than κ many extensions in T , we shall identify T with a *prolific* tree, i.e., a tree satisfying that for every $\alpha < \kappa$, for every $t \in T_\alpha$, for every $\tau < \alpha$, $t^\frown \langle \tau \rangle$ is in T (see [RS23, Proposition 2.16]).

Fix a surjection $d : \kappa \rightarrow \kappa$ that satisfies $|\{\delta < \kappa \mid d(\delta) = \tau\}| = \kappa$ for every $\tau < \kappa$. To see that this works, fix a nonzero $\sigma < \nu$, a ht-rapid $S \subseteq {}^\sigma T$ of size κ , and a prescribed colour $\tau < \kappa$. Fix a regular cardinal $\lambda \in [\sigma, \nu]$. For each $\alpha \in E_\lambda^\kappa$, we do the following:

- fix $\langle x_i^\alpha \mid i < \sigma \rangle$ in S such that $\text{ht}(x_i^\alpha) > \alpha$ for every $i < \sigma$;
- let $I_\alpha \subseteq \sigma$ be a maximal set on which $i \mapsto x_i^\alpha \restriction \alpha$ is injective;
- let $\epsilon_\alpha < \alpha$ be large enough such that $i \mapsto x_i^\alpha \restriction \epsilon_\alpha$ is injective over I_α .

Since T is ν -free, it follows from [BR17, Lemma 7.7(1)] that $\mu^{<\nu} < \kappa$ for every $\mu < \kappa$. So since T is a κ -tree we may fix a stationary $U \subseteq E_\lambda^\kappa$, $\epsilon < \kappa$, $I \subseteq \sigma$ and $\langle x_i \mid i \in I \rangle \in {}^I(T_\epsilon)$ such that:

- (1) for every $\alpha \in U$, $\langle x_i^\alpha \restriction \epsilon_\alpha \mid i \in I_\alpha \rangle = \langle x_i \mid i \in I \rangle$;
- (2) for all $\alpha, \beta \in U$, $i \in I$ and $j < \sigma$, $x_i^\alpha \restriction \alpha \subseteq x_j^\alpha$ iff $x_i^\beta \restriction \alpha \subseteq x_j^\beta$;
- (3) for every pair $\alpha < \beta$ of ordinals from U , $\sup\{\text{ht}(x_i^\alpha) \mid i < \sigma\} < \beta$.

As $|I| \leq \sigma < \nu$ and T is ν -free, $T^* := \bigotimes_{i \in I} x_i^\uparrow$ is a κ -Souslin tree (recall [BR17, Definition 1.3]). As $S' := \{\langle x_i^\alpha \restriction \alpha \mid i \in I \rangle \mid \alpha \in U\}$ is a κ -sized subset of T^* , a standard fact (see [RS23, Lemma 2.14]) yields a node $\langle t_i \mid i \in I \rangle$ in T^* that has κ many extensions in T^* and such that every extension of it in T^* is extended by some node from S' . By possibly passing to an extension of $\langle t_i \mid i \in I \rangle$ in T^* , we may assume that $\delta := \text{dom}(t_0)$ satisfies $d(\delta) = \tau$. Fix an $\alpha \in U$ such that $x_i^\alpha \restriction \alpha$ extends $t_i \restriction \langle 0 \rangle$ for every $i \in I$, and then fix a $\beta \in U \setminus \alpha$ such that $x_i^\beta \restriction \beta$ extends $t_i \restriction \langle 1 \rangle$ for every $i \in I$. For every $j < \sigma$, by Clause (2), there exists some $i \in I$ such that $d(\text{ht}(x_j^\alpha \wedge x_j^\beta)) = d(\text{ht}(x_i^\alpha \wedge x_i^\beta)) = d(\delta) = \tau$, and

$$x_j^\alpha(\delta) = (t_i \restriction \langle 0 \rangle)(\delta) = 0 < 1 = (t_i \restriction \langle 1 \rangle)(\delta) = x_j^\beta(\delta),$$

so that $x_j^\alpha <_{\text{lex}} x_j^\beta$, and by Clause (3), $\text{ht}(x_j^\alpha) < \text{ht}(x_j^\beta)$. $\square$

Proposition 3.2. *Suppose:*

- $\nu \leq \lambda < \kappa$ are infinite regular cardinals such that $\mu^{<\nu} < \kappa$ for every $\mu < \kappa$;
- T is a streamlined λ -coherent κ -Souslin tree.

Then $T \xrightarrow{\Delta} [\kappa]_\kappa^{<\nu, 1}$ holds.

Proof. By a standard fact (see [BR17, Lemma 2.4]), the restriction of T to some club is normal and splitting. For notational simplicity, and in order to avoid relativizing all of our calculations to such a club, we hereafter assume that every node of T admits κ many successors and at least two immediate successors. Fix a surjection $d : \kappa \rightarrow \kappa$ that satisfies $|\{\delta < \kappa \mid d(\delta) = \tau\}| = \kappa$ for every $\tau < \kappa$. To see that this works, fix a nonzero $\sigma < \nu$, a ht-rapid $S \subseteq {}^\sigma T$ of size κ , and a prescribed colour $\tau < \kappa$. As S is ht-rapid, for every $\alpha < \kappa$, we may fix $\langle x_i^\alpha \mid i < \sigma \rangle$ in S such that $\text{ht}(x_i^\alpha) > \alpha$ for every $i < \sigma$. As T is λ -coherent, we may fix a stationary $U \subseteq E_\lambda^\kappa$ and some $\epsilon < \kappa$ such that the following hold:

- (1) for every $\alpha \in U$, for every $\beta \in [\epsilon, \alpha)$, the map $i \mapsto x_i^\alpha(\beta)$ is constant over σ .

As T is a κ -tree and using the fact that $\mu^{<\nu} < \kappa$ for every $\mu < \kappa$, we may then find a stationary $U' \subseteq U$ such that the following hold as well:

- (2) the map $\alpha \mapsto \langle x_i^\alpha \restriction \epsilon \mid i < \sigma \rangle$ is constant over U' ;
- (3) for every pair $\alpha < \beta$ of ordinals from U' , $\sup\{\text{ht}(x_i^\alpha) \mid i < \sigma\} < \beta$.

As $A := \{x_0^\alpha \restriction \alpha \mid \alpha \in U'\}$ is a κ -sized subset of the κ -Souslin tree T , another standard fact (see [RS23, Lemma 2.14]), yields a node $t \in T$ such that every extension of t in T is extended by some node from A . By possibly extending t , we may assume that $\delta := \text{ht}(t)$ is bigger than ϵ and satisfying $d(\delta) = \tau$. Let $t_0 <_{\text{lex}} t_1$ be some pair of immediate successors of t . Now fix $\alpha \in U'$ such that $x_0^\alpha \restriction \alpha$ extends t_0 , and then fix $\beta \in U'$ above α such that $x_0^\beta \restriction \beta$ extends t_1 . So $\Delta(x_0^\alpha, x_0^\beta) = \Delta(t_0, t_1) = \delta < \alpha$, $x_0^\alpha(\delta) = t_0(\delta)$ and $x_0^\beta(\delta) = t_1(\delta)$. By Clause (2), $\Delta(x_i^\alpha, x_i^\beta) \geq \epsilon$ for every $i < \sigma$. By Clause (1), then, for every $i < \sigma$, $\Delta(x_i^\alpha, x_i^\beta) = \delta$, $x_i^\alpha(\delta) = t_0(\delta)$ and $x_i^\beta(\delta) = t_1(\delta)$, so that $d(\text{ht}(x_i^\alpha \wedge x_i^\beta)) = d(\delta) = \tau$, $x_i^\alpha <_{\text{lex}} x_i^\beta$ and (by Clause (3)) $\text{ht}(x_i^\alpha) < \text{ht}(x_i^\beta)$. $\square$

In Todorćević's note [Tod18] that was written before the introduction of Definition 2.9, it is proved that CH implies the existence of some streamlined $\aleph_2$ -Aronszajn tree T for which Clauses (a) and (c) of $T \xrightarrow{\Delta} [\aleph_2]_{\aleph_1}^{\aleph_0,1}$ hold. The note also contains a discussion on how this may be generalized to higher cardinals. Here we present two additional generalizations that make use of recent results from [IR26].

Proposition 3.3. *Suppose that $\lambda = \lambda^{<\lambda}$ is an uncountable cardinal admitting a nontrivial C -sequence. Then any streamlined λ^+ -Aronszajn tree T as in Fact 2.6 satisfies $T \xrightarrow{\Delta} [\lambda^+]_\lambda^{<\lambda,1}$.*

Proof. Denote $\kappa := \lambda^+$, so that $T \subseteq {}^{<\kappa}\lambda$. For every nonzero $\sigma < \lambda$, for every rapid family $S \subseteq {}^\sigma T$, let $D(S)$ be the collection of all $\delta < \kappa$ for which there are $x \neq y$ in S such that the following two hold:

- (a) for every $i < \sigma$, $\Delta(x(i), y(i)) = \delta$;
- (b) for every $i < \sigma$, $x(i) <_{\text{lex}} y(i)$ iff $\text{dom}(x(i)) < \text{dom}(y(i))$.

Now let $\mathcal{F}$ be the collection of all sets $E \subseteq \kappa$ for which there are nonzero $\sigma < \lambda$ and a rapid $S \subseteq {}^\sigma T$ such that $D(S) \subseteq E$. Trivially, $\mathcal{F}$ is closed under taking supersets. We claim it is a λ -complete filter. To verify, suppose that we are given $\nu < \lambda$, and for every $j < \nu$, we are given a nonzero $\sigma_j < \lambda$ and a rapid $S_j \subseteq {}^{\sigma_j} T$ of size κ . For all $j < \nu$ and $\alpha < \kappa$, fix $\langle x_{j,i}^\alpha \mid i < \sigma_j \rangle$ in S_j such that $\text{ht}(x_{j,i}^\alpha) > \alpha$ for every $i < \sigma_j$. Fix an ordinal $\sigma < \lambda$ and some bijection $\pi : \sigma \leftrightarrow \bigcup_{j < \nu} \{j\} \times \sigma_j$. For every $\alpha < \kappa$, define $\langle x_i^\alpha \mid i < \sigma \rangle$ via $x_i^\alpha := x_{\pi(i)}^\alpha$. As T is a κ -tree, we may find some club $C \subseteq \kappa$ for which $S := \{\langle x_i^\alpha \mid i < \sigma \rangle \mid \alpha \in C\}$ is a rapid subfamily of ${}^\sigma T$. Clearly, $D(S) \subseteq \bigcap_{j < \nu} D(S_j)$.

Claim 3.3.1. $\mathcal{F}$ is a uniform nonempty filter.

Proof. Fix a nonzero $\sigma < \lambda$, a rapid $S \subseteq {}^\sigma T$ of size κ , and an $\varepsilon < \kappa$, and we shall show that $D(S) \setminus \varepsilon \neq \emptyset$. As S is rapid, for every $\alpha < \kappa$, we may fix $\langle x_i^\alpha \mid i < \sigma \rangle$ in S such that $\text{ht}(x_i^\alpha) > \alpha$ for every $i < \sigma$. As in the proof of Proposition 3.2, since T is a λ -coherent λ^+ -tree and $\lambda^{<\lambda} = \lambda$, we may fix a stationary $U' \subseteq E_\lambda^\kappa$ and some $\epsilon \in [\varepsilon, \kappa)$ such that all of the following hold:

- (1) for every $\alpha \in U'$, for every $\beta \in [\epsilon, \alpha)$, the map $i \mapsto x_i^\alpha(\beta)$ is constant over σ ;
- (2) the map $\alpha \mapsto \langle x_i^\alpha \restriction \epsilon \mid i < \sigma \rangle$ is constant over U' ;
- (3) for every pair $\alpha < \beta$ of ordinals from U' , $\sup\{\text{dom}(x_i^\alpha) \mid i < \sigma\} < \beta$.

By Remark 2.4, $(T, <_{\text{lex}})$ contains no copies of $(\kappa, \exists)$, so we may pick a pair $\alpha < \beta$ of ordinals in U' such that $x_0^\alpha \restriction \alpha <_{\text{lex}} x_0^\beta \restriction \beta$. Recalling Definition 2.3, this means that $\delta := \Delta(x_0^\alpha \restriction \alpha, x_0^\beta \restriction \beta)$ is smaller than α , and $x_0^\alpha(\delta) < x_0^\beta(\delta)$. By Clause (2), $\delta \geq \epsilon$ and, more generally, $\Delta(x_i^\alpha, x_i^\beta) \geq \epsilon$ for every $i < \sigma$. By Clause (1), then, for every $i < \sigma$, $\Delta(x_i^\alpha, x_i^\beta) = \delta$ and $x_i^\alpha(\delta) = x_0^\alpha(\delta) < x_0^\beta(\delta) = x_i^\beta(\delta)$, so that $x_i^\alpha <_{\text{lex}} x_i^\beta$. $\square$

Altogether, $\mathcal{F}$ is a λ -complete uniform nonempty filter over λ^+ . If it is λ^+ -complete, then by Ulam's theorem, it is not weakly λ^+ -saturated. If it is λ^+ -incomplete, then since λ admits a nontrivial C -sequence, [IR26, Corollary 2.7] implies that $\mathcal{F}$ is not weakly λ -saturated. So, in both cases, we may fix a map $d : \kappa \rightarrow \lambda$ such that $d[E] = \lambda$ for every $E \in \mathcal{F}$. Then d witnesses that $T \xrightarrow{\Delta} [\kappa]_\lambda^{<\lambda,1}$ holds, and we are done. $\square$

Proposition 3.4. *Suppose that:*

- $\nu \leq \lambda < \kappa$ are regular uncountable cardinals such that $\mu^{<\nu} < \kappa$ for every $\mu < \kappa$;
- T is a streamlined λ -coherent κ -Aronszajn tree;
- $\square(\kappa, <\nu)$ holds.

Then $T \xrightarrow{\Delta} [\kappa]_\kappa^{<\nu,1}$ holds.

Proof. A proof similar to that of Proposition 3.3 yields that the collection $\mathcal{F}$ of all sets $E \subseteq \kappa$ for which there are nonzero $\sigma < \nu$ and a rapid $S \subseteq {}^\sigma T$ such that $D(S) \subseteq E$ is a ν -complete uniform nonempty filter over κ . Since $\square(\kappa, <\nu)$ holds, [IR26, Theorem D] implies that $\mathcal{F}$ is not weakly κ -saturated. As in the proof of Proposition 3.3, it follows that $T \xrightarrow{\Delta} [\kappa]_\kappa^{<\nu,1}$ holds. $\square$

Remark 3.5. By [IR25, Theorem E], for every regular $\kappa \geq \aleph_2$, if $\square(\kappa)$ holds then there exists a streamlined κ -Aronszajn tree T such that $T \xrightarrow{\Delta} [\kappa]_\kappa^{n,1}$ holds for every $n < \omega$. By the preceding proposition and Fact 2.7, assuming additionally $\mu^{\aleph_0} < \kappa$ for every $\mu < \kappa$, moreover $T \xrightarrow{\Delta} [\kappa]_\kappa^{\aleph_0,1}$ holds.

4. A FAMILY OF INCOMPARABLE TREES

The next lemma is inspired by the proof of [Tod07a, Proposition 8.4].

Lemma 4.1. *Suppose that T is a χ -complete streamlined κ -tree and weak $T \xrightarrow{\Delta} [\kappa]_\theta^{2,1}$ holds, with $\theta \geq 2$. Then there is a sequence $\langle T^A \mid A \in \mathcal{P}(\theta) \rangle$ of χ -complete streamlined κ -Aronszajn trees such that for all $A, B \in \mathcal{P}(\theta)$:*

- (1) $(T \restriction \text{acc}(\kappa), \subseteq)$ and $(T^A \restriction \text{acc}(\kappa), \subseteq)$ are order-isomorphic, there is a weak embedding from T to T^A , and if T is normal, then so is T^A ;

- (2) if $A \subseteq B$, then there is a level-preserving weak embedding from T^A to T^B ;
- (3) if $A \not\subseteq B$, then there is no Lipschitz map from a κ -sized subset of T^A to T^B ;
- (4) if $A \neq \emptyset$, then there is no Lipschitz map from a κ -sized subset of T^A to T .

If $T \xrightarrow{\Delta} [\kappa]_{\theta}^{2,1}$ holds, then (1)–(4) can be achieved in addition to the following:

- (5) For every $S \subseteq \text{acc}(\kappa)$ such that T is S -coherent,² all trees in $\langle T^A \mid A \in \mathcal{P}(\theta) \rangle$ are S -coherent.

Proof. By possibly applying an injection from $\{t(\alpha) \mid t \in T, \alpha \in \text{dom}(t)\}$ to κ , we may assume that $T \subseteq {}^{<\kappa}\kappa$. Define a map $\pi : {}^{<\kappa}\kappa \rightarrow {}^{<\kappa}\kappa$, as follows. Given $\alpha < \kappa$, $i < 2$ and $s : (2 \cdot \alpha) + i \rightarrow \kappa$, let $\pi(s)$ be the unique function from α to κ to satisfy

$$(\pi(s))(\beta) = s((2 \cdot \beta) + 1), \quad (\beta < \alpha).$$

Fix a witness c for weak $T \xrightarrow{\Delta} [\kappa]_{\theta}^{2,1}$. For each $A \subseteq \theta$, we shall define a map $\mathfrak{h}_A : T \rightarrow {}^{<\kappa}\kappa$ that will satisfy $\pi(\mathfrak{h}_A(t)) = t$ for every $t \in T$. Indeed, given $\alpha < \kappa$ and $t \in T_\alpha$, let $\mathfrak{h}_A(t)$ be the unique function from $(2 \cdot \alpha)$ to κ to satisfy that for all $\beta < \alpha$ and $i < 2$:

$$(\mathfrak{h}_A(t))(2 \cdot \beta + i) = \begin{cases} 0, & \text{if } i = 0 \text{ and } c(t \upharpoonright \beta) \in A; \\ t(\beta), & \text{otherwise.} \end{cases}$$

Consider $T^A := \{s \upharpoonright \epsilon \mid s \in \text{Im}(\mathfrak{h}_A), \epsilon \leq \text{dom}(s)\}$, and note that:

- (i) $\pi[T^A] \subseteq T$;
- (ii) for all $t, t' \in T$ with $\text{dom}(t) < \text{dom}(t')$:
 - $t \subseteq t'$ iff $\mathfrak{h}_A(t) \subseteq \mathfrak{h}_A(t')$;
 - $\pi(\mathfrak{h}_A(t)) = t$.
- (iii) for all $s, s' \in T^A$ with $\text{dom}(s) < \text{dom}(s')$:
 - $s \subseteq s'$ iff $\pi(s) \subseteq \pi(s')$;
 - if $\text{dom}(s) = 2 \cdot \alpha$ for some α , then $\mathfrak{h}_A(\pi(s)) = s$;
 - if $\text{dom}(s) = 2 \cdot \alpha + 1$ for some α , then $\mathfrak{h}_A(\pi(s)) = s \upharpoonright (2 \cdot \alpha)$.

Consequently, $(T \upharpoonright \text{acc}(\kappa), \subseteq)$ and $(T^A \upharpoonright \text{acc}(\kappa), \subseteq)$ are order-isomorphic, and if T is normal then so is T^A . By Lemma 2.11, T is a κ -Aronszajn tree. So, by the said existence of isomorphism, so is T^A . It also follows from Clauses (ii) and (iii) that T^A is χ -complete, and that $\mathfrak{h}_A$ is a weak embedding from T to T^A .

Claim 4.1.1. *Let $A \subseteq B \subseteq \theta$. Then there is a level-preserving weak embedding from T^A to T^B .*

²This is a refinement of Definition 2.5: T is S -coherent iff for all $\alpha \in S$ and $t_0, t_1 \in T_\alpha$, there is an $\epsilon < \alpha$ such that $t_0(\beta) = t_1(\beta)$ for every $\beta \in [\epsilon, \alpha)$.

Proof. Define $g : T^A \rightarrow {}^{<\kappa}\kappa$ as follows. Given $s \in T^A$, set $\alpha := \text{dom}(s)$, and define $g(s) : \alpha \rightarrow \kappa$ by letting for all $\beta < \kappa$ and $i < 2$ with $2 \cdot \beta + i < \alpha$:

$$g(s)(2 \cdot \beta + i) = \begin{cases} 0, & \text{if } i = 0 \text{ and } c(\pi(s) \upharpoonright \beta) \in B; \\ s(2 \cdot \beta + i), & \text{otherwise.} \end{cases}$$

By Clause (iii), for every pair $s \subsetneq s'$ of nodes in T^A , we have $g(s) \subsetneq g(s')$. Thus, we are left with showing that $\text{Im}(g) \subseteq T^B$. To this end, let $s \in T^A$. Recalling the definition of T^A , let us pick $t \in T$ such that $s \subseteq \dot{\cup}_A(t)$. As $g(s) \subseteq g(\dot{\cup}_A(t))$ and $\dot{\cup}_B(t) \in T^B$, it thus suffices to prove that $g(\dot{\cup}_A(t)) = \dot{\cup}_B(t)$.

Let $\beta < \kappa$ and $i < 2$ be such that $2 \cdot \beta + i < \text{dom}(t)$. By expanding the definition of the map g and then the definition of the map $\dot{\cup}_A$, and recalling Clause (ii), we see that

$$\begin{aligned} g(\dot{\cup}_A(t))(2 \cdot \beta + i) &= \begin{cases} 0, & \text{if } i = 0 \text{ and } c(t \upharpoonright \beta) \in B; \\ (\dot{\cup}_A(t))(2 \cdot \beta + i), & \text{otherwise.} \end{cases} \\ &= \begin{cases} 0, & \text{if } i = 0 \text{ and } c(t \upharpoonright \beta) \in B; \\ 0, & \text{if } i = 0 \text{ and } c(t \upharpoonright \beta) \in A \setminus B; \\ t(\beta), & \text{otherwise.} \end{cases} \end{aligned}$$

However, $A \subseteq B$, which means that the middle case does not exist. So, there are only the top and bottom cases and they coincide with the definition of $(\dot{\cup}_B(t))(2 \cdot \beta + i)$, so we are done. $\square$

Claim 4.1.2. *Let $A, B \in \mathcal{P}(\theta)$ with $A \not\subseteq B$. Then there is no Lipschitz map from a κ -sized subset of T^A to T^B . Furthermore, for every $X \subseteq T^A$ of size κ and every $(<\kappa)$ -to-one map $g : X \rightarrow T^B$, there are $s \neq s'$ in X such that $\Delta(g(s), g(s')) < \Delta(s, s')$.*

Proof. Fix $\tau \in A \setminus B$. Since T^A and T^B are κ -trees, given a map $g : X \rightarrow T^B$ as above, we may fix a κ -sized dom-rapid subfamily S of $\{(\pi(s), \pi(g(s))) \mid s \in X\}$. As c witnesses weak $T \xrightarrow{\wedge} [\kappa]_{\theta}^{2,1}$, we may pick $(x, y) \neq (x', y')$ in S and $\delta < \kappa$ such that:

- $c(x \wedge x') = \tau = c(y \wedge y')$;
- if $x <_{\text{lex}} x'$, then $\text{dom}(x) < \text{dom}(x')$;
- if $y <_{\text{lex}} y'$, then $\text{dom}(y) < \text{dom}(y')$;
- $\text{dom}(x \wedge x') = \delta = \text{dom}(y \wedge y')$.

Without loss of generality, $\text{dom}(x) < \text{dom}(x')$. So, since S is dom-rapid, also $\text{dom}(y) < \text{dom}(y')$. It thus follows that $x <_{\text{lex}} x'$ and $y <_{\text{lex}} y'$. Recalling Definition 2.3, this means that x is incomparable with x' , and y is incomparable with y' . Altogether, $\delta \in \text{dom}(x) \cap \text{dom}(x') \cap \text{dom}(y) \cap \text{dom}(y')$.

Fix $s, s' \in X$ such that $(x, y) = (\pi(s), \pi(g(s)))$ and $(x', y') = (\pi(s'), \pi(g(s')))$. By Clause (iii), $\dot{\cup}_A(x) \subseteq s$, $\dot{\cup}_A(x') \subseteq s'$, $\dot{\cup}_B(y) \subseteq g(s)$ and $\dot{\cup}_B(y') \subseteq g(s')$. We now analyze $s \wedge s'$ and $g(s) \wedge g(s')$:

- As $x \upharpoonright \delta = x' \upharpoonright \delta$, we have $\dot{\cap}_A(x) \upharpoonright (2 \cdot \delta) = \dot{\cap}_A(x') \upharpoonright (2 \cdot \delta)$. As $c(x \wedge x') = \tau \in A$, we have $\dot{\cap}_A(x)(2 \cdot \delta) = 0 = \dot{\cap}_A(x')(2 \cdot \delta)$. Finally, $\dot{\cap}_A(x)(2 \cdot \delta + 1) = x(\delta)$ and $\dot{\cap}_A(x')(2 \cdot \delta + 1) = x'(\delta)$. Thus, altogether, $\text{dom}(s \wedge s') = \text{dom}(\dot{\cap}_A(x) \wedge \dot{\cap}_A(x')) = 2 \cdot \delta + 1$.
- As $y \upharpoonright \delta = y' \upharpoonright \delta$, we have $\dot{\cap}_B(y) \upharpoonright (2 \cdot \delta) = \dot{\cap}_B(y') \upharpoonright (2 \cdot \delta)$. As $c(y \wedge y') = \tau \notin B$, we have $\dot{\cap}_B(y)(2 \cdot \delta) = y(\delta)$ and $\dot{\cap}_B(y')(2 \cdot \delta) = y'(\delta)$. Altogether, $\text{dom}(g(s) \wedge g(s')) = \text{dom}(\dot{\cap}_B(y) \wedge \dot{\cap}_B(y')) = 2 \cdot \delta$.

Altogether, $\Delta(g(s), g(s')) < \Delta(s, s')$, as sought. $\square$

Claim 4.1.3. *Let $A \in \mathcal{P}(\theta) \setminus \{\emptyset\}$. Then there is no Lipschitz map from a κ -sized subset of T^A to T . Furthermore, for every $X \subseteq T^A$ of size κ and every $(<\kappa)$ -to-one map $g : X \rightarrow T$, there are $s \neq s'$ in X such that $\Delta(g(s), g(s')) < \Delta(s, s')$.*

Proof. This is very similar to the proof of Claim 4.1.2. Fix $\tau \in A$. Given a map $g : X \rightarrow T$ as above, fix a κ -sized dom-rapid subfamily S of $\{(\pi(s), g(s)) \mid s \in X\}$. Pick $(x, y) \neq (x', y')$ in S and $\delta < \kappa$ such that:

- $c(x \wedge x') = \tau = c(y \wedge y')$;
- $\text{dom}(x \wedge x') = \delta = \text{dom}(y \wedge y')$;
- $\delta \in \text{dom}(x) \cap \text{dom}(x') \cap \text{dom}(y) \cap \text{dom}(y')$.

Fix $s, s' \in X$ such that $(x, y) = (\pi(s), g(s))$ and $(x', y') = (\pi(s'), g(s'))$. So $\Delta(g(s), g(s')) = \delta$, and since $\tau \in A$, we have $\Delta(s, s') = 2 \cdot \delta + 1$. $\square$

Suppose now that $T \xrightarrow{\Delta} [\kappa]_\theta^{2,1}$ holds, as witnessed by a colouring $d : \kappa \rightarrow \theta$. In particular, the map $t \mapsto d(\text{dom}(t))$ witnesses weak $T \xrightarrow{\Delta} [\kappa]_\theta^{2,1}$, so we may assume that our initial c is derived from d in this very way.

Let S denote the collection of all $\alpha \in \text{acc}(\kappa)$ such that all $t_0, t_1 \in T_\alpha$ agree on a tail, i.e., there is an $\epsilon < \alpha$ such that $t_0(\beta) = t_1(\beta)$ for every $\beta \in [\epsilon, \alpha)$.

Claim 4.1.4. *Let $A \subseteq \theta$. Then T^A is S -coherent.*

Proof. Let $\alpha \in S$. Let s_0, s_1 be two elements of the α^{th} -level of T^A . It follows that we may fix t_0, t_1 from the α^{th} -level of T such that $s_j = \dot{\cap}_A(t_j)$ for each $j < 2$. Fix a large enough $\epsilon < \alpha$ such that $t_0(\beta) = t_1(\beta)$ for every

$\beta \in [\epsilon, \alpha)$. Then, for every $\beta \in [\epsilon, \alpha)$, for every $i < 2$,

$$\begin{aligned}
s_0(2 \cdot \beta + i) &= (\psi_A(t_0))(2 \cdot \beta + i) \\
&= \begin{cases} 0, & \text{if } i = 0 \text{ and } c(t_0 \upharpoonright \beta) \in A; \\ t_0(\beta), & \text{otherwise.} \end{cases} \\
&= \begin{cases} 0, & \text{if } i = 0 \text{ and } d(\beta) \in A; \\ t_0(\beta), & \text{otherwise.} \end{cases} \\
&= \begin{cases} 0, & \text{if } i = 0 \text{ and } d(\beta) \in A; \\ t_1(\beta), & \text{otherwise.} \end{cases} \\
&= \begin{cases} 0, & \text{if } i = 0 \text{ and } c(t_1 \upharpoonright \beta) \in A; \\ t_0(\beta), & \text{otherwise.} \end{cases} \\
&= (\psi_A(t_1))(2 \cdot \beta + i) = s_1(2 \cdot \beta + i),
\end{aligned}$$

so that s_0 and s_1 agree on a tail. $\square$

This completes the proof $\square$

We now arrive at Theorem B.

Corollary 4.2. *Suppose that $\lambda = \lambda^{<\lambda}$ is an uncountable cardinal, and any of the following:*

- $\square(\lambda^+, <\lambda)$ holds and $\theta = \lambda^+$;
- λ admits a nontrivial C -sequence and $\theta = \lambda$.

Then there is a family of 2^θ many streamlined λ -coherent λ -complete λ^+ -Aronszajn trees such that no two of them admit a Lipschitz map from a λ^+ -sized subset of one to the other.

Proof. Using Fact 2.6, let T be a streamlined λ -coherent λ -complete λ^+ -Aronszajn tree T . By Propositions 3.3 and 3.4, $T \xrightarrow{\Delta} [\lambda^+]_\theta^{<\lambda,1}$ holds. By Lemma 4.1, then, there is a sequence $\langle T^A \mid A \in \mathcal{P}(\theta) \rangle$ of λ -coherent, λ -complete streamlined λ^+ -Aronszajn trees such that for all $A, B \in \mathcal{P}(\theta)$, if $A \not\subseteq B$, then there is no Lipschitz map from a λ^+ -sized subset of T^A to T^B . Let $\mathcal{A}$ be an antichain in $\mathcal{P}(\theta)$ of size 2^θ . Then $\langle T^A \mid A \in \mathcal{A} \rangle$ is as sought. $\square$

By feeding Lemma 4.1 with [IR25, Theorem A], we get another conclusion:

Corollary 4.3. *Suppose that κ is the successor of a regular uncountable cardinal. Then the following are equivalent:*

- *There is a special κ -Aronszajn tree;*
- *There is a family of 2^κ many streamlined special κ -Aronszajn trees such that no two of them admit a Lipschitz map from a κ -sized subset of one to the other.* $\square$

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