A failed attempt to preserve unbounded families

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Abstract

I describe how an attempt to simplify the method of *ultrafilter limits* to control the unbounded number in finite support iterations of ccc posets led me to prove that, whenever there is a new real over a model V_0 of ZFC, one can construct an infinite set of natural numbers which is null with respect to all non-principal ultrafilters over ω from V_0.

Introduction

When forcing different values to many cardinal characteristics, as those in Cichoń's diagram, one of the most challenging tasks seems to be separating the unbounding number $\mathfrak{b}$ and its dual, the dominating number $\mathfrak{d}$, from everything else. In the context of FS (finite support) iterations of ccc posets, goodness arguments [JS90, Bre91, CM19] and dualization methods [KTT18, GKS19, GKMS22] are very useful to force different values to cardinal characteristics, but controlling the values of $\mathfrak{b}$ and $\mathfrak{d}$ requires sophisticated methods such as *iterations with ultrafilter limits* [GMS16, Mej24, Yam25] and *iterations with fam-limits* [She00, CMU].[1]

In an attempt to simplify the methods to control $\mathfrak{b}$ and $\mathfrak{d}$, I revisited the theory and tried to improve it by weakening its dependency on ultrafilters and fams as much as possible. Although I failed in this task, I ended up proving that, whenever there is a new real over a model V_0 of ZFC, one can construct an infinite set of natural numbers which is null with respect to all non-principal ultrafilters over ω from V_0 (Thm. 9), which is the main result of this paper.

This paper is divided into two sections. Section 1 reviews methods to control $\mathfrak{b}$ and $\mathfrak{d}$ in FS iterations of ccc posets and describes my attempts to simplify the known framework. This motivates Question 7, which only concerns ultrafilters over ω. We answer this question in Section 2 with the main result (Thm. 9).

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[1] Here, "fam" abbreviates "finitely additive measure".

The purpose of Section 1 is to describe the line of thought leading to the main question and main result of this paper, and the latter is presented in Section 2. The reader can safely jump to Section 2 without any trouble.

1 Attempts to preserve unbounded families

The core of these methods are the following properties, which guarantee that certain strong unbounded families can be preserved after forcing. They generalize other methods such as Miller's *compactness argument* to preserve unbounded families [Mil81].

Definition 1 (Mejía [Mej19, BCM21]). Let $\mathbb{P}$ be a poset and $Q \subseteq \mathbb{P}$. We say that Q is Fr-*linked* in $\mathbb{P}$ if, for any $\bar{p} = \{p_k \mid k < \omega\} \in {}^{\omega}Q$, there is some $q \in \mathbb{P}$ forcing that $\{k < \omega \mid p_k \in \dot{G}\}$ is infinite.[2]

For an infinite cardinal θ, define:

(1) $\mathbb{P}$ is θ-Fr-*linked* if $\mathbb{P}$ can be covered by $\leq \theta$ many Fr-linked subsets. When $\theta = \aleph_0$, we just write σ-Fr-*linked*.

(2) $\mathbb{P}$ is θ-Fr-*Knaster* if any subset of $\mathbb{P}$ of size θ contains an Fr-linked subset of the same size.

It is clear that any θ-Fr-linked forcing is θ'-Fr-Knaster for any cardinal θ' with $\mathsf{cf}(\theta') > \theta$.

We look at the following type of unbounded families.

Definition 2 (See e.g. [BCM21, CM22]). Let θ be an infinite cardinal. A set $\{x_i \mid i \in I\} \subseteq {}^{\omega}\omega$ is θ-*unbounded* if $|I| \geq \theta$ and, for any $y \in {}^{\omega}\omega$, $|\{i \in I \mid x_i \leq^* y\}| < \theta$.

Fact 3. *If there exists a θ-unbounded family indexed by a set I (of size $\geq \theta$), then $\mathfrak{b} \leq \theta$ and $\mathsf{cov}([I]^{<\theta}) \leq \mathfrak{d}$.*[3]

The following facts illustrate the connection between Fr-linkedness and the preservation of unbounded families.

Lemma 4 (Cardona and Mejía [Mej19, Mej24]). *Let $\mathbb{P}$ be a poset. A set $Q \subseteq \mathbb{P}$ is Fr-linked in $\mathbb{P}$ iff, for any $\mathbb{P}$-name $\dot{k}$ of a natural number, there is some $k' < \omega$ (in the ground model) such that no $p \in Q$ forces $k' \leq \dot{k}$.*

Theorem 5 (Brendle, Cardona, and Mejía [BCM21]). *If θ is a regular cardinal and $\mathbb{P}$ is a θ-Fr-Knaster poset, then $\mathbb{P}$ preserves θ-unbounded families.*[4]

From now on, we fix an uncountable regular cardinal θ. The typical way to force $\mathfrak{b} = \theta$ in a FS iteration of ccc posets is:

[2] Fr stands for *Fréchet*, in allusion to the Fréchet filter, in the sense that $\{k < \omega \mid p_k \in \dot{G}\}$ is forced positive with respect to that filter.

[3] The covering number $\mathsf{cov}([I]^{<\theta})$ equals $|I|$ if either θ is regular or $\theta < |I|$, otherwise it equals $\mathsf{cf}(\theta)$.

[4] A variation of the Knaster property, called *sequentially* θ-Fr-*Knaster*, works to eliminate the regularity of θ in the hypothesis [MU].

(B1) Force $\mathfrak{b} \geq \theta$ by book-keeping, i.e., take care of any $F \subseteq {}^{\omega}\omega$ of size $< \theta$ by adding a dominating real over F. This is done by using Hechler forcing restricted to models of size $< \theta$.

(B2) Add a θ-unbounded family and preserve it until the final step of the iteration.

This unbounded family can be added with Cohen reals at some intermediate stage, and its preservation is (morally) guaranteed if our iteration is θ-Fr-Knaster.

The task (B1) can be dealt with without difficulties. Concerning (B2), as the Cohen reals indeed add this θ-unbounded family, the remaining problem is to show that the iteration is θ-Fr-Knaster.

How can we ensure a θ-Fr-Knaster iteration? It is known that the FS iteration of θ-Fr-Knaster posets is θ-Fr-Knaster [Mej19], however, in practice it is unclear whether the iterands of an iteration such as the one forcing different values to the left side of Cichoń's diagram [GMS16] is θ-Fr-Knaster.

To explain the difficulty, let us review a couple of examples.

(E1) Any poset $\mathbb{Q}$ of size $< \theta$ is $|\mathbb{Q}|$-Fr-linked (witnessed by its singletons), and hence θ-Fr-Knaster.

(E2) The standard forcing $\mathbb{E}$ adding an eventually different real is σ-Fr-linked ([Mej19]). Let $\langle E_n \mid n < \omega \rangle$ be the sequence of Fr-linked sets covering $\mathbb{E}$.

Assume that θ' is another regular cardinal and λ a cardinal such that $\theta < \theta' < \lambda = \lambda^{\aleph_0}$ and $\mathsf{cof}([\lambda]^{<\theta}) = \mathsf{cof}([\lambda]^{<\theta'}) = \lambda$ (this latter assumption is required for a successful book-keeping). To force $\mathfrak{b} = \theta < \mathsf{non}(\mathcal{M}) = \theta' < \mathsf{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$, one first add λ-many Cohen reals to get a θ-unbounded family of size λ, and afterward alternate Hechler forcing restricted to models of size $< \theta$, and $\mathbb{E}$ restricted to models of size $< \theta'$. The Cohen and Hechler iterands fit (E1), however, it is not guaranteed that the restricted $\mathbb{E}$ is σ-Fr-linked, in spite that the full forcing has the property by (E2).

To simplify the discussion, let us think about the case $\theta = \aleph_1$, i.e. we look at the problem of forcing $\mathfrak{b} = \aleph_1 < \mathsf{non}(\mathcal{M}) = \theta' < \mathsf{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$. We still look at an iteration as above, but we can forget about Hechler forcing as there is no need to increase $\mathfrak{b}$ anymore, instead, we aim to preserve an $\aleph_1$-unbounded family composed of Cohen reals. So the iteration is structured as first adding λ many Cohen reals, and then iterating with $\mathbb{E}$ restricted to models of size $< \theta'$. Book-keeping and goodness work to ensure the values of $\mathsf{non}(\mathcal{M})$ and $\mathsf{cov}(\mathcal{M})$, so the remaining issue is to ensure that the iteration is $\aleph_1$-Fr-Knaster.

Let us fix, in detail, the type of iteration we are dealing with. Consider a FS iteration $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi \mid \xi < \pi \rangle$ satisfying:

(P1) $\pi \geq \lambda$.

(P2) $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_\xi$-name of Cohen forcing for $\xi < \lambda$.

(P3) $\mathbb{P}^\bullet_\xi$ ($\xi \leq \pi$) is a dense subset of $\mathbb{P}_\xi$, which we describe in (P5).

(P4) For $\lambda \leq \xi < \pi$, $\mathbb{P}^-_\xi$ is a complete subposet of $\mathbb{P}^\bullet_\xi$, $|\mathbb{P}^-_\xi| < \theta'$, and $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}^-_\xi$-name of $\mathbb{E}$.

(P5) $\mathbb{P}^\bullet_\xi$ is the set of conditions $p \in \mathbb{P}_\xi$ such that, for $\alpha \in \mathsf{dom}\, p$, if $\alpha < \lambda$ then $p(\alpha)$ is decided (as a condition in Cohen forcing); and if $\alpha \geq \lambda$ then there is an $n < \omega$ such that $p(\alpha)$ is a $\mathbb{P}^-_\alpha$-name of a member of E_n (recall from (E2) that $\mathbb{E} = \bigcup_{n<\omega} E_n$).

Such an iteration is constructed as follows: First construct $\mathbb{P}_\lambda$, from which $\mathbb{P}^\bullet_\lambda$ can be defined; at a further step ξ with $\lambda \leq \xi < \pi$, once $\mathbb{P}_\xi$ is constructed, $\mathbb{P}^\bullet_\xi$ gets defined, so we pick $\mathbb{P}^-_\xi$ (at will), which allows to define $\dot{\mathbb{Q}}_\xi$ and $\mathbb{P}_{\xi+1}$ (as the two-step iteration of $\mathbb{P}_\xi$ and $\dot{\mathbb{Q}}_\xi$).

Remark 6. Fix $\lambda \leq \xi < \pi$ and let G_ξ be a $\mathbb{P}_\xi$-generic over V. Denote $G^-_\xi := \mathbb{P}^-_\xi \cap G_\xi$, which is $\mathbb{P}^-_\xi$-generic over V. Then, the iterand to reach step $\xi + 1$ is $\mathbb{Q}_\xi = \mathbb{E}^{V[G^-_\xi]}$. Notice that we can write $\mathbb{Q}_\xi = \mathbb{E}^{N_\xi}$ where N_ξ is a model (in $V[G_\xi]$) of (a large enough finite fragment of) ZFC of size $|{}^\omega\omega \cap V[G^-_\xi]|$ which contains ${}^\omega\omega \cap V[G^-_\xi]$.

In general, $|\mathbb{P}^-_\xi| < \theta'$ does not imply that $|{}^\omega\omega \cap V[G^-_\xi]| < \theta'$. To ensure the implication, one can assume in the ground model that θ' is $\aleph_1$-*inaccessible*, which means that $\mu^{\aleph_0} < \theta'$ for any cardinal $\mu < \theta'$.

Typically, the definition of $\mathbb{P}^-_\xi$ is very concrete. In the ground model, one picks a model $M_\xi \prec H_\chi$ (where χ is a large enough regular cardinal), containing relevant information, such that ${}^\omega M_\xi \subseteq M_\xi$ and $|M_\xi| < \theta'$ (this is possible if θ' is $\aleph_1$-inaccessible). So set $\mathbb{P}^-_\xi := \mathbb{P}_\xi \cap M_\xi$. Then $M_\xi[G_\xi] \prec H^{V[G_\xi]}_\chi$, $\mathbb{Q}_\xi = \mathbb{E}^{V[G^-_\xi]} = \mathbb{E}^{M_\xi[G_\xi]}$, and $|M_\xi[G_\xi]| = |M_\xi| < \theta'$.

We now attempt to prove that the iteration is $\aleph_1$-Fr-Knaster. Let $\{p_\alpha \mid \alpha < \omega_1\} \subseteq \mathbb{P}^\bullet_\pi$. We may assume that the domains of the conditions form a Δ-system with root Δ and, by further shrinking the set, we may assume that, for any $\xi \in \Delta$, there is some $n_\xi < \omega$ such that $p_\alpha(\xi)$ is forced in E_{n_ξ} for all $\alpha < \omega_1$. This is what we call a *uniform* Δ*-system*. The aim is to show that any uniform Δ-system is Fr-linked, for which it is enough to prove that

(♡) For any uniform Δ-system $\{p_k \mid k < \omega\} \subseteq \mathbb{P}^\bullet_\xi$, there is some $q \in \mathbb{P}_\pi$ forcing that $\{k < \omega \mid p_k \in \dot{G}\}$ is infinite.

So let $\{p_k \mid k < \omega\} \subseteq \mathbb{P}^\bullet_\xi$ be a uniform Δ-system. Enumerate $\Delta = \{\xi^*_i \mid i < m^*\}$ increasingly. As $\langle p_k(\xi^*_0) \mid k < \omega\rangle$ is a sequence of $\mathbb{P}^-_{\xi^*_0}$-names of members of $E_{n_{\xi^*_0}}$, they define a $\mathbb{P}^-_{\xi^*_0}$-name of a countable sequence in $E_{n_{\xi^*_0}}$, so we can find a $\mathbb{P}^-_{\xi^*_0}$-name $q(\xi^*_0)$ of a member of $\dot{\mathbb{Q}}_\xi$ forcing that infinitely many $p_k(\xi^*_0)$ are in the generic set. It is easy to show that the condition in $\mathbb{P}_\pi$ composed of only $q(\xi^*_0)$ forces that infinitely many $p_k{\restriction}\xi^*_1$ are in the generic set.

The real issue appears when we look at ξ^*_1. In the same way, we can find a $\mathbb{P}^-_{\xi^*_1}$-name $q(\xi^*_1)$ of a member of $\dot{\mathbb{Q}}_\xi$ forcing that infinitely many $p_k(\xi^*_1)$ are in the generic set. However, it cannot be ensured that the sets $\{k < \omega \mid p_k(\xi^*_0) \in \dot{G}(\xi^*_0)\}$ and $\{k < \omega \mid p_k(\xi^*_1) \in \dot{G}(\xi^*_1)\}$ intersect (and we want them to have infinite intersection). Alternatively, we can look at the $\mathbb{P}_{\xi^*_1}$-name $\dot{w}_1$ of the set $\{k < \omega \mid p_k{\restriction}\xi^*_1 \in \dot{G}\}$. If $\dot{w}_1$ could be forced in $V[G^-_{\xi^*_1}]$, one could apply Fr-linkedness to $\langle p_k(\xi^*_1) \mid k \in \dot{w}_1\rangle$ and find some $\mathbb{P}^-_{\xi^*_1}$-name $q(\xi^*_1)$ of a member of $\dot{\mathbb{Q}}_\xi$ forcing that $\dot{w}_1 \cap \{k < \omega \mid p_k(\xi^*_1) \in \dot{G}(\xi^*_1)\}$ is infinite, which would let us continue with the construction of q. However, nothing guarantees that $\dot{w}_1$ can be forced in $V[G^-_{\xi^*_1}]$.

This is the point where ultrafilters come into play to guarantee the construction of q and prove (♡). This considers sequences of names of the form $\langle \dot{D}_\xi \mid \xi \leq \pi\rangle$ where each $\dot{D}_\xi$ is a

$\mathbb{P}_\xi$-name of a non-principal ultrafilter over ω, and they form an increasing chain. If we can force $\dot{w}_1 \in \dot{D}_{\xi_1^*}$ and $\{k < \omega \mid p_k(\xi_1^*) \in \dot{G}(\xi_1^*)\} \in \dot{D}_{\xi_1^*+1}$, we then get that their intersection is infinite, and even in $\dot{D}_{\xi_1^*+1}$, so there is hope to continue with the construction of q.

To have $\{k < \omega \mid p_k(\xi_1^*) \in \dot{G}(\xi_1^*)\} \in \dot{D}_{\xi_1^*+1}$, it is not enough that $\mathbb{E}$ is Fr-linked, but one needs a stronger notion allowing the so called *ultrafilter limits*. Fortunately, $\mathbb{E}$ has this property, but to apply it in our argument, one needs an ultrafilter in $V[G^-_{\xi_1^*}]$ (not just in $V[G_{\xi_1^*}]$), so an additional requirement in the construction is to force that $\dot{D}_\xi \cap V[G^-_\xi] \in V[G^-_\xi]$ for all $\xi < \pi$.

Things will get messy when setting up the framework that guarantees (♡). The sequences of names of ultrafilters have to be constructed simultaneously with the whole iteration, and all ultrafilters at ξ must reflect down to $V[G^-_\xi]$ (as explained above). Since we want that $|\mathbb{P}^-_\xi| < \theta'$, the number of sequences of names of ultrafilters must be $< \theta'$. Here is where the technical method of *iterations with ultrafilter limits* arises.

I have been trying to get rid of the sequences of ultrafilters and find a simpler way to prove (♡) and, as a consequence, that the iteration is $\aleph_1$-Fr-Knaster. One can still rely on ultrafilter limits on $\mathbb{E}$, which do not make things more difficult, on the contrary, it is good to keep them with the hope to find the desired simpler approach.

My last attempt relies on the reflection of ultrafilters from $V[G_\xi]$ to $V[G^-_\xi]$. We come back to the construction of q above and assume we have constructed $q{\restriction}\xi_1^*$. In $V[G_{\xi_1^*}]$, if we are able to find a non-principal ultrafilter D over ω such that $\dot{w}_1 \in D$ and $D^- := D \cap V[G^-_{\xi_1^*}] \in V[G^-_{\xi_1^*}]$, then the ultrafilter-limit property of $\mathbb{E}$ allows to find a $\mathbb{P}^-_{\xi_1^*} * \dot{\mathbb{Q}}_{\xi_1^*}$-name of an ultrafilter $\dot{D}^+$ extending D^-, and a $\mathbb{P}^-_{\xi_1^*}$-name $q(\xi_1^*)$ of a member of $\mathbb{Q}_\xi$ forcing that $\{k < \omega \mid p_k(\xi_1^*) \in \dot{G}(\xi_1^*)\} \in \dot{D}^+$. Then, the filters D and $\dot{D}^+$ can be amalgamated [BCM21, Lem. 3.20]: it is forced that they have the finite intersection property. Therefore, $\dot{w}_1 \cap \{k < \omega \mid p_k(\xi_1^*) \in \dot{G}(\xi_1^*)\}$ is forced infinite and we can continue with the construction of q.

The argument above would be successful if we solve the following problem in the positive.

Question 7. *Let $\mathbb{P}$ be a ccc poset and $\dot{w}$ a $\mathbb{P}$-name of an infinite subset of ω. Is there a non-principal ultrafilter D over ω (in the ground model) such that $\mathbb{P}$ forces that $\dot{w}$ is D-positive?*

2 A theorem about ultrafilters

First, we fix some terminology.

Definition 8. Let V_0 be a model of ZFC and w an infinite set of natural numbers. We say that w is *uf-null over* V_0 if it is null with respect of all non-principal ultrafilters over ω in V_0.

When trying to solve Question 7, I came up with a very non-ccc poset adding a uf-null set over V_0. But I was not able to see whether some ccc forcing would add such a thing.

In September 2025, during the 18th Asian Logic Conference in Kyoto, I asked several colleagues about Question 7. I also posted the question in mathoverflow and got some feedback, but I was still away from finding the answer to this problem.

In a desperate attempt, I bothered ChatGPT, but his reply was a lot of discouraging robotic hallucination. But among this pile of non-sense, the AI suggested to increase the continuum way above the number of ground-model ultrafilters over ω. This idea was meaningful, not just to find a ccc counter-example to Question 7, but for something worse. I then asked myself, "how do we know how many ultrafilters are there?", and remembered that independent families are used to show that there are $2^{2^{\aleph_0}}$ different ultrafilters over the ground model. In a very similar way, I could use independent families to prove that the answer to Question 7 is as bad as one can expect, and to reply my own question in mathoverflow.[5]

Theorem 9. *Let $V_0 \subseteq V_1$ be transitive models of* ZFC*. If there is a real $r \in V_1 \smallsetminus V_0$, then there is some uf-null $z \in [\omega]^{\aleph_0} \cap V_1$ over V_0.*

Proof. Let $r \in {}^{\omega}2 \cap V_1 \smallsetminus V_0$. Fix a countable independent family $\langle a_n \mid n < \omega \rangle \in V_0$ of infinite subsets of ω. For $a \subseteq \omega$, denote $a^0 := a$ and $a^1 := \omega \smallsetminus a$. Then, for any $x \in {}^{\omega}2$, $B_x := \{a_n^{x(n)} \mid n < \omega\}$ has the strong finite intersection property (i.e. the intersection of finitely many sets from B_x is infinite).

If $U \in V_0$ is a non-principal ultrafilter over ω (in V_0), then there is a unique $x^U \in {}^{\omega}2$ such that $B_{x^U} \subseteq U$ and $x^U \in V_0$. Since $r \notin V_0$, $r(m) \neq x^U(m)$ for some $m < \omega$, thus B_r contains a U-null set (namely, $a_m^{r(m)}$).

Since B_r is countable, it has a pseudo-intersection $z \in [\omega]^{\aleph_0} \cap V_1$, which is clearly uf-null over V_0. □

Corollary 10. *If $\mathbb{P}$ is a poset adding a new real, then it adds a uf-null set over the ground model.*

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References

[BCM21] Jörg Brendle, Miguel A. Cardona, and Diego A. Mejía. Filter-linkedness and its effect on preservation of cardinal characteristics. *Ann. Pure Appl. Logic*, 172(1):102856, 2021.

[Bre91] Jörg Brendle. Larger cardinals in Cichoń's diagram. *J. Symbolic Logic*, 56(3):795–810, 1991.

[5] https://mathoverflow.net/questions/499480/ultrafilters-of-the-ground-model-in-ccc-generic-extensions

[CM19] Miguel A. Cardona and Diego A. Mejía. On cardinal characteristics of Yorioka ideals. *Math. Log. Q.*, 65(2):170–199, 2019.

[CM22] Miguel A. Cardona and Diego A. Mejía. Forcing constellations of Cichoń's diagram by using the Tukey order. *Kyōto Daigaku Sūrikaiseki Kenkyūsho Kōkyūroku*, 2213:14–47, 2022. `http://hdl.handle.net/2433/275447`.

[CMU] Miguel A. Cardona, Diego A. Mejía, and Andrés Uribe-Zapata. A general theory of iterated forcing using finitely additive measures. Preprint, arXiv:2406.09978, 2024.

[GKMS22] Martin Goldstern, Jakob Kellner, Diego A. Mejía, and Saharon Shelah. Cichoń's maximum without large cardinals. *J. Eur. Math. Soc. (JEMS)*, 24(11):3951–3967, 2022.

[GKS19] Martin Goldstern, Jakob Kellner, and Saharon Shelah. Cichoń's maximum. *Ann. of Math. (2)*, 190(1):113–143, 2019.

[GMS16] Martin Goldstern, Diego Alejandro Mejía, and Saharon Shelah. The left side of Cichoń's diagram. *Proc. Amer. Math. Soc.*, 144(9):4025–4042, 2016.

[JS90] Haim Judah and Saharon Shelah. The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing). *J. Symbolic Logic*, 55(3):909–927, 1990.

[KTT18] Jakob Kellner, Anda Ramona Tănasie, and Fabio Elio Tonti. Compact cardinals and eight values in Cichoń's diagram. *J. Symb. Log.*, 83(2):790–803, 2018.

[Mej19] Diego A. Mejía. Matrix iterations with vertical support restrictions. In *Proceedings of the 14th and 15th Asian Logic Conferences*, pages 213–248. World Sci. Publ., Hackensack, NJ, 2019.

[Mej24] Diego A. Mejia. Forcing techniques for Cichoń's maximum: lecture notes for the mini-course at the University of Vienna. *Bol. Mat.*, 31(1):1–48, 2024.

[Mil81] Arnold W. Miller. Some properties of measure and category. *Trans. Amer. Math. Soc.*, 266(1):93–114, 1981.

[MU] Diego A. Mejía and Andrés F. Uribe-Zapata. Knaster-like properties at singular cardinals. In preparation.

[She00] Saharon Shelah. Covering of the null ideal may have countable cofinality. *Fund. Math.*, 166(1-2):109–136, 2000.

[Yam25] Takashi Yamazoe. Cichoń's maximum with evasion number. *J. Symb. Log.*, pages 1–31, 2025. Published online, doi:10.1017/jsl.2024.65.