NAMBA FORCINGS ON INACCESSIBLE CARDINALS

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Abstract. In this paper, we show that if κ is inaccessible and $\mathrm{Nm}(\kappa, \lambda, F)$ is semiproper for all κ-complete fine filter F over $\mathcal{P}_\kappa\lambda$ then $\mathrm{Nm}(\kappa, 2^\lambda)$ is locally semiproper.

1. Introduction

It is known that the strong compactness of κ has many characterizations. Moreover, level-by-level characterizations are also known, as follows.

Theorem 1.1 (Hayut [3] for (1)$\leftrightarrow$ (2) $\leftrightarrow$ (3), Jech [4] for (2)$\leftrightarrow$ (4))**.** *For cardinals $\kappa < \lambda = \lambda^{<\kappa}$, the following are equivalent:*

(1) *For every transitive model M, if $2^\lambda \subseteq M$ and $|M| = 2^\lambda$, then there are a transitive model N and an elementary embedding $i : M \to N$ with the following properties:*
- $\mathrm{crit}(i) = \kappa$.
- *there is an $s \in N$ such that $i``2^\lambda \subseteq s$ and $N \models |s| < i(\kappa)$*

(2) *For every $<\kappa$-satisfiable theory T of $\mathcal{L}_{\kappa\kappa}$, if $|T| \leq 2^\lambda$, then T is satisfiable.*
(3) *κ is λ-compact. That is, every κ-complete fine filter over λ can be extended to a κ-complete ultrafilter.*
(4) *Every $\mathcal{P}_\kappa 2^\lambda$-list has a cofinal branch.*
(5) *Every $\mathcal{P}_\kappa 2^\lambda$-thin list has a branch, and κ is inaccessible.*

We also know further implications related to the definition of strong compactness, as follows.

$$\begin{aligned}\kappa \text{ is strongly compact } &\Rightarrow \cdots \\ &\Rightarrow \kappa \text{ is } 2^\lambda\text{-strongly compact} \\ &\Rightarrow \kappa \text{ is } \lambda\text{-compact} \\ &\Rightarrow \kappa \text{ is } \lambda\text{-strongly compact} \\ &\Rightarrow \cdots\end{aligned}$$

Here, we say that κ is λ-strongly compact if $\mathcal{P}_\kappa\lambda$ carries a κ-complete fine ultrafilter.

We introduced a similar hierarchy of the (†) principle, which we call the dissection of (†), via two-cardinal Namba forcings (Figure 2). Through the dissection of (†), we can identify level-by-level implications between the above hierarchy of

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strong compactness and the dissection of (†). Figure 1 gives one such example and naturally suggests Question 1.2.

$$\begin{tikzcd}
\text{(1) of Theorem 1.1} \arrow[r] \arrow[d, leftrightarrow] & \mathrm{Nm}(\kappa, 2^{\lambda}) \text{ is loc. sp.} \arrow[d, leftrightarrow] \\
\text{(2) of Theorem 1.1} \arrow[r, "\text{Shelah [8]}"'] \arrow[d, leftrightarrow] & \mathrm{SSR}([2^{\lambda}]^{\omega}, {<}\kappa) \arrow[d] \\
\text{(3) of Theorem 1.1} \arrow[r] & \forall F (\mathrm{Nm}(\kappa, \lambda, F) \text{ is sp. })
\end{tikzcd}$$

FIGURE 1. It seems that strong compactness and (†) have level-by-level implications.

Question 1.2. *Suppose that* $\mathrm{Nm}(\kappa, \lambda, F)$ *is semiproper for all* κ*-complete fine filter* F *over* $\mathcal{P}_{\kappa}\lambda$*. Does* $\mathrm{SSR}([2^{\lambda}]^{\omega}, {<}\kappa)$ *hold?*

For the definitions on the right-hand side of Figure 1, see Section 2. In this paper, we show that

Theorem 1.3. *If* κ *is inaccessible, then for every regular* $\lambda = \lambda^{<\kappa} \geq \kappa$*, the following are equivalent:*

1. $\mathrm{Nm}(\kappa, 2^{\lambda})$ *is locally semiproper.*
2. $\mathrm{SSR}([2^{\lambda}]^{<\kappa}, {<}\kappa)$*.*
3. $\mathrm{Nm}(\kappa, \lambda, F)$ *is semiproper for all* κ*-complete fine filters over* $\mathcal{P}_{\kappa}\lambda$*.*

2. PRELIMINARIES

In this section, we recall basic facts about Namba forcing with filters and a specific filter. We use [5] as a general reference for set theory. For more on the topic of filters, we refer to [2].

2.1. **Semiproperness of Namba forcings.** Namba forcing, introduced by Namba [6], is a poset that forces $\mathrm{cf}(\aleph_2) = \omega$ while preserving $\aleph_1$. We begin by introducing a class, which includes Namba's original poset, denoted by $\mathrm{Nm}(Z, F)$, indexed by filters F and its base set Z. For an $\aleph_2$-complete fine filter F over $Z \subseteq \mathcal{P}(X)$, an F-Namba tree p is a set $p \subseteq [Z]^{<\omega}$ satisfying the following conditions:

- p is a tree, i.e., each $s \in p$ is $\subseteq$-increasing, and p is closed under initial segments.
- There exists a maximal $s \in p$ such that $\forall t \in p(s \subseteq t \vee t \subseteq s)$. This s is called the trunk, denoted by $\mathrm{tr}(p)$.
- For each $s \in p$, if $s \sqsupseteq \mathrm{tr}(p)$, then $\mathrm{Suc}(s) = \{a \in Z \mid s^\frown\langle a \rangle \in p\} \in F^{+}$.

If $Z = \mathcal{P}_{\kappa}\lambda$, we write $\mathrm{Nm}(\kappa, \lambda, F)$ to refer to $\mathrm{Nm}(Z, F)$ (to emphasize the context of two-cardinal combinatorics). For a filter F, $\mathrm{Nm}(Z, F)$ is defined as $\mathrm{Nm}(Z, F^{*})$. For an ideal I, we consider its dual filter, i.e., $\mathrm{Nm}(Z, I)$ is $\mathrm{Nm}(Z, I^{*})$. We write $\mathrm{Nm}(\kappa)$ and $\mathrm{Nm}(\kappa, \lambda)$ to refer to $\mathrm{Nm}(\kappa, I_{\kappa})$ and $\mathrm{Nm}(\mathcal{P}_{\kappa}\lambda, I_{\kappa\lambda})$, respectively. Here, I_{κ} and $I_{\kappa\lambda}$ are bounded ideals over κ and $\mathcal{P}_{\kappa}\lambda$, respectively. $\mathrm{Nm}(\aleph_2)$ is the original Namba forcing. We now examine the basic properties of $\mathrm{Nm}(Z, F)$.

Lemma 2.1. *If* F *is an* $\aleph_2$*-complete fine filter over* $Z \subseteq \mathcal{P}(X)$ *then the following holds:*

(1) *If F is uniformly and exactly κ-complete*[1] *then* $\mathrm{Nm}(Z,F) \Vdash \dot{\mathrm{cf}}(\kappa) = \omega$.
(2) *If F is uniformly and exactly κ-complete then, for all $\delta \in [\kappa, |X|^V] \cap \mathrm{Reg}$, if $\{z \in Z \mid |X| < z\} \in F$, then* $\mathrm{Nm}(Z,F) \Vdash \dot{\mathrm{cf}}(\delta) = \omega$.
(3) *If F is κ-complete then, for all $\delta \in [\aleph_1, \kappa) \cap \mathrm{Reg}$, if $\delta^\omega < \kappa$ or $\delta = \aleph_1$, then for every semistationary (or stationary, respectively) subset $S \subseteq [\delta]^\omega$,* $\mathrm{Nm}(Z,F) \Vdash$ *S is semistationary (or stationary, respectively). In particular,* $\mathrm{Nm}(Z,F) \Vdash \dot{\mathrm{cf}}(\delta) > \omega$.
(4) $\mathrm{Nm}(Z,F)$ *is ω_1-stationary preserving.*

Namba forcings may be semiproper. Indeed, the following are well-known.

Theorem 2.2.
(1) *(Shelah [8] for (a) $\leftrightarrow$ (b); Doebler–Schindler [1] for (b) $\to$ (c)) The following are equivalent:*
  (a) ($\dagger$) *holds, that is, every ω_1-stationary preserving poset is semiproper.*
  (b) $\mathrm{Nm}(\kappa)$ *is semiproper for all regular $\kappa \geq \aleph_2$.*
  (c) SSR *holds, that is, for every $\lambda \geq \aleph_2$ and a semistationary subset $S \subseteq [\lambda]^\omega$, there exists an $x \in \mathcal{P}_{\aleph_2}\lambda$ such that $S \cap [x]^\omega$ is semistationary.*
(2) *(Shelah [8]) For all regular κ, the following are equivalent:*
  (a) $\mathrm{Nm}(\kappa)$ *is semiproper.*
  (b) *There exists a semiproper poset that forces* $\dot{\mathrm{cf}}(\kappa) = \omega$.

Note that semiproperness of Namba forcings and semistationary reflection principles have a level-by-level characterization.

Theorem 2.3 (Tsukuura [11])**.** *Suppose that $\mu^\omega < \kappa$ for all $\mu < \kappa$ or that $\kappa = \aleph_2$ and $\aleph_2 \leq \kappa \leq \lambda = \lambda^{<\kappa}$ are regular cardinals. For a semistationary subset $S \subseteq [\lambda]^\omega$, the following are equivalent:*
(1) $\mathrm{Nm}(\kappa, \lambda)$ *forces S to be semistationary.*
(2) *There exists an $a \in \mathcal{P}_\kappa\lambda$ such that $S \cap [a]^\omega$ is semistationary.*
(3) *For every κ-complete fine filter F over $\mathcal{P}_\kappa\lambda$,* $\mathrm{Nm}(\kappa,\lambda,F)$ *forces S to be semistationary.*

Note that the implication from (1) to (2) holds even if $\mu^\omega \geq \kappa$ for some $\mu < \kappa$.

We say that $\mathrm{Nm}(\kappa,\lambda,F)$ is *locally semiproper* if, for every semistationary subset $S \subseteq [\lambda]^\omega$, $\mathrm{Nm}(\kappa,\lambda,F) \Vdash S$ is semistationary. The local semiproperness of $\mathrm{Nm}(\lambda,F)$ is defined analogously. We get a diagram as in Figure 2

Lastly, we conclude this section with a Lemma that used in killing the semiproperness of a Namba forcing. For a set $S \subseteq [W]^\omega$, we introduce the specific set $\mathrm{Gal}_\theta(Z,F,S,A)$, the set of all $M \in [\mathcal{H}_\theta]^\omega$ such that $M \cap W \in S$ and $\bigcap\{f^{-1}M \cap \omega_1 \mid f : A \to \omega_1 \in M\} \in F^*$. The idea behind this definition comes from [9]. Note that the *non*-stationarity of $\mathrm{Gal}_\theta(\aleph_2, I_{\aleph_2}, [\aleph_2]^\omega, \aleph_2)$ is equivalent to the strong Chang's conjecture at $\aleph_2$. We will use the following:

Lemma 2.4. *For a stationary subset $S \subseteq [W]^\omega$, the following are equivalent:*
(1) $\mathrm{Gal}_\theta(Z,F,S,A)$ *is stationary.*
(2) *There exist a stationary subset $T \subseteq S$ and $p_A \in \mathrm{Nm}(Z,F)$ such that $p_A \Vdash T$ is* non-*semistationary.*

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[1] That is $\{A \in F^+ \mid F \restriction A$ is κ-complete and *not* κ^+-complete $\}$ is dense in $\langle F^+, \subseteq\rangle$

$$
\begin{tikzcd}
\mathrm{SSR} \arrow[r, leftrightarrow] \arrow[dd, dotted] & (\dagger) \arrow[d, dotted] & \\
& \mathrm{Nm}(\kappa, 2^\lambda) \text{ is sp.} \arrow[r] \arrow[d] & \mathrm{Nm}(2^\lambda) \text{ is sp.} \arrow[d] \\
\mathrm{SSR}([2^\lambda]^\omega, {<}\kappa) \arrow[r, leftrightarrow] \arrow[ddd] & \mathrm{Nm}(\kappa, 2^\lambda) \text{ is loc. sp.} \arrow[r] \arrow[d] & \mathrm{Nm}(2^\lambda) \text{ is loc. sp.} \arrow[dd, "/" marking] \\
& \text{All } \mathrm{Nm}(\kappa, \lambda, F) \text{ are sp.} \arrow[d, shift right] & \\
& \mathrm{Nm}(\kappa, \lambda) \text{ is sp.} \arrow[u, shift right, "/" marking] \arrow[r] \arrow[d] & \mathrm{Nm}(\lambda) \text{ is sp.} \arrow[d] \\
\mathrm{SSR}([\lambda]^\omega, {<}\kappa) \arrow[r, leftrightarrow] \arrow[ddd, dotted] & \mathrm{Nm}(\kappa, \lambda) \text{ is loc. sp.} \arrow[r] \arrow[dd, dotted] & \mathrm{Nm}(\lambda) \text{ is loc. sp.} \\
& & \\
& \mathrm{Nm}(\kappa, \kappa) \text{ is sp.} \arrow[r, leftrightarrow] \arrow[d] & \mathrm{Nm}(\kappa) \text{ is sp.} \arrow[d] \\
\mathrm{SSR}([\kappa]^\omega, {<}\kappa) \arrow[r, leftrightarrow] & \mathrm{Nm}(\kappa, \kappa) \text{ is loc. sp.} \arrow[r, leftrightarrow] & \mathrm{Nm}(\kappa) \text{ is loc. sp.}
\end{tikzcd}
$$

FIGURE 2. The dissection of $(\dagger)$.

2.2. **Filters assocated with thin lists.** In the proof of Theorem 1.3, we use a specific filter $F_{\overline{d}}$ over $\mathcal{P}_\kappa\lambda$ associated with a $\mathcal{P}_\kappa 2^\lambda$-thin list $\overline{d}$. For regular cardinals $\kappa \leq \lambda = \lambda^{<\kappa} \leq 2^\lambda$, a thin $\mathcal{P}_\kappa 2^\lambda$-list $\overline{d} = \langle d_x \mid x \in \mathcal{P}_\kappa 2^\lambda \rangle$ is a sequence of subsets $d_x \subseteq x$ such that $\mathrm{Lev}_x(\overline{d}) = \{d_y \cap x \mid x \subseteq y\}$ has size $< \kappa$ for all $x \in \mathcal{P}_\kappa 2^\lambda$. We fix such a list $\overline{d}$.

To define $F_{\overline{d}}$, the key ingredient is the following lemma.

Lemma 2.5. *For regular cardinals $\kappa \leq \lambda = \lambda^{<\kappa}$, there exists a family $\langle A_\alpha \mid \alpha < 2^\lambda \rangle$ of subsets of $\mathcal{P}_\kappa\lambda$ such that, for every $x, y \in \mathcal{P}_\kappa 2^\lambda$ with $x \cap y = \emptyset$,*

$$\bigcap_{\alpha \in x} A_\alpha \setminus \bigcup_{\alpha \in y} A_\alpha \textit{ is unbounded in } \mathcal{P}_\kappa\lambda.$$

For $x \in \mathcal{P}_\kappa 2^\lambda$, define B_x by

$$B_x = \bigcup_{d \in \mathrm{Lev}_x(\overline{d})} \left(\bigcap_{\alpha \in d \cap x} A_\alpha \setminus \bigcup_{\alpha \in x \setminus d} A_\alpha \right).$$

Here, $\langle A_\alpha \mid \alpha < 2^\lambda \rangle$ is given by Lemma 2.5. By Lemma 2.5, B_x is unbounded in $\mathcal{P}_\kappa\lambda$. Let $F_{\overline{d}}$ be the κ-complete fine filter generated by $\{B_x \mid x \in \mathcal{P}_\kappa 2^\lambda\}$.

$F_{\overline{d}}$ is a filter that directly connects λ-compactness and $\mathrm{TP}(\kappa, \lambda)$[2]. For details, we refer to [12] and [10].

[2] This filter can be used in a proof of (3) $\to$ (5) in Theorem 1.1.

3. Proof of Theorem 1.3

We begin with the following lemma.

Lemma 3.1. *For a set X with $\omega_1 \subseteq X$ and $S \subseteq [X]^\omega$, the following are equivalent:*

1. *S is a semistationary subset.*
2. *For every $F : [X]^{<\omega} \to \omega_1$, there is an $x \in S$ such that x is closed under F.*

Proof. (1) $\to$ (2) is easy. Indeed, for each function $F : [X]^{<\omega} \to \omega_1$, we can choose x and y such that $x \in S$, $x \subseteq y$, $x \cap \omega_1 = y \cap \omega_1$, and y is closed under F. Then, for every $\xi_0, ..., \xi_n \in x$, $F(\xi_0, ..., \xi_n) \in y \cap \omega_1 = x \cap \omega_1$, as desired.

Let us show (2) $\to$ (1). Fix a function $F : [X]^{<\omega} \to X$. We want to find an $x \in S$ and a y such that $x \subseteq y$, $y \cap \omega_1 = x \cap \omega_1$, and y is closed under F. For each $a = \{a_0, ..., a_n\}$, define finite sets X^a_n by

- $X^a_0 = a$.
- $X^a_{i+1} = \{F(b) \mid b \in [X^a_i]^{<\omega}\} \cup X^a_i$.

Of course, each X^a_n is finite. Define $H : [X]^{<\omega} \to \omega_1$ by

$$H(a_0, ..., a_n) = \begin{cases} \text{the } k\text{-th element of } X_l^{\{a_0,...,a_m\}} \cap \omega_1 & n = 2^k 3^l 5^m \\ 0 & \text{otherwise} \end{cases}$$

By (2), we can choose an $x \in S$ such that x is closed under H. Pick the least subset y that is closed under F and contains x. By the definition, $y \cap \omega_1$ can be written as the hull of x under H. So $y \cap \omega_1 = x \cap \omega_1$, as desired. □

Lemma 3.2. *For an inaccessible cardinal κ and a regular cardinal $\lambda = \lambda^{<\kappa} \leq \kappa$, suppose that for every κ-complete fine filter F over $\mathcal{P}_\kappa\lambda$, $\mathrm{Nm}(\kappa, \lambda, F)$ is semiproper. Then $\mathrm{SSR}([2^\lambda]^\omega, {<}\kappa)$ holds.*

Proof. We show the contrapositive. Fix a *non*-reflecting semistationary subset $S \subseteq [2^\lambda]^\omega$. For every $x \in \mathcal{P}_\kappa 2^\lambda$, $S \cap [x]^\omega$ is *not* semistationary. By Lemma 3.1, we can choose $F_x : [x]^{<\omega} \to \omega_1$. Let $\pi : [2^\lambda]^{<\omega} \times \omega_1 \to 2^\lambda$ be a bijection. Define a $\mathcal{P}_\kappa 2^\lambda$-list $\overline{d} = \langle d_x \mid x \in \mathcal{P}_\kappa 2^\lambda\rangle$ by $d_x = (\pi``[x]^{<\omega} \times \omega_1) \cap x$. Since κ is inaccessible, $\overline{d}$ is a thin list.

Fix an independent family $\langle A_\xi \mid \xi < 2^\lambda\rangle$ on $\mathcal{P}_\kappa\lambda$. We can define $F_{\overline{d}}$ as in Section 2.2. Let θ be a sufficiently large regular cardinal. We show that $\mathrm{Gal}_\theta(\mathcal{P}_\kappa\lambda, F_{\overline{d}}, \mathcal{P}_\kappa\lambda, S^{\mathrm{cl}})$ is a stationary subset.

By Lemma 2.4, it is enough to show that $\mathrm{Gal}_\theta(\mathcal{P}_\kappa\lambda, F_{\overline{d}}, \mathcal{P}_\kappa\lambda, S^{\mathrm{cl}})$ is stationary.

For each club $C \subseteq [\mathcal{H}_\theta]^\omega$, pick a countable elementary substructure $M \prec \mathcal{H}_\theta$ with the following properties:

- $M \cap 2^\lambda \in S^{\mathrm{cl}}$,
- $\kappa, \lambda, \langle A_\xi \mid \xi < 2^\lambda\rangle, F_{\overline{d}}, S \in M$, and
- $M \in C$.

We show that $M \in \mathrm{Gal}_\theta(\mathcal{P}_\kappa\lambda, F_{\overline{d}}, \mathcal{P}_\kappa\lambda, S^{\mathrm{cl}})$. That is, $\bigcap\{f^{-1}M \cap \omega_1 \mid f : \mathcal{P}_\kappa\lambda \to \omega_1 \in M\} \in F^*_{\overline{d}}$ is our claim.

Let $x \in \mathcal{P}_\kappa 2^\lambda$ be a subset with $\pi``([x]^{<\omega} \times \omega_1) = x$ and $\bigcup(M \cap \mathcal{P}_\kappa 2^\lambda) \subseteq x$. Note that $B_x = \bigcup_{d \in \mathrm{Lev}_{\overline{d}}(x)}(\bigcap_{\xi \in d} A_\xi \setminus \bigcup_{\xi \in x \setminus d} A_\xi) \in F_{\overline{d}}$. For each $a \in B_x$, we claim that $M \cap \omega_1 < \mathrm{Sk}(M \cup \{a\}) \cap \omega_1$.

We work in $N = \mathrm{Sk}(M \cup \{a\})$. Define a function $f : [\mathcal{H}_\theta]^{<\omega} \to \omega_1$ by

$$f(a) = \begin{cases} \xi & a \in A_{\pi(a,\xi)} \text{ and there is no other } \zeta \text{ such that } a \in A_{\pi(a,\zeta)} \\ 0 & \text{otherwise} \end{cases}$$

By $a \in B_x$, we have $d \in \mathrm{Lev}_x(\vec{d})$ with $a \in \bigcap_{\xi \in d} A_\xi \setminus \bigcup_{\xi \in x \setminus d} A_\xi$. We have y with the following properties:

- $x \subseteq y$,
- $d_y \cap x = d$, and
- $f \upharpoonright [x]^{<\omega} = f_y \upharpoonright [x]^{<\omega}$.

Since $M \cap 2^\lambda = M \cap x \in S^{\mathrm{cl}} \cap [x]^\omega \subseteq S^{\mathrm{cl}} \cap [y]^\omega$ and there is no $X \in S \cap [y]^\omega$ that is closed under f_y, new countable ordinals are defined by f in N. In particular, $a \notin \bigcap\{f^{-1}M \cap \omega_1 \mid f : \mathcal{P}_\kappa\lambda \to \omega_1 \in M\}$. Therefore $B_x \subseteq \mathcal{P}_\kappa\lambda \setminus \bigcap\{f^{-1}M \cap \omega_1 \mid f : \mathcal{P}_\kappa\lambda \to \omega_1 \in M\}$, and thus, $\bigcap\{f^{-1}M \cap \omega_1 \mid f : \mathcal{P}_\kappa\lambda \to \omega_1 \in M\} \in F^*_{\vec{d}}$, as desired. □

Proof of Theorem 1.3. (2) ↔ (1) and (2) → (3) follow from Theorem 2.3 and [11], respectively. (3) → (2) follows from Lemma 3.2. □

Theorem 1.3 gives a simplified model in which there are pair of κ-complete fine filters $\langle F_0, F_1\rangle$ over $\mathcal{P}_\kappa\lambda$ such that $\mathrm{Nm}(\kappa, \lambda, F_0)$ is semiproper but $\mathrm{Nm}(\kappa, \lambda, F_1)$ is *not* semiproper.

Corollary 3.3. *If κ is λ-strongly compact, then a standard poset which adds a $\square(2^\lambda)$-sequence forces the following properties:*

1. *$\mathcal{P}_\kappa\lambda$ is unchanged.*
2. *$\mathrm{Nm}(\kappa, \lambda)$ is semiproper.*
3. *There is a κ-complete fine filter F such that $\mathrm{Nm}(\kappa, \lambda, F)$ is* not *semiproper.*

Proof. Let P be a standard poset which adds a $\square(2^\lambda)$-sequence. It is known that P is $\lambda + 1$-strategically closed.

Since κ is λ-strongly compact, $\mathrm{Nm}(\kappa, \lambda)$ is semiproper. By [12, (2) of Lemma 2.11], $\mathrm{Nm}(\kappa, \lambda)$ is forced to be semiproper.

On the other hand, by [7, Theorem 2.1], $\square(2^\lambda)$ implies that $\mathrm{SSR}([2^\lambda]^\omega, {<}\kappa)$ fails. Since κ is still inaccessible in the extension, by Theorem 1.3, it is forced that $\mathcal{P}_\kappa\lambda$ carries a κ-complete fine filter F such that $\mathrm{Nm}(\kappa, \lambda, F)$ is *not* semiproper. □

Note that we proved that if $\square(2^\lambda)$ holds, then $\mathcal{P}_\kappa\lambda$ carries a κ-complete fine filter F such that $\mathrm{Nm}(\kappa, \lambda, F)$ is *not* semiproper even if κ is not inaccessible [12]. The ideas are similar to those in the proof of Lemma 3.2; on the other hand, to define a $\mathcal{P}_\kappa\lambda$-thin list from a $\square(2^\lambda)$-sequence, we need the minimal walk method. This is somewhat difficult.

A $\square(2^\lambda)$-sequence defines an explicit *non*-reflecting semistationary subset $S \subseteq [2^\lambda]^\omega$ and a thin list $\vec{d}$. As in the proof of Lemma 3.2, if we can show the thinness of $\vec{d}$ (starting from a *non*-reflecting stationary subset S, where S is not explicit), then we can omit the inaccessibility assumption from Theorem 1.3.

We conclude this paper with the following question.

Question 3.4. *Can we omit the inaccessibility assumption from Theorem 1.3?*

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