Strong Chain Forced by Bubbly Path

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Abstract

This note presents a proper poset to force a strong chain of length ω_3, an original result by Aspero and Rodriguez. We borrow some of their key ideas.

Introduction

Let $\langle f_\alpha \mid \alpha < \omega_3\rangle$ be a sequence of functions from ω_1 into 2. We call it a *strong chain* of length ω_3, if for each $\alpha < \beta < \omega_3$, we have $\{\delta < \omega_1 \mid f_\alpha(\delta) = 1, f_\beta(\delta) = 0\}$ is finite, $\{\delta < \omega_1 \mid f_\alpha(\delta) = 0, f_\beta(\delta) = 1\}$ is cofinal in ω_1. Let us denote $SC(\omega_1, \omega_3)$, if there exists a strong chain of length ω_3. By [AR], $SC(\omega_1, \omega_3)$ is consistent. We present our proper poset. Due to space limitations, we only mention relevant lemmas (comparison and descend), (merge), and (amalgamation) that are specific to the side conditions of two-type. The second lemma is almost verbatim to [M]. Accordingly, we provide some detail on the first and third lemmas. No higher version of properness nor the ω_3-cc are discussed, as they are identical to the case of properness.

Preliminary

Notation. (GCH) Let X, Y, Z be sets. Since we take intersections quite often, we use abbreviations.

$$XY := X \cap Y, \quad XYZ := X \cap Y \cap Z.$$

Similarly for more sets. Let us write H_{ω_3} to mean

$$H_{\omega_3} = \{x \mid \text{the transitive closure of } x \text{ is of a size strictly less than } \omega_3\}.$$

Now a transitive relational structure $(H_{\omega_3}, \in, \cdots)$ is fixed, where let the dots $\cdots$ represent your favorite predicates. For example, a bijection $\Phi : \omega_3 \longrightarrow H_{\omega_3}$ as, say, a binary relation s.t. for any elementary substructures $(X, \in, \cdots)$ and $(Y, \in, \cdots)$ of $(H_{\omega_3}, \in, \cdots)$, we have the set of images $\Phi[\omega_3 X] = X$ and the intersection XY induces an elementary substructure of $(H_{\omega_3}, \in, \cdots)$. Let us write $\mathcal{C}_{\omega_0}$ and $\mathcal{C}_{\omega_1}$ to mean

$$\mathcal{C}_{\omega_0} = \{S \in [H_{\omega_3}]^{\omega_0} \mid S \text{ induces an elementary substructure of } (H_{\omega_3}, \in, \cdots)\},$$

$$\mathcal{C}_{\omega_1} = \{L \in [H_{\omega_3}]^{\omega_1} \mid L^\omega \subset L, L \text{ induces an elementary substructure of } (H_{\omega_3}, \in, \cdots)\}.$$

Hence, $\omega_1 S = \omega_1 \cap S < \omega_1$ and $\omega_2 L = \omega_2 \cap L < \omega_2$. We are usually sloppy in distingushing the base set of a structure and the structure itself.

Let $X, Y \in \mathcal{C}_{\omega_0}$. Write $X <_{\omega_1} Y$, if $\omega_1 X < \omega_1 Y$. Write $X =_{\omega_1} Y$, if $\omega_1 X = \omega_1 Y$.
Let $I, J \in \mathcal{C}_{\omega_1}$. Write $I <_{\omega_2} J$, if $\omega_2 I < \omega_2 J$. Write $I =_{\omega_2} J$, if $\omega_2 I = \omega_2 J$.

For any finite $\mathcal{S} \subset \mathcal{C}_{\omega_0}$ and $A \subseteq \omega_1$, write

$$\omega_1 \mid \mathcal{S} := \{\omega_1 X \mid X \in \mathcal{S}\}, \quad \mathcal{S} \,|\, A := \{X \in \mathcal{S} \mid \omega_1 X \in A\}.$$

For any finite $\mathcal{L} \subset \mathcal{C}_{\omega_1}$ and $B \subseteq \omega_2$, write

$$\omega_2 \mid \mathcal{L} := \{\omega_2 I \mid I \in \mathcal{L}\}, \quad \mathcal{L} \,|\, B := \{I \in \mathcal{L} \mid \omega_2 I \in B\}.$$

Let $X \in \mathcal{C}_{\omega_0}$ and $I \in \mathcal{C}_{\omega_1}$ with $I \in X$. Let us assume that a filtration $\langle I_\delta \mid \delta < \omega_1 \rangle$ of I belongs to X. Hence the intersection XI satisfies

$$XI = I_{\omega_1 X}.$$

Thus, when we take XI, we deal with the $\omega_1 X$-th member of the filtration of I.

In typical situations, $X \in \mathcal{C}_{\omega_0}$ has an associated $t_X \in H_{\omega_1}$ that consists of relevant collapses, i.e., types of countable nature. For example, the transitive collapse

$$\overline{(X, \in, a, \omega_3 X, XI, \omega_3 XI)}$$

of a countable structure $(X, \in, a, \omega_3 X, XI, \omega_3 XI)$, where a specific constant $a \in X$ and $I \in X \cap \mathcal{C}_{\omega_1}$, is a member of t_X and t_X may contain other types of collapses. If $X <_{\omega_1} S$, we demand $t_X \in S$. This is going to be a typical "fast function" to get an accurate S^*-copy X' of X, where $S = H_{\omega_3} S^*$ and $P \in S^*$, in showing a relevant condition $q \in P$ is (P, S^*)-generic.

Lemma. Let $X, Y, X', Y' \in \mathcal{C}_{\omega_0}$. Let $g_X : X \longrightarrow X'$ and $g_Y : Y \longrightarrow Y'$ be isomorphisms. Let the pointwise images satisfy $g_X[\omega_3 XY] = g_Y[\omega_3 XY] = \omega_3 X'Y'$. Then not only $g_X \lceil \omega_3 XY = g_Y \lceil \omega_3 XY$ as maps but

(extended) $g_X \lceil XY = g_Y \lceil XY$ and $g_X[XY] = g_Y[XY] = X'Y'$.

In particular, the two intersections XY and $X'Y'$ are isomorphic members of $\mathcal{C}_{\omega_0}$.

Proof. By the bijection $\Phi : \omega_3 \longrightarrow H_{\omega_3}$ that is present in the relational structure H_{ω_3}.

□

Bubbly Path

Here is the main concept due to Aspero and Rodriguez.

Definition. (Aspero and Rodriguez, modified a bit) Let $\mathcal{X}$ be a finite subset of $\mathcal{C}_{\omega_0}$. Let $n < \omega$ and $\delta < \omega_1$. We write

$$\gamma_0 B_0 \gamma_1 \cdots \gamma_k B_k \gamma_{k+1} \cdots \gamma_n B_n \gamma_{n+1}$$

and call it a *bubbly path* (that connects $\gamma_0 < \gamma_{n+1}$) on δ in $\mathcal{X}$, if

- $\{B_0, \cdots, B_k, \cdots, B_n\} \subseteq \mathcal{X}$.
- $\omega_1 B_0, \cdots, \omega_1 B_k, \cdots, \omega_1 B_n \leq \delta$.
- $\{\gamma_0 < \gamma_1\} \subset B_0, \cdots, \{\gamma_k < \gamma_{k+1}\} \subset B_k, \cdots, \{\gamma_n < \gamma_{n+1}\} \subset B_n$.

Proposition. (tight bubbly path) Let $\alpha X_0 \gamma_1 X_1 \cdots \gamma_n X_n \beta$ be a bubbly path on δ. Then it can be replaced by a bubbly path $\alpha X_{i_0} \gamma^1 X_{i_1} \cdots \gamma^m X_{i_m} \beta$ on δ s.t. $\{X_{i_0}, X_{i_1}, \cdots X_{i_m}\}$ is of the size $1 + m$ and is a subset of $\{X_0, X_1, \cdots, X_n\}$.

The Poset

We consider symmetric systems where $\mathcal{C}_{\omega_1}$ with $=_{\omega_2}$ is exactly the same as the usual $\mathcal{C}_{\omega_0}$ with $=_{\omega_1}$.

Definition. Let $\mathcal{L} \in [\mathcal{C}_{\omega_1}]^{<\omega}$. We call $\mathcal{L}$ is a *symmetric system* w.r.t. $\mathcal{C}_{\omega_1}$, if

(iso) If $I, J \in \mathcal{L}$ with $I =_{\omega_2} J$, meaning $\omega_2 I = \omega_2 J$, then there exists a unique isomorphism

$$\phi_{I,J} : (I, \in, \cdots) \longrightarrow (J, \in, \cdots)$$

s.t. $\phi_{I,J}(x) = x$ for all $x \in IJ$.

(up) If $I_3, I_2 \in \mathcal{L}$ with $I_3 <_{\omega_2} I_2$, then there exists $I_1 \in \mathcal{L}$ s.t. $I_3 \in I_1 =_{\omega_2} I_2$.

(down) If $I_3, I_1, I_2 \in \mathcal{L}$ with $I_3 \in I_1 =_{\omega_2} I_2$, then $\phi_{I_1,I_2}(I_3) \in \mathcal{L}$.

Let $\mathcal{L}$ be a symmetric system w.r.t. $\mathcal{C}_{\omega_1}$. Let $I \in \mathcal{L}$ and δ be an immediate predecessor of $\omega_2 I$ in $\omega_2 \mid \mathcal{L} = \{\omega_2 L \mid L \in \mathcal{L}\}$. Let us write

$$\mathcal{L}^I := \{J \in \mathcal{L}I \mid \omega_2 J = \delta\}.$$

Hence $\mathcal{L}^I$ denotes the set of immediately $\in$-below elements of I in $\mathcal{L}$. We extend $\mathcal{L}^I = \emptyset$, when $\omega_2 I$ is the least member of $\omega_2 \mid \mathcal{L}$. We do not assume but may image $\mathcal{L}^I$ is either $\emptyset$, a singleton set, or a two-element set aiming at an $(\omega_2, 1)$-morass, though it takes more work. If $I_1 =_{\omega_2} I_2$ in $\mathcal{L}$, then $\phi_{I_1,I_2}[\mathcal{L}^{I_1}] = \mathcal{L}^{I_2}$.

Definition. Let $\mathcal{L}$ be a symmetric system w.r.t. $\mathcal{C}_{\omega_1}$ and $\mathcal{S} \in [\mathcal{C}_{\omega_0}]^{<\omega}$. We say $(\mathcal{L}, \mathcal{S})$ is a *two-type*, if there exists $\langle \mathcal{S}^I \mid I \in \mathcal{L}\rangle$ s.t.

(suspend) $\mathcal{S}^I = \{X \in \mathcal{S} \mid \mathcal{L}^I \in X \in I\}$ that is $\in$-linear.

(iso) If $I_1, I_2 \in \mathcal{L}$ with $I_1 =_{\omega_2} I_2$, then $\phi_{I_1,I_2}[\mathcal{S}^{I_1}] = \mathcal{S}^{I_2}$.

(sorted) $\mathcal{S} = \bigcup\{\mathcal{S}^I \mid I \in \mathcal{L}\}$.

(intersection) If $I \in \mathcal{L}$, $X \in \mathcal{S}^I$, and $L \in \mathcal{L}^I$, then $XL \in \mathcal{S}$.

Hence, given $I \in \mathcal{L}$, since $|\,I\,| = \omega_1$, we are going to climb up within $I \cap \mathcal{C}_{\omega_0}$ to cover the whole I. Note that in (sorted), it might happen $S^I S^J \neq \emptyset$ with $I \neq J$.

Here is our proposed poset to force $SC(\omega_1, \omega_3)$.

Definition. Let $p = (\mathcal{L}_p, \mathcal{S}_p, \mathcal{X}_p, f_p) \in P$, if

(two-type) $(\mathcal{L}_p, \mathcal{S}_p)$ is a two-type.

(fast fun) $\mathcal{X}_p \subseteq \mathcal{S}_p$ s.t. for each $X \in \mathcal{X}_p$, associated is some $t_X \in H_{\omega_1}$, where, say,

$$t_X := \{\overline{(X, \in, \cdots)}, \overline{(X, \in, x)}, \overline{(X, \in, XI)} \mid I \in \mathcal{C}_{\omega_1} X, x \in X\}$$

and for any $X, Y \in \mathcal{X}_p$ with $X <_{\omega_1} Y$, we demand $t_X \in Y$.

(bubble) $f_p : a_p \times (\omega_1 \mid \mathcal{X}_p) \longrightarrow 2$ with $a_p \in [\omega_3]^{<\omega}$ s.t. for any $\delta \in \omega_1 \mid \mathcal{X}_p$ and $\{\alpha < \beta\} \subseteq a_p$ that happen to be connected by a bubbly path on δ in $\mathcal{X}_p$, we demand

$$f_p(\alpha, \delta) \leq f_p(\beta, \delta).$$

For $p, q \in P$, let $q \leq p$ in P, if

(as usual) $\mathcal{L}_q \supseteq \mathcal{L}_p$, $\mathcal{S}_q \supseteq \mathcal{S}_p$, $\mathcal{X}_q \supseteq \mathcal{X}_p$, and $f_q \supseteq f_p$.

(1-0 frozen) For any $\{\alpha < \beta\} \subseteq a_p$ (old horizontal coordinates) and $\delta \in (\omega_1 \mid \mathcal{X}_q) \setminus (\omega_1 \mid \mathcal{X}_p)$ (new vertical coordinate), we demand

$$f_q(\alpha, \delta) \leq f_q(\beta, \delta).$$

Claim. (GCH) Force with P, then the cofinalities and so cardinalities are all preserved and we get $SC(\omega_1, \omega_3)$.

Comparison and Descend

Let $q \in P$. We may assume $S \in \mathcal{X}_q$, where $S := S^* H_{\omega_3}$, S^* is a countable elementary substructure, containing relevant constants, of H_θ, and θ is a sufficiently large regular cardinal as usual.

Lemma. (X descends as Y indicated) Let $X, Y \in \mathcal{S}_q$. Let $I_X, J_Y \in \mathcal{L}_q$ s.t. $I_X =_{\omega_2} J_Y$, $X \in S_q^{I_X}$, $J_0 \in \mathcal{L}_q^{J_Y}$, and $Y \in J_0$. Let $I_0 = \phi_{J_Y, I_X}(J_0)$. Then $I_0 \in \mathcal{L}_q^{I_X}$, $I_0 \in X$, and $XY = (XI_0)Y$.

Proof. Let $a \in XY$. Since $a \in X \in I_X$ and $a \in Y \in J_0 \in J_Y$, we have $a \in I_X J_Y$. Hence $a = \phi_{I_X, J_Y}(a)$. Since $a = \phi_{I_X, J_Y}(a) \in J_0$, we have $a \in \phi_{J_Y, I_X}(J_0) = I_0$.

□

Lemma. (length ω_3 in business) Let $I, J \in \mathcal{L}_q$ with $I =_{\omega_2} J$. Let $X \in \mathcal{S}_q^I$.

(initial) $\omega_3 IJ$ is an initial segment of both $\omega_3 I$ and $\omega_3 J$.

(cross)

$$XJ = \phi_{I,J}(X) I = X \phi_{I,J}(X).$$

In particular,

$$\omega_3 XJ = \omega_3 \phi_{I,J}(X) I = \phi_{I,J}[\omega_3 X] I = \omega_3 X \phi_{I,J}(X) = \omega_3 X \phi_{I,J}[\omega_3 X].$$

(initial) $\omega_3 XJ = (\omega_3 X)(\omega_3 \phi_{I,J}(X))$ is an initial segment of both $\omega_3 X$ and $\omega_3 \phi_{I,J}(X)$.

□

Corollary. (XY-initial) Let $I, J \in \mathcal{L}_q$ with $I =_{\omega_2} J$, $X \in \mathcal{S}_q^I$, and $Y \in \mathcal{S}_q^J$ with $X \leq_{\omega_1} Y$. Then

(tame) $\omega_3 XY = \omega_3 XJ$ and so $\omega_3 XY$ is an intial segment of both $\omega_3 X$ and $\omega_3 \phi_{I,J}(X)$.

(tame) If $X <_{\omega_1} Y$, then $\omega_3 XY \in Y$ and so $XY \in Y$.

□

Lemma. (comparison and descend) Let $X, Y \in \mathcal{S}_q$, $X \leq_{\omega_1} Y$, $I_X, J_Y \in \mathcal{L}_q$, $X \in \mathcal{S}_q^{I_X}$, and $Y \in \mathcal{S}_q^{J_Y}$. Depending on how $\omega_2 X$ and $\omega_2 Y$ alternate above before they begin to have common elements in $\omega_2 \,|\, \mathcal{L}_q$, we have 4 cases, i.e., either (no decsend), (X-alone descend), (Y-alone descend), or (mutually descend) applies.

1. (no descend) $I_X =_{\omega_2} J_Y$.

   (tame) $\omega_3 XY$ is an initial segment of $\omega_3 X$.

   (tame) If $X <_{\omega_1} Y$, then $\omega_3 XY \in Y$ and so $XY \in Y$.

2. (X-alone descend) There exists a set $\{I_1, I\} \subseteq \mathcal{L}_q$ s.t. $XI = XI_1$, $I_1 \in X$, $I = I_1$ or $I \in I_1$.

   ($(XI)(Y)$-initial) $I =_{\omega_2} J_Y$, $XI \in \mathcal{S}_q^I$, and $XY = (XI)(Y)$.

   (tame) $\omega_3 XY$ is an initial segment of $\omega_3 XI$.

   (tame) If $X <_{\omega_1} Y$, then $\omega_3 XY \in Y$ and so $XY \in Y$.

3. (Y-alone descend) There exists a set $\{J_1, J\} \subseteq \mathcal{L}_q$ s.t. $YJ = YJ_1$, $J_1 \in Y$, $J = J_1$ or $J \in J_1$.

   ($(X)(YJ)$-initial) $I_X =_{\omega_2} J$, $YJ \in \mathcal{S}_q^J$, and $XY = (X)(YJ)$.

   (tame) $\omega_3 XY$ is an initial segment of $\omega_3 X$.

   (tame) If $X <_{\omega_1} Y$, then $\omega_3 XY \in YJ$ and so $XY \in YJ$.

4. (mutually descend) There exists a set $\{I_1, I, J_1, J\} \subseteq \mathcal{L}_q$ s.t.

   (X-descend) $XI = XI_1$, $I_1 \in X$, $I = I_1$ or $I \in I_1$.

(Y-decsend) $YJ = YJ_1$, $J_1 \in Y$, $J = J_1$ or $J \in J_1$.

($(XI)(YJ)$-initial) $I =_{\omega_2} J$, $XI \in \mathcal{S}_q^I$, $YJ \in \mathcal{S}_q^J$, and $XY = (XI)(YJ)$.

(tame) $\omega_3 XY$ is an initial segment of $\omega_3 XI$.

(tame) If $X <_{\omega_1} Y$, then $\omega_3 XY \in YJ$ and so $XY \in YJ$.

□

Merge

We continue to very out-line that P is proper. Let $q \in P$ with $S \in \mathcal{X}_q$, where S and S^* as before.

Lemma. (merge: the finite union M of elementary substructures below S) Let $1 \leq n < \omega$. Let $a_1, a_2, \cdots, a_n \in H_{\omega_3} \setminus S$ be any n-many constants and $t_1, t_2, \cdots, t_n \in S$ be any another n-many constants. Let

$$H_\theta \models \text{“}\phi(a_1, a_2, \cdots, a_n, q, S, t_1, t_2, \cdots, t_n, P)\text{”},$$

where let ϕ be any $\in$-formula. Then there exist $a'_1, a'_2, \cdots, a'_n \in S$, $q' \in SP$, and $S' \in \mathcal{X}_{q'}$ s.t.

(S^*-copying $'$) $H_\theta \models$ “$\phi(a'_1, a'_2, \cdots, a'_n, q', S', t_1, t_2, \cdots, t_n, P)$”.

(S^*-copying $'$) In particular, $\mathcal{X}_q = \{S, X, Y, \cdots, Z\}$ have their S^*-copies $\mathcal{X}_{q'} = \{S', X', Y', \cdots, Z'\}$ s.t.

(visible) For each $X <_{\omega_1} S$, $XS = X'S' = XX' \in SS' = S'$.

(iso) For each $X <_{\omega_1} S$, there exists a unique isomorphism $g_X : (X, \in, \cdots) \longrightarrow (X', \in, \cdots)$.

(pin) For each $X <_{\omega_1} S$, $g_X[XS] = X'S'$ and so $g_X \lceil XX' = \mathrm{id}_{XX'}$.

(cohere-mutually) For $X, Y \in \mathcal{X}_q$ with $X, Y <_{\omega_1} S$, $g_X[XY] = g_Y[XY] = X'Y'$.

(cohere-constants) If $1 \leq i \leq n$ and $a_i \in X \in \mathcal{X}_q$ with $X <_{\omega_1} S$, then $g_X(a_i) = a'_i$.

(merge below S) Let

$$M := \bigcup\{X \mid X \in \mathcal{X}_q, X <_{\omega_1} S\},$$

$$M' := \bigcup\{X' \mid X' \in \mathcal{X}_{q'}, X' <_{\omega_1} S'\},$$

$$g := \bigcup\{g_X \mid X \in \mathcal{X}_q, X <_{\omega_1} S\}.$$

Then the glued $g : M \longrightarrow M'$ is a bijection s.t. g is locally an isomorphism, i.e., for any $X \in \mathcal{X}_q$ with $X <_{\omega_1} S$,

$$g\lceil X = g_X : (X, \in, \cdots) \longrightarrow (X', \in, \cdots).$$

□

Claim. q is (P, S^*)-generic.

Proof. (very out-line) Let $D \in S^*$ be predense in P. We assume $q \leq d \in D$ for some d. It suffices to show that q and its S^*-copy q' are compatible in P. Let us set $(\mathcal{L}, \mathcal{S}, \mathcal{X}, f)$ as follows:

(two-type) $(\mathcal{L}, \mathcal{S})$ is a two-type s.t. $\mathcal{L} \supseteq \mathcal{L}_q, \mathcal{L}_{q'}, \omega_2 \mid \mathcal{L} = (\omega_2 \mid \mathcal{L}_q) \cup (\omega_2 \mid \mathcal{L}_{q'})$ and $\mathcal{S} \supseteq \mathcal{S}_q, \mathcal{S}_{q'}$. The construction of $(\mathcal{L}, \mathcal{S})$ is routine, though requires exact descriptions. This dealt in lemma (amalgamation) below.

(fast fun) $\mathcal{X} = \mathcal{X}_q \cup \mathcal{X}_{q'}$. Thus just take the union to form $\mathcal{X}$.

We are left with constructing f s.t. $f \supseteq f_q, f_{q'}$. Let $a_{q'}$ denote the S^*-copy of a_q. We have $a_q S = a_{q'} S'$. Introduce the constants α_A, α_B, α_C, $\alpha_{A|B}$, and $\alpha_{A|C}$ as follows:

$$a_q = (a_q S) \cup (a_q \setminus S) = \{\cdots, \alpha_A, \cdots\} \cup \{\cdots, \alpha_C, \cdots\} = \{\cdots, \alpha_{A|C}, \cdots\},$$

$$a_{q'} = (a_{q'}S') \cup (a_{q'} \setminus S') = \{\cdots, \alpha_A, \cdots\} \cup \{\cdots, \alpha_B(=\alpha'_C), \cdots\} = \{\cdots, \alpha_{A|B}, \cdots\}.$$

Two finite sets $\omega_1 \mid \mathcal{X}_q$ and $\omega_1 \mid \mathcal{X}_{q'}$ are located like common *HEAD* + *Tail* $'$ + *Tail* in ω_1 vertically, i.e., a Δ-system. Introduce the constants δ_H in the *HEAD*, the constants δ in *Tail*, and its $'$, i.e., δ' in *Tail* $'$ as follows:

$$\omega_1 \mid \mathcal{X}_q = HEAD + Tail = \{\cdots, \delta_H, \cdots\} \cup \{\omega_1 S, \cdots, \delta, \cdots\},$$

$$\omega_1 \mid \mathcal{X}_{q'} = HEAD + Tail\ ' = \{\cdots, \delta_H, \cdots\} \cup \{\omega_1 S', \cdots', \delta', \cdots'\},$$

$$\{\cdots, \delta_H, \cdots\} < \{\omega_1 S', \cdots', \delta', \cdots'\} < \{\omega_1 S, \cdots, \delta, \cdots\} < \omega_1.$$

At this moment, we have a rectangle divided into 9-parts.

$$[(a_q S) \times (HEAD + Tail\ ' + Tail)] \cup [(a_{q'} \setminus S') \times (HEAD + Tail\ ' + Tail)] \cup [(a_q \setminus S) \times (HEAD + Tail\ ' + Tail)].$$

(Head) $f_q(\alpha_A, \delta_H) = f_{q'}(\alpha_A, \delta_H)$.

(Head) $f_q(\alpha_C, \delta_H) = f_{q'}(\alpha'_C, \delta_H)$.

(Tail-Tail $'$) $f_q(\alpha_A, \delta) = f_{q'}(\alpha_A, \delta')$.

(Tail-Tail $'$) $f_q(\alpha_C, \delta) = f_{q'}(\alpha'_C, \delta')$.

We need to set values of f on this rectangle. Hence, it suffices to define the following:

(Tail: α_B) $f(\alpha_B, \delta) = 1$ iff there exists $\alpha_{A|C} < \alpha_B$ s.t.

$$f_q(\alpha_{A|C}, \delta) = 1, \alpha_{A|C} \text{ and } \alpha_B \text{ get connected by a bubbly path on } \delta \text{ in } \mathcal{X}_q.$$

(Tail $'$: α_C) $f(\alpha_C, \delta') = 1$ iff $f(\alpha'_C, \delta) = 1$.

Note in (Tail: α_B), we know $\{S', X', Y', \cdots, Z'\} = \mathcal{X}_{q'} \in S \in \mathcal{X}_q$. Hence for example, X' can be replaced by S in the bubbles on δ.

We must argue that $(\mathcal{L}, \mathcal{S}, \mathcal{X}, f) \in P$. In particular, f must satisfy (bubble) dealing with $\mathcal{X}$. However, we claim that it is almost verbatim as in [M].

□

Amalgamation

Lemma. (amalgamation: characterization of $\mathcal{L}$) There exists the $\subseteq$-least $\mathcal{L}$ s.t.

- (sym) $\mathcal{L}$ is a symmetric system w.r.t. $\mathcal{C}_{\omega_1}$.
- (projection to ω_2) $\omega_2 \mid \mathcal{L} = (\omega_2 \mid \mathcal{L}_q) \cup (\omega_2 \mid \mathcal{L}_{q'})$.
- (extension) $\mathcal{L}_{q'}, \mathcal{L}_q \subseteq \mathcal{L}$.

Lemma. (amalgamation: characterization of $\mathcal{S}$ relative to $\mathcal{L}$) Fix $\mathcal{L}$ as above. Then there exists the $\subseteq$-least $\mathcal{S}$ s.t.

- (two-type) $(\mathcal{L}, \mathcal{S})$ is a two-type.
- (extension) $\mathcal{S}_{q'}, \mathcal{S}_q \subseteq \mathcal{S}$.

We call $(\mathcal{L}, \mathcal{S})$ as above the S^*-amalgamation of $(\mathcal{L}_q, \mathcal{S}_q)$ and its S^*-copy $(\mathcal{L}_{q'}, \mathcal{S}_{q'})$.

S^*-path

A summary of proof follows: We assume $|\,(\omega_2\,|\,\mathcal{L}_q)S\,| \geq 2$ and denote

$$(\omega_2\,|\,\mathcal{L}_q)S = \{\delta_0 > \cdots > \delta_k > \cdots > \delta_n\}.$$

Thus

$$(\omega_2\,|\,\mathcal{L}_q)S = \{\delta_0 = \delta'_0 > \cdots > \delta_k = \delta'_k > \cdots > \delta_n = \delta'_n\} = (\omega_2\,|\,\mathcal{L}_{q'})S'.$$

Fix any

$$I_0, I_1, \cdots, I_k, I_{k+1}, \cdots, I_n, I_{n+1},$$

$$L_0, \cdots, L_k, \cdots, L_n \,(\text{no } L_{n+1})$$

such that

- $I_0 \in \mathcal{L}_q$, $S \in \mathcal{S}_q^{I_0}$, $L_0 \in \mathcal{L}_q^{I_0} \subseteq (\mathcal{L}_q\,|\,\{\delta_0\})S$, and so $\delta_0 = \omega_2 L_0$.
- $I_k \in \mathcal{L}_q$, $I_k S \in \mathcal{S}_q^{I_k}$, $L_k \in \mathcal{L}_q^{I_k} \subseteq (\mathcal{L}_q\,|\,\{\delta_k\})S$, and so $\delta_k = \omega_2 L_k$.
- $I_{k+1} \in \mathcal{L}_q(L_k \cup \{L_k\})$ and $L_k S = I_{k+1} S \in \mathcal{S}_q^{I_{k+1}}$.
- If $k < n$, then $L_{k+1} \in \mathcal{L}_q^{I_{k+1}} \subseteq (\mathcal{L}_q\,|\,\{\delta_{k+1}\})S$.
- If $k = n$, then $L_{n+1}\uparrow$, i.e., $\mathcal{L}_q^{I_{k+1}} = \mathcal{L}_q^{I_{n+1}} = \emptyset$.

We call it an (partial) S^*-path.

Guided by the S^*-path, we construct a two-type $(\mathcal{L}, \mathcal{S})$ which shall be our S^*-amalgamation of $(\mathcal{L}_q, \mathcal{S}_q)$ and $(\mathcal{L}_{q'}, \mathcal{S}_{q'})$, i.e., they are independent of S^*-path chosen. Some assumptions on q and its S^*-copy q^*:

- $\{\delta_0\} < (\delta_0, \omega_2)\,|\,\mathcal{L}_{q'} < (\delta_0, \omega_2)\,|\,\mathcal{L}_q < \{\omega_2\}$.
- $\big((\delta_0, \omega_2)\,|\,\mathcal{L}_{q'}, <\big) \sim \big((\delta_0, \omega_2)\,|\,\mathcal{L}_q, <\big)$.
- $\mathcal{S}_q^{I_0} S = \mathcal{S}_{q'}^{I'_0} S'$ and $\mathcal{L}_q^{I_0} = \mathcal{L}_{q'}^{I'_0}$.
- $\{\delta_{k+1}\} < (\delta_{k+1}, \delta_k)\,|\,\mathcal{L}_{q'} < (\delta_{k+1}, \delta_k)\,|\,\mathcal{L}_q < \{\delta_k\}$.
- $\big((\delta_{k+1}, \delta_k)\,|\,\mathcal{L}_{q'}, <\big) \sim \big((\delta_{k+1}, \delta_k)\,|\,\mathcal{L}_q, <\big)$.
- $\mathcal{S}_q^{I_{k+1}} S = \mathcal{S}_{q'}^{I'_{k+1}} S'$ and $\mathcal{L}_q^{I_{k+1}} = \mathcal{L}_{q'}^{I'_{k+1}}$.
- $\{0\} < (0, \delta_n)\,|\,\mathcal{L}_{q'} < (0, \delta_n)\,|\,\mathcal{L}_q < \{\delta_n\}$.
- $\big((0, \delta_n)\,|\,\mathcal{L}_{q'}, <\big) \sim \big((0, \delta_n)\,|\,\mathcal{L}_q, <\big)$.
- $\mathcal{S}_q^{I_{n+1}} S = \mathcal{S}_{q'}^{I'_{n+1}} S'$ and $\mathcal{L}_q^{I_{n+1}} = \mathcal{L}_{q'}^{I'_{n+1}} = \emptyset$.

There is a chance that we need more.

Construction

Construction of $\mathcal{L}$ and $\mathcal{S}^J$ for $J \in \mathcal{L}$ classified into (SKY), (MID), and (GRD) as follows:

- (SKY) $\mathcal{L}\,|\,[\delta_0, \omega_2)$ with $\mathcal{L}_q\,|\,[\delta_0, \omega_2)$ and $\mathcal{L}_{q'}\,|\,[\delta_0, \omega_2)$.
- (SKY) $\mathcal{S}^J$ for $J \in \mathcal{L}\,|\,(\delta_0, \omega_2)$ with $\langle S_q^L \mid L \in \mathcal{L}_q\,|\,(\delta_0, \omega_2)\rangle$ and $\langle S_{q'}^{L'} \mid L' \in \mathcal{L}_{q'}\,|\,(\delta_0, \omega_2)\rangle$.
- (MID) $\mathcal{L}\,|\,[\delta_{k+1}, \delta_k)$ with $\mathcal{L}_q\,|\,[\delta_{k+1}, \delta_k)$ and $\mathcal{L}_{q'}\,|\,[\delta_{k+1}, \delta_k)$.
- (MID) $\mathcal{S}^J$ for $J \in \mathcal{L}\,|\,(\delta_{k+1}, \delta_k]$ with $\langle S_q^L \mid L \in \mathcal{L}_q\,|\,(\delta_{k+1}, \delta_k]\rangle$ and $\langle S_{q'}^{L'} \mid L' \in \mathcal{L}_{q'}\,|\,(\delta_{k+1}, \delta_k]\rangle$.
- (GRD) $\mathcal{L}\,|\,[0, \delta_n)$ with $\mathcal{L}_q\,|\,[0, \delta_n)$ and $\mathcal{L}_{q'}\,|\,[0, \delta_n)$.

- (GRD) $\mathcal{S}^J$ for $J \in \mathcal{L}\,|\,(0,\delta_n]$ with $\langle S_q^L \mid L \in \mathcal{L}_q\,|\,(0,\delta_n]\rangle$ and $\langle S_{q'}^{L'} \mid L' \in \mathcal{L}_{q'}\,|\,(0,\delta_n]\rangle$.

  Since (SKY), (MID), and (GRD) are identical, we concentrate on (MID), omitting (SKY) and (GRD).

Claim 1. (induction hypothesis, (8) is a bit sloppy)

(1) (sym) $\mathcal{L}\,|\,[\delta_k,\omega_2)$ is a symmetric system w.r.t. $\mathcal{C}_{\omega_1}$.

(2) (projection to ω_2) $\{\omega_2 J \mid J \in \mathcal{L}\,|\,[\delta_k,\omega_2)\} = ([\delta_k,\omega_2)\,|\,\mathcal{L}_{q'}) \cup ([\delta_k,\omega_2)\,|\,\mathcal{L}_q)$.

(3) (extend $\mathcal{L}_q$) $\mathcal{L}_q\,|\,[\delta_k,\omega_2) \subseteq \mathcal{L}\,|\,[\delta_k,\omega_2)$.

(4) (extend $\mathcal{L}_{q'}$) $\mathcal{L}_{q'}\,|\,[\delta_k,\omega_2) \subseteq \mathcal{L}\,|\,[\delta_k,\omega_2)$.

(5) (iso) For $I, J \in \mathcal{L}\,|\,(\delta_k,\omega_2)$ with $I =_{\omega_2} J$, we have $\phi_{I,J}[\mathcal{S}^I] = \mathcal{S}^J$.

(6) (extend $\mathcal{S}_q$) For $L \in \mathcal{L}_q\,|\,(\delta_k,\omega_2)$, we have $\mathcal{S}_q^L \subseteq \bigcup\{\mathcal{S}^J \mid J \in \mathcal{L}\,|\,(\delta_k,\omega_2)\}$.

(7) (extend $\mathcal{S}_{q'}$) For $L' \in \mathcal{L}_{q'}\,|\,(\delta_k,\omega_2)$, we have $\mathcal{S}_{q'}^{L'} \subseteq \bigcup\{\mathcal{S}^J \mid J \in \mathcal{L}\,|\,(\delta_k,\omega_2)\}$.

(8) (intersection) Assume that $\mathcal{S}_q, \mathcal{S}_{q'} \subseteq \mathcal{S} := \bigcup\{\mathcal{S}^J \mid J \in \mathcal{L}\}$ and that for any $I, J \in \mathcal{L}$ with $I =_{\omega_2} J$, we have $\phi_{I,J}[\mathcal{S}] \subseteq \mathcal{S}$. Then the $\mathcal{S}^J$'s constructed so far satisfy (intersection).

(MID)

Let $k < n$ and suppose we have $\mathcal{L}\,|\,[\delta_k,\omega_2)$ and $\langle \mathcal{S}^J \mid J \in \mathcal{L}\,|\,(\delta_k,\omega_2)\rangle$ s.t. (1) through (8) in claim 1 with k hold. We construct $\mathcal{L}\,|\,[\delta_{k+1},\delta_k)$ and $\langle S^J \mid J \in \mathcal{L}\,|\,(\delta_{k+1},\delta_k]\rangle$.

CASE N. $I_{k+1} \in L_k$:

- (MID 0) For $J \in \mathcal{L}\,|\,\{\delta_k\}$, let $\mathcal{S}^J := \phi_{L_k,J}[\mathcal{S}_q^{L_k}]$.
- (MID 1) Let $\mathcal{L}\,|\,[\omega_2 I_{k+1},\delta_k) := \bigcup\big\{\phi_{L_k,J}\big[(\mathcal{L}_q L_k)\,|\,[\omega_2 I_{k+1},\delta_k)\big] \mid J \in \mathcal{L}\,|\,\{\delta_k\}\big\}$.

  Let $J \in \mathcal{L}\,|\,[\omega_2 I_{k+1},\delta_k)$.

  Case 1. $I_{k+1} <_{\omega_2} J <_{\omega_2} L_k$: Pick any $L \in \mathcal{L}_q L_k$ with $L =_{\omega_2} J$ and let $\mathcal{S}^J := \phi_{L,J}[\mathcal{S}_q^L]$.

  Case 2. $J =_{\omega_2} I_{k+1}$: Let $\mathcal{S}^J := \phi_{I_{k+1},J}\big[\mathcal{S}_{q'}^{L_k} \cup (\mathcal{S}_q^{I_{k+1}} \setminus S)\big]$.
- (MID 2) Let $\mathcal{L}\,|\,[\delta_{k+1},\omega_2 I_{k+1}) := \bigcup\big\{\phi_{I_{k+1},J}\big[(\mathcal{L}_{q'} L_k)\,|\,[\delta_{k+1},\delta_k)\big] \mid J \in \mathcal{L}\,|\,\{\omega_2 I_{k+1}\}\big\}$.

  Let $J \in \mathcal{L}\,|\,(\delta_{k+1},\omega_2 I_{k+1})$.

  Case 1. $I'_{k+1} <_{\omega_2} J <_{\omega_2} I_{k+1}$: Pick any $L' \in \mathcal{L}_{q'} L_k$ with $L' =_{\omega_2} J$ and let

$$\mathcal{S}^J := \phi_{L',J}\big[\mathcal{S}_{q'}^{L'} \cup \{L'X \mid X \in \mathcal{S}_q^{I_{k+1}} \setminus S\}\big].$$

  Case 2. $I'_{k+1} =_{\omega_2} J$: Let

$$\mathcal{S}^J := \phi_{I'_{k+1},J}\big[\mathcal{S}_{q'}^{I'_{k+1}} \cup \{I'_{k+1}X \mid X \in \mathcal{S}_q^{I_{k+1}} \setminus S\}\big]$$

$$= \phi_{I'_{k+1},J}\big[(\mathcal{S}_q^{I_{k+1}} S) \cup (\mathcal{S}_{q'}^{I'_{k+1}} \setminus S') \cup \{I'_{k+1}X \mid X \in \mathcal{S}_q^{I_{k+1}} \setminus S\}\big].$$

CASE D. $L_k = I_{k+1}$: Let $\mathcal{L}\,|\,[\delta_{k+1},\delta_k) = \mathcal{L}\,|\,\{\delta_{k+1}\} := \bigcup\{\phi_{L_k,J}[\mathcal{L}_q^{L_k}] \mid J \in \mathcal{L}\,|\,\{\delta_k\}\}$.

For $J \in \mathcal{L}\,|\,\{\delta_k\}$, let

$$\mathcal{S}^J := \phi_{L_k,J}[\mathcal{S}_{q'}^{L_k} \cup \mathcal{S}_q^{L_k}] = \phi_{L_k,J}\big[(\mathcal{S}_q^{L_k} S) \cup (\mathcal{S}_{q'}^{L_k} \setminus S') \cup (\mathcal{S}_q^{L_k} \setminus S)\big].$$

Claim 2. Let $\mathcal{L}\,|\,[\delta_{k+1},\omega_2) := \mathcal{L}\,|\,[\delta_{k+1},\delta_k) \cup \mathcal{L}\,|\,[\delta_k,\omega_2)$ and let $\langle S^J \mid J \in \mathcal{L}\,|\,(\delta_{k+1},\omega_2)\rangle := \langle S^J \mid J \in \mathcal{L}\,|\,(\delta_{k+1},\delta_k]\rangle \cup \langle S^J \mid J \in \mathcal{L}\,|\,(\delta_k,\omega_2)\rangle$.

(1) (sym) $\mathcal{L}\,|\,[\delta_{k+1},\omega_2)$ is a symmetric system w.r.t. $\mathcal{C}_{\omega_1}$.

(2) (projection to ω_2) $\{\omega_2 J \mid J \in \mathcal{L}\,|\,[\delta_{k+1},\omega_2)\} = ([\delta_{k+1},\omega_2)\,|\,\mathcal{L}_{q'}) \cup ([\delta_{k+1},\omega_2)\,|\,\mathcal{L}_q)$.

(3) (extend $\mathcal{L}_q$) $\mathcal{L}_q\,|\,[\delta_{k+1},\omega_2) \subseteq \mathcal{L}\,|\,[\delta_{k+1},\omega_2)$.

(4) (extend $\mathcal{L}_{q'}$) $\mathcal{L}_{q'}\,|\,[\delta_{k+1},\omega_2) \subseteq \mathcal{L}\,|\,[\delta_{k+1},\omega_2)$.

(5) (iso) For $I, J \in \mathcal{L}\,|\,(\delta_{k+1},\omega_2)$ with $I =_{\omega_2} J$, we have $\phi_{I,J}[\mathcal{S}^I] = \mathcal{S}^J$.

(6) (extend $\mathcal{S}_q$) For $L \in \mathcal{L}_q\,|\,(\delta_{k+1},\omega_2)$, we have $\mathcal{S}_q^L \subseteq \bigcup\{\mathcal{S}^J \mid J \in \mathcal{L}\,|\,(\delta_{k+1},\omega_2)\}$.

(7) (extend $\mathcal{S}_{q'}$) For $L' \in \mathcal{L}_{q'}\,|\,(\delta_{k+1},\omega_2)$, we have $\mathcal{S}_{q'}^{L'} \subseteq \bigcup\{\mathcal{S}^J \mid J \in \mathcal{L}\,|\,(\delta_{k+1},\omega_2)\}$.

(8) (intersection) Assume that $\mathcal{S}_q, \mathcal{S}_{q'} \subseteq \mathcal{S} := \bigcup\{\mathcal{S}^J \mid J \in \mathcal{L}\}$ and that for any $I, J \in \mathcal{L}$ with $I =_{\omega_2} J$, we have $\phi_{I,J}[\mathcal{S}] \subseteq \mathcal{S}$. Then the $\mathcal{S}^J$'s constructed so far satisfy (intersection).

Proof. We just treat case N (normal) and leave case D (degenerate).

CASE N:

(1), (2): It is basic that $\mathcal{L}\,|\,[\omega_2 I_{k+1},\omega_2)$ is a symmetric system w.r.t. $\mathcal{C}_{\omega_1}$. So is $\mathcal{L}\,|\,[\delta_{k+1},\omega_2)$. We have

$$\{\omega_2 J \mid J \in \mathcal{L}\,|\,[\delta_{k+1},\delta_k)\} = ([\delta_{k+1},\delta_k)\,|\,\mathcal{L}_{q'}) \cup ([\delta_{k+1},\delta_k)\,|\,\mathcal{L}_q).$$

Thus

$$\{\omega_2 J \mid J \in \mathcal{L}\,|\,[\delta_{k+1},\omega_2)\} = ([\delta_{k+1},\omega_2)\,|\,\mathcal{L}_{q'}) \cup ([\delta_{k+1},\omega_2)\,|\,\mathcal{L}_q).$$

(3), (4): Because $\mathcal{L}_q^{I_{k+1}} = \mathcal{L}_{q'}^{I'_{k+1}} \subseteq (\mathcal{L}_{q'}L_k)\,|\,[\delta_{k+1},\delta_k)$, we have

$$\begin{aligned}(\mathcal{L}_q L_k)\,|\,[\delta_{k+1},\delta_k) &= (\mathcal{L}_q L_k)\,|\,[\omega_2 I_{k+1},\delta_k) \cup \bigcup\left\{\phi_{I_{k+1},L}[\mathcal{L}_q^{I_{k+1}}] \mid L \in (\mathcal{L}_q L_k)\,|\,\{\omega_2 I_{k+1}\}\right\}\\ &\subseteq (\mathcal{L}_q L_k)\,|\,[\omega_2 I_{k+1},\delta_k) \cup \bigcup\left\{\phi_{I_{k+1},L}[(\mathcal{L}_{q'}L_k)\,|\,[\delta_{k+1},\delta_k)] \mid L \in (\mathcal{L}_q L_k)\,|\,\{\omega_2 I_{k+1}\}\right\}\\ &\subseteq \mathcal{L}\,|\,[\delta_{k+1},\delta_k).\end{aligned}$$

Also

$$(\mathcal{L}_{q'}L_k)\,|\,[\delta_{k+1},\delta_k) \subseteq \mathcal{L}\,|\,[\delta_{k+1},\delta_k).$$

Thus

$$\begin{aligned}\mathcal{L}_q\,|\,[\delta_{k+1},\delta_k) &= \bigcup\left\{\phi_{L_k,L}\big[(\mathcal{L}_q L_k)\,|\,[\delta_{k+1},\delta_k)\big] \mid L \in \mathcal{L}_q\,|\,\{\delta_k\}\right\}\\ &\subseteq \bigcup\left\{\phi_{L_k,J}\big[\mathcal{L}\,|\,[\delta_{k+1},\delta_k)\big] \mid J \in \mathcal{L}\,|\,\{\delta_k\}\right\} = \mathcal{L}\,|\,[\delta_{k+1},\delta_k).\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{q'}\,|\,[\delta_{k+1},\delta_k) &= \bigcup\left\{\phi_{L_k,L'}\big[(\mathcal{L}_{q'} L_k)\,|\,[\delta_{k+1},\delta_k)\big] \mid L' \in \mathcal{L}_{q'}\,|\,\{\delta_k\}\right\}\\ &\subseteq \bigcup\left\{\phi_{L_k,J}\big[\mathcal{L}\,|\,[\delta_{k+1},\delta_k)\big] \mid J \in \mathcal{L}\,|\,\{\delta_k\}\right\} = \mathcal{L}\,|\,[\delta_{k+1},\delta_k).\end{aligned}$$

$$\mathcal{L}_q\,|\,[\delta_{k+1},\omega_2) = \big(\mathcal{L}_q\,|\,[\delta_{k+1},\delta_k)\big) \cup \big(\mathcal{L}_q\,|\,[\delta_k,\omega_2)\big) \subseteq \big(\mathcal{L}\,|\,[\delta_{k+1},\delta_k)\big) \cup \big(\mathcal{L}\,|\,[\delta_k,\omega_2)\big) = \mathcal{L}\,|\,[\delta_{k+1},\omega_2).$$

$$\mathcal{L}_{q'}\,|\,[\delta_{k+1},\omega_2) = \big(\mathcal{L}_{q'}\,|\,[\delta_{k+1},\delta_k)\big) \cup \big(\mathcal{L}_{q'}\,|\,[\delta_k,\omega_2)\big) \subseteq \big(\mathcal{L}\,|\,[\delta_{k+1},\delta_k)\big) \cup \big(\mathcal{L}\,|\,[\delta_k,\omega_2)\big) = \mathcal{L}\,|\,[\delta_{k+1},\omega_2).$$

(5):

Case 0. $I =_{\omega_2} J =_{\omega_2} L_k$: $\phi_{I,J}[\mathcal{S}^I] = \phi_{I,J} \circ \phi_{L_k,I}[\mathcal{S}_q^{L_k}] = \phi_{L_k,J}[\mathcal{S}_q^{L_k}] = \mathcal{S}^J$.

Case 1. $I_{k+1} <_{\omega_2} I =_{\omega_2} J <_{\omega_2} L_k$: Let $L_A, L_B \in \mathcal{L}_q L_k$ with $L_A =_{\omega_2} L_B =_{\omega_2} I =_{\omega_2} J$. Then

$$\phi_{I,J}[\mathcal{S}^I] = \phi_{I,J} \circ \phi_{L_A,I}[\mathcal{S}_q^{L_A}] = \phi_{I,J} \circ \phi_{L_A,I} \circ \phi_{L_B,L_A}[\mathcal{S}_q^{L_B}] = \phi_{L_B,J}[\mathcal{S}_q^{L_B}] = \mathcal{S}^J.$$

Case 2. $I_{k+1} =_{\omega_2} I =_{\omega_2} J$:

$$\phi_{I,J}[\mathcal{S}^I] = \phi_{I,J} \circ \phi_{I_{k+1},I}[\mathcal{S}_{q'}^{L_k} \cup (\mathcal{S}_q^{I_{k+1}} \setminus S)] = \phi_{I_{k+1},J}[\mathcal{S}_{q'}^{L_k} \cup (\mathcal{S}_q^{I_{k+1}} \setminus S)] = \mathcal{S}^J.$$

Case 3. $L_{k+1} <_{\omega_2} I =_{\omega_2} J <_{\omega_2} I_{k+1}$: Let $L'_A, L'_B \in \mathcal{L}_{q'} L_k$ with $L'_A =_{\omega_2} L'_B =_{\omega_2} I =_{\omega_2} J$. Then $\phi_{L'_A,L'_B} \in S \subseteq X \in \mathcal{S}_q^{I_{k+1}} \setminus S$ and so

$$\phi_{L'_B,L'_A}\big[\mathcal{S}_{q'}^{L'_B} \cup \{L'_B X \mid X \in \mathcal{S}_q^{I_{k+1}} \setminus S\}\big] = \mathcal{S}_{q'}^{L'_A} \cup \{L'_A X \mid X \in \mathcal{S}_q^{I_{k+1}} \setminus S\}.$$

Thus

$$\begin{aligned}
\phi_{I,J}[\mathcal{S}^I] &= \phi_{I,J} \circ \phi_{L'_A,I}[\mathcal{S}_{q'}^{L'_A} \cup \{L'_A X \mid X \in \mathcal{S}_q^{I_{k+1}} \setminus S\}] \\
&= \phi_{I,J} \circ \phi_{L'_A,I} \circ \phi_{L'_B,L'_A}[\mathcal{S}_{q'}^{L'_B} \cup \{L'_B X \mid X \in \mathcal{S}_q^{I_{k+1}} \setminus S\}] \\
&= \phi_{L'_B,J}[\mathcal{S}_{q'}^{L'_B} \cup \{L'_B X \mid X \in \mathcal{S}_q^{I_{k+1}} \setminus S\}] = \mathcal{S}^J.
\end{aligned}$$

(6), (7): For $L \in \mathcal{L}_q \,|\, (\delta_{k+1}, \delta_k]$, want to show $\mathcal{S}_q^L \subseteq \bigcup\{\mathcal{S}^J \mid J \in \mathcal{L} \,|\, (\delta_{k+1}, \delta_k]\}$.

Case 0. $L =_{\omega_2} L_k$: $\mathcal{S}_q^L = \phi_{L_k,L}[\mathcal{S}_q^{L_k}] = \mathcal{S}^L \subseteq \bigcup\{\mathcal{S}^J \mid J \in \mathcal{L} \,|\, (\delta_{k+1}, \delta_k]\}$.

Case 1. $I_{k+1} <_{\omega_2} L <_{\omega_2} L_k$: Pick $J_2 \in \mathcal{L}_q \,|\, \{\delta_k\} \subseteq \mathcal{L} \,|\, \{\delta_k\}$ s.t. $L \in J_2 =_{\omega_2} L_k$. Let $J_3 = \phi_{J_2,L_k}(L) \in \mathcal{L}_q L_k$. Then

$$\mathcal{S}_q^L = \phi_{J_3,L}[\mathcal{S}_q^{J_3}] = \mathcal{S}^L \subseteq \bigcup\{\mathcal{S}^J \mid J \in \mathcal{L} \,|\, (\delta_{k+1}, \delta_k]\}.$$

Case 2. $L =_{\omega_2} I_{k+1}$: First

$$\mathcal{S}_q^{I_{k+1}} = (\mathcal{S}_q^{I_{k+1}} S) \cup (\mathcal{S}_q^{I_{k+1}} \setminus S) = (\mathcal{S}_{q'}^{I'_{k+1}} S') \cup (\mathcal{S}_q^{I_{k+1}} \setminus S) \subseteq \mathcal{S}^{I'_{k+1}} \cup \mathcal{S}^{I_{k+1}}.$$

Let $J_4 = \phi_{I_{k+1},L}(I'_{k+1}) \in \mathcal{L} \,|\, (\delta_{k+1}, \delta_k)$. Then $\phi_{I'_{k+1},J_4} = \phi_{I_{k+1},L} \lceil I'_{k+1}$ and so

$$\mathcal{S}_q^L = \phi_{I_{k+1},L}[\mathcal{S}_q^{I_{k+1}}] \subseteq \phi_{I'_{k+1},J_4}[\mathcal{S}^{I'_{k+1}}] \cup \phi_{I_{k+1},L}[\mathcal{S}^{I_{k+1}}] = \mathcal{S}^{J_4} \cup \mathcal{S}^L \subseteq \bigcup\{\mathcal{S}^J \mid J \in \mathcal{L} \,|\, (\delta_{k+1}, \delta_k]\}.$$

For $L' \in \mathcal{L}_{q'} \,|\, (\delta_{k+1}, \delta_k]$, suffices to show $\mathcal{S}_{q'}^{L'} \subseteq \bigcup\{\mathcal{S}^J \mid J \in \mathcal{L}_{q'} \,|\, (\delta_{k+1}, \delta_k]\}$. Pick any $L'_A \in \mathcal{L}_{q'}(L_k \cup \{L_k\})$ with $L' =_{\omega_2} L'_A$. When $L'_A = L_k$, we have

$$\mathcal{S}_{q'}^{L'} = \phi_{L_k,L'}[\mathcal{S}_{q'}^{L_k}] \subseteq \phi_{L_k,L'}[\mathcal{S}^{I_{k+1}}] = \mathcal{S}^{\phi_{L_k,L'}(I_{k+1})} \subseteq \bigcup\{\mathcal{S}^J \mid J \in \mathcal{L} \,|\, (\delta_{k+1}, \delta_k]\}.$$

When $L'_A \in L_k$, we have

$$\mathcal{S}_{q'}^{L'} = \phi_{L'_A,L'}[\mathcal{S}_{q'}^{L'_A}] \subseteq \phi_{L'_A,L'}[\mathcal{S}^{L'_A}] = \mathcal{S}^{L'} \subseteq \bigcup\{\mathcal{S}^J \mid J \in \mathcal{L} \,|\, (\delta_{k+1}, \delta_k]\}.$$

(8): Let $I \in \mathcal{L} \,|\, (\delta_{k+1}, \delta_k]$, $X \in \mathcal{S}^I$, and $J \in \mathcal{L}^I$. We want $JX \in \mathcal{S}$.

Case 0. $I =_{\omega_2} L_k$: $\mathcal{S}^I = \phi_{L_k,I}[\mathcal{S}_q^{L_k}]$ and $\mathcal{L}^I = \phi_{L_k,I}[\mathcal{L}_q^{L_k}]$. Pick $Y \in \mathcal{S}_q^{L_k}$ and $L \in \mathcal{L}_q^{L_k}$ s.t. $X = \phi_{L_k,I}(Y)$ and $J = \phi_{L_k,I}(L)$. We have $LY \in \mathcal{S}_q \subseteq \mathcal{S}$. Thus

$$JX = \phi_{L_k,I}(L)\phi_{L_k,I}(Y) = \phi_{L_k,I}(LY) \in \phi_{L_k,I}[\mathcal{S}] \subseteq \mathcal{S}.$$

Case 1. $I_{k+1} <_{\omega_2} I <_{\omega_2} L_k$: Pick any $J_1 \in \mathcal{L}_q L_k$ s.t. $J_1 =_{\omega_2} I$. Let $Y \in \mathcal{S}_q^{J_1}$ s.t. $\phi_{J_1,I}(Y) = X$. Let $J_3 \in \mathcal{L}_q^{J_1}$ s.t. $\phi_{J_1,I}(J_3) = J$. We have $J_3 Y \in \mathcal{S}_q \subseteq \mathcal{S}$. Thus

$$JX = \phi_{J_1,I}(J_3)\phi_{J_1,I}(Y) = \phi_{J_1,I}(J_3 Y) \in \phi_{J_1,I}[\mathcal{S}] \subseteq \mathcal{S}.$$

Case 2. $I =_{\omega_2} I_{k+1}$: $\mathcal{S}^I = \phi_{I_{k+1},I}[\mathcal{S}_{q'}^{L_k} \cup (\mathcal{S}_q^{I_{k+1}} \setminus S)]$ and $\mathcal{L}^I = \phi_{I_{k+1},I}[\mathcal{L}_{q'}^{L_k}]$. Let $J_3' \in \mathcal{L}_{q'}^{L_k}$ with $\phi_{I_{k+1},I}(J_3') = J$.

Subcase 1. $X = \phi_{I_{k+1},I}(Y')$ with $Y' \in \mathcal{S}_{q'}^{L_k}$: We have $J_3'Y' \in \mathcal{S}_{q'}$. Thus

$$JX = \phi_{I_{k+1},I}(J_3')\phi_{I_{k+1},I}(Y') = \phi_{I_{k+1},I}(J_3'Y') \in \phi_{I_{k+1},I}[\mathcal{S}_{q'}] \subseteq \phi_{I_{k+1},I}[\mathcal{S}] \subseteq \mathcal{S}.$$

Subcase 2. $X = \phi_{I_{k+1},I}(Y)$ with $Y \in \mathcal{S}_q^{I_{k+1}} \setminus S$: By definition, $J_3'Y \in \mathcal{S}^{J_3'}$. Thus

$$JX = \phi_{I_{k+1},I}(J_3')\phi_{I_{k+1},I}(Y) = \phi_{I_{k+1},I}(J_3'Y) \in \phi_{I_{k+1},I}[\mathcal{S}^{J_3'}] \subseteq \phi_{I_{k+1},I}[\mathcal{S}] \subseteq \mathcal{S}.$$

Case 3. $I_{k+1}' <_{\omega_2} I <_{\omega_2} I_{k+1}$: Pick $J_1' \in \mathcal{L}_{q'} L_k$ with $J_1' =_{\omega_2} I$. We have $\mathcal{S}^I = \phi_{J_1',I}[\mathcal{S}^{J_1'}] = \phi_{J_1',I}\big[\mathcal{S}_{q'}^{J_1'} \cup \{J_1'Y \mid Y \in \mathcal{S}_q^{I_{k+1}} \setminus S\}\big]$ and $\mathcal{L}^I = \phi_{J_1',I}[\mathcal{L}_{q'}^{J_1'}]$. Let $J_3' \in \mathcal{L}_{q'}^{J_1'}$ with $\phi_{J_1',I}(J_3') = J$.

Subcase 1. $X = \phi_{J_1',I}(Y')$ with $Y' \in \mathcal{S}_{q'}^{J_1'}$: We have $J_3'Y' \in \mathcal{S}_{q'} \subseteq \mathcal{S}$. Thus

$$JX = \phi_{J_1',I}(J_3')\phi_{J_1',I}(Y') = \phi_{J_1',I}(J_3'Y') \in \phi_{J_1',I}[\mathcal{S}] \subseteq \mathcal{S}.$$

Subcase 2. $X = \phi_{J_1',I}(J_1'Y)$ with $Y \in \mathcal{S}_q^{I_{k+1}} \setminus S$: We have $J_3'J_1'Y = J_3'Y \in \mathcal{S}^{J_3'} \subseteq \mathcal{S}$. Thus

$$JX = \phi_{J_1',I}(J_3')\phi_{J_1',I}(J_1'Y) = \phi_{J_1',I}(J_3'Y) \in \phi_{J_1',I}[\mathcal{S}] \subseteq \mathcal{S}.$$

Case 4. $I =_{\omega_2} I_{k+1}'$: $\mathcal{S}^I = \phi_{I_{k+1}',I}[\mathcal{S}^{I_{k+1}'}] = \phi_{I_{k+1}',I}\big[\mathcal{S}_{q'}^{I_{k+1}'} \cup \{I_{k+1}'Y \mid Y \in \mathcal{S}_q^{I_{k+1}} \setminus S\}\big]$ and $\mathcal{L}^I = \phi_{I_{k+1}',I}[\mathcal{L}_{q'}^{I_{k+1}'}] = \phi_{I_{k+1}',I}[\mathcal{L}_q^{I_{k+1}}]$. Let $J = \phi_{I_{k+1}',I}(J_3)$ with $J_3' = J_3 \in \mathcal{L}_{q'}^{I_{k+1}'} = \mathcal{L}_q^{I_{k+1}}$.

Subcase 1. $X = \phi_{I_{k+1}',I}(Y')$ with $Y' \in \mathcal{S}_{q'}^{I_{k+1}'}$: We have $J_3'Y' \in \mathcal{S}_{q'} \subseteq \mathcal{S}$. Thus

$$JX = \phi_{I_{k+1}',I}(J_3')\phi_{I_{k+1}',I}(Y') = \phi_{I_{k+1}',I}(J_3'Y') \in \phi_{I_{k+1}',I}[\mathcal{S}] \subseteq \mathcal{S}.$$

Subcase 2. $X = \phi_{I_{k+1}',I}(I_{k+1}'Y)$ with $Y \in \mathcal{S}_q^{I_{k+1}} \setminus S$: We have $J_3 I_{k+1}'Y = J_3 Y \in \mathcal{S}_q \subseteq \mathcal{S}$. Thus

$$JX = \phi_{I_{k+1}',I}(J_3)\phi_{I_{k+1}',I}(I_{k+1}'Y) = \phi_{I_{k+1}',I}(J_3 Y) \in \phi_{I_{k+1}',I}[\mathcal{S}] \subseteq \mathcal{S}.$$

□

Suppose we are done with (SKY), (MID), and (GRD).

Claim. Let

$$\mathcal{L} := \big(\mathcal{L} \restriction [0,\delta_n)\big) \cup \Big(\bigcup \big\{\mathcal{L} \restriction [\delta_{k+1},\delta_k) \mid k+1 \le n\big\}\Big) \cup \big(\mathcal{L} \restriction [\delta_0,\omega_2)\big).$$

$$\mathcal{S} := \bigcup \big\{\mathcal{S}^J \mid J \in \mathcal{L}\big\}.$$

Then $(\mathcal{L}, \mathcal{S})$ is a two-type s.t.

(projection to ω_2) $\omega_2 \,|\, \mathcal{L} = (\omega_2 \,|\, \mathcal{L}_q) \cup (\omega_2 \,|\, \mathcal{L}_{q'})$.

(extend) $\mathcal{L}_q, \mathcal{L}_{q'} \subseteq \mathcal{L}$.

(extend) $\mathcal{S}_q, \mathcal{S}_{q'} \subseteq \mathcal{S}$.

In particular,

(iso) For any $I =_{\omega_2} J$ in $\mathcal{L}$, we have $\phi_{I.J}[\mathcal{S}] \subseteq \mathcal{S}$.

Proof. We discuss (iso) as follows: Let $X \in \mathcal{S}I$. Because $\mathcal{S} = \bigcup \{\mathcal{S}^{J_3} \mid J_3 \in \mathcal{L}\}$, there exists $J_3 \in \mathcal{L}$ s.t. $X \in \mathcal{S}^{J_3}$. We have $J_3 \leq_{\omega_2} I$. Otherwise, $I <_{\omega_2} J_3$ and so $\sup(\omega_2 X) \in I \leq_{\omega_2} \mathcal{L}^{J_3} < \sup(\omega_2 X)$. This is absurd.

Case 1. $J_3 <_{\omega_2} I$: Take $I_1 \in \mathcal{L}$ s.t. $J_3 \in I_1 =_{\omega_2} I$. Let $J_4 = \phi_{I_1,I}(J_3)$. Then $\phi_{J_3,J_4} = \phi_{I_1,I} \lceil J_3$. Thus $X = \phi_{I_1,I}(X) \in \phi_{J_3,J_4}[\mathcal{S}^{J_3}] = \mathcal{S}^{J_4}$ and so $\phi_{I,J}(X) \in \mathcal{S}^{\phi_{I,J}(J_4)} \subseteq \mathcal{S}$.

Case 2. $J_3 =_{\omega_2} I$: Then $X = \phi_{J_3,I}(X) \in \phi_{J_3,I}[\mathcal{S}^{J_3}] = \mathcal{S}^I$ and so $\phi_{I,J}(X) \in \phi_{I,J}[\mathcal{S}^I] = \mathcal{S}^J \subseteq \mathcal{S}$.

□

Corollary. There exists the $\subseteq$-least $\mathcal{L}$ s.t.

(sym) $\mathcal{L}$ is a symmetric system w.r.t. $\mathcal{C}_{\omega_1}$.

(projection) $\omega_2 \,|\, \mathcal{L} = (\omega_2 \,|\, \mathcal{L}_{q'}) \cup (\omega_2 \,|\, \mathcal{L}_q)$.

(extend) $\mathcal{L}_{q'}, \mathcal{L}_q \subseteq \mathcal{L}$.

□

Corollary. There exists the $\subseteq$-least $\mathcal{S} \subseteq \mathcal{C}_\omega$ s.t.

(two-type) $(\mathcal{L}, \mathcal{S})$ is a two-type.

(extend) $\mathcal{S}_{q'}, \mathcal{S}_q \subseteq \mathcal{S}$.

□

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