EVASION NUMBERS VIA ZERO-PREDICTION

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Abstract. Cruz Chapital, Goto, Hayashi and the author showed that the game-theoretic variants $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^{**}}$ of the splitting number $\mathfrak{s}$ are consistently different, although the corresponding two games differ only in a minor case. This result suggests that even if two relational systems $\mathbf{R} = \langle X, Y, \sqsubset\rangle$, $\mathbf{R}' = \langle X, Y, \sqsubset'\rangle$ are the same modulo a countable set $C \subseteq X$, the associated cardinal invariants might be different. We study this phenomenon for the standard relational system of evasion and prediction and for a variation of it. We show that such a difference occurs for the standard one, but not for the variation.

1. Introduction

In [CGHY26], Cruz Chapital, Goto, Hayashi and the author introduced six game-theoretic variants of the splitting number $\mathfrak{s}$. We deal with two of them, $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^{**}}$, defined as follows:

We define two kinds of games of length ω played by Player I and Player II. Fix a set $\mathcal{A} \subseteq \mathcal{P}(\omega)$. We call the game indicated by the following Table 1 the *splitting* game* with respect to $\mathcal{A}$.

Table 1. The splitting* game

Player I	i_0		i_1		$\ldots$	
Player II		j_0		j_1		$\ldots$

Here, $\langle i_k : k \in \omega\rangle$ and $\langle j_k : k \in \omega\rangle$ are in 2^ω. Player II wins when either Player I did not play 1 infinitely often or

$$\{k \in \omega : j_k = 1\} \in \mathcal{A} \text{ and splits } \{k \in \omega : i_k = 1\}.$$

In the splitting* game, when both $\{k \in \omega : j_k = 1\}$ and $\{k \in \omega : i_k = 1\}$ are finite, the winner is defined as Player II, which is not necessarily obvious. Hence, we just define another game by switching the winner in this case. Namely, the *splitting** game* with respect to $\mathcal{A}$ follows the same rule as the splitting* game, but only the winning condition of Player II is replaced by: Player II wins when <u>either</u> Player I did not play 1 infinitely often *and Player II did*, <u>or</u>

$$\{k \in \omega : j_k = 1\} \in \mathcal{A} \text{ and splits } \{k \in \omega : i_k = 1\}.$$

Let:

$$\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*} = \min\{|\mathcal{A}| : \text{Player I has no winning strategy for the splitting* game with respect to } \mathcal{A}\}, \text{ and}$$

$$\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^{**}} = \min\{|\mathcal{A}| : \text{Player I has no winning strategy for the splitting** game with respect to } \mathcal{A}\}.$$

Thus, the splitting* game and the splitting** game are basically the same: the only difference appears when both players play finite sets, in which case the winner is Player II in the splitting* game but is Player I in the splitting** game. However, the cardinal invariants of the two games are different:

Theorem 1.1 ([CGHY26, Theorem 3.17])**.** $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*} < \mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^{**}}$ *is consistent.*

To analyze this separation result in more detail, recall the concept of relational systems:

Definition 1.2.
- $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ is a *relational system* if X and Y are non-empty sets and $\sqsubset \subseteq X \times Y$.
- We call an element of X a *challenge* and an element of Y a *response*, and say that x *is met by* y when $x \sqsubset y$.
- $F \subseteq X$ is $\mathbf{R}$-*unbounded* if no response meets all challenges in F.
- $F \subseteq Y$ is $\mathbf{R}$-*dominating* if every challenge is met by some response in F.
- $\mathbf{R}$ is non-trivial if X is $\mathbf{R}$-unbounded and Y is $\mathbf{R}$-dominating. For non-trivial $\mathbf{R}$, define
  - $\mathfrak{b}(\mathbf{R}) \coloneqq \min\{|F| : F \subseteq X \text{ is } \mathbf{R}\text{-unbounded}\}$, and
  - $\mathfrak{d}(\mathbf{R}) \coloneqq \min\{|F| : F \subseteq Y \text{ is } \mathbf{R}\text{-dominating}\}$.

Let us reformulate $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^{**}}$ in the framework of relational systems.

Definition 1.3. Let $x, y \in 2^\omega$.

(1) $\mathbb{0}$ denotes the set of all sequences $z \in 2^\omega$ such that $z(n) = 0$ for almost all n.

(2) $x \sqsubset^{\mathrm{s}} y$ if $x(n) = y(n) = 1$ and $x(m) = 1$ and $y(m) = 0$ for infinitely many $n, m < \omega$, i.e., $x^{-1}(\{1\})$ and $y^{-1}(\{1\})$ are infinite and $x^{-1}(\{1\})$ is split by $y^{-1}(\{1\})$. $y \sqsubset^{\mathrm{r}} x$ if $\neg(x \sqsubset^{\mathrm{s}} y)$. Note that $y \sqsubset^{\mathrm{r}} x$ holds whenever $y \in \mathbb{0}$.

(3) $y \vartriangleleft_* x$ if $x \in 2^\omega \setminus \mathbb{0}$ and $y \sqsubset^{\mathrm{r}} x$, i.e., I wins with the play x against the play y of II in the splitting* game. $y \vartriangleleft_{**} x$ if either $x \in 2^\omega \setminus \mathbb{0}$ and $y \sqsubset^{\mathrm{r}} x$ or $y \in \mathbb{0}$, i.e., I wins with the play x against the play y of II in the splitting** game.

(4) Str denotes the set of all I's strategies, namely, $\mathrm{Str} \coloneqq 2^{(2^{<\omega})}$. For $\sigma \in \mathrm{Str}$, $\sigma * y$ denotes the play of I according to the strategy

σ and the play y of II, namely, $\sigma * y(n) := \sigma(y \restriction n)$ for $n < \omega$. $y \lhd_* \sigma$ denotes $y \lhd_* \sigma * y$ and analogously for $\lhd_{**}$.

(5) Define relational systems $\mathbf{R}^*$ and $\mathbf{R}^{**}$ by $\mathbf{R}^* := \langle 2^\omega, \mathrm{Str}, \lhd_* \rangle$ and $\mathbf{R}^{**} := \langle 2^\omega, \mathrm{Str}, \lhd_{**} \rangle$.

Thus, $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*} = \mathfrak{b}(\mathbf{R}^*)$ and $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^{**}} = \mathfrak{b}(\mathbf{R}^{**})$ hold.

The two relational systems $\mathbf{R}^*$ and $\mathbf{R}^{**}$ behave the same way in most cases: for any $y \in 2^\omega$ and $\sigma \in \mathrm{Str}$, *whenever* $y \notin \mathbb{0}$, $y \lhd_* \sigma$ if and only if $y \lhd_{**} \sigma$. Nevertheless, their bounding numbers $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^{**}}$ might be consistently different.

We investigate whether a similar phenomenon can occur for other relational systems. More precisely, we study how the bounding number of $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ is affected by changing the relation $x \sqsubset y$ only when x belongs to some fixed countable set $C \subseteq X$.

As a test case, we focus on the evasion number $\mathfrak{e}$ since the two relational systems $\mathbf{R}^*$ and $\mathbf{R}^{**}$ are similar to the combinatorial concept of evasion and prediction, in that both a strategy and a predictor decide their value by seeing the previous values of a function. In what follows, we naturally extend the notation $\mathbb{0}$ to ω^ω, so that $\mathbb{0}$ denotes the set of all eventually zero functions in ω^ω.

For several relational systems $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ associated with evasion and prediction, we define the $\mathbb{0}$-version $\mathbf{R}_\mathbb{0}$ by switching the prediction relation $x \sqsubset y$ only when $x \in \mathbb{0}$. We compare the bounding numbers $\mathfrak{b}(\mathbf{R})$ and $\mathfrak{b}(\mathbf{R})_\mathbb{0} := \mathfrak{b}(\mathbf{R}_\mathbb{0})$.

On the one hand, we show that this "$\mathbb{0}$-operation" changes the standard evasion number $\mathfrak{e}$:

Theorem A (Theorem 3.15)**.** *$\mathfrak{e}_\mathbb{0} < \mathfrak{e}$ is consistent.*

On the other hand, we find a variation $\mathfrak{e}^{\mathrm{G}}_k$ of the evasion number $\mathfrak{e}$, which is not changed by the $\mathbb{0}$-operation:

Theorem B (Theorem 2.9)**.** *For any $k \geq 1$, $(\mathfrak{e}^{\mathrm{G}}_k)_\mathbb{0} = \mathfrak{e}^{\mathrm{G}}_k$.*

2. $\mathbb{0}$-PREDICTION

First we clarify the notation of the standard evasion and prediction.

Definition 2.1.
- A pair $\pi = (D, \{\pi_n : n \in D\})$ is a *predictor* if $D \in [\omega]^\omega$ and $\pi_n \colon \omega^n \to \omega$ for each $n \in D$.
- For a predictor $\pi = (D, \{\pi_n : n \in D\})$ and a function $f \in \omega^\omega$, π *predicts* f if $f(n) = \pi_n(f \restriction n)$ for almost all $n \in D$.
- Pred denotes the set of all predictors.
- $\mathbf{PR} := \langle \omega^\omega, \mathrm{Pred}, \sqsubset^{\mathrm{p}} \rangle$, where $f \sqsubset^{\mathrm{p}} \pi \colon \Leftrightarrow f$ is predicted by π.

We introduce $\mathbb{0}$-*prediction.*

Definition 2.2. Let π be a predictor and $f \in \omega^\omega$.

(1) π $\mathbb{0}$-*predicts* f, denoted by $f \sqsubset^{\mathrm{p}}_\mathbb{0} \pi$, if either:

- $f \notin \mathbb{0}$ and $f \sqsubset^{\mathrm{p}} \pi$, or
- $f \in \mathbb{0}$ and $\neg(f \sqsubset^{\mathrm{p}} \pi)$.

(2) Put $\mathbf{PR}_{\mathbb{0}} \coloneqq \langle \omega^\omega, \mathrm{Pred}, \sqsubset^{\mathrm{p}}_{\mathbb{0}} \rangle$ and $\mathfrak{e}_{\mathbb{0}} \coloneqq \mathfrak{b}(\mathbf{PR}_{\mathbb{0}})$.

Lemma 2.3. $\mathfrak{e}_{\mathbb{0}} \leq \mathfrak{e}$.

Proof. By identifying ω and $\omega \setminus 1$, we can take an evading family $F \subseteq (\omega \setminus 1)^\omega$ of size $\mathfrak{e}$. Then, F is also a $\mathbb{0}$-evading family. □

For interval partitions $I = \langle I_n : n < \omega \rangle$ and $J = \langle J_m : m < \omega \rangle$ of ω, we say I *is dominated by* J, denoted by $I \leq^* J$, if for almost all m there is n such that $I_n \subseteq J_m$. It is known that $\mathfrak{b}(\langle \mathsf{IP}, \mathsf{IP}, \leq^* \rangle) = \mathfrak{b}$ where IP denotes the set of all interval partitions of ω (see [Bla10]).

Lemma 2.4. $\mathfrak{e}_{\mathbb{0}} \geq \min\{\mathfrak{e}, \mathfrak{b}\}$.

Proof. Let $F \subseteq \omega^\omega$ with size $< \min\{\mathfrak{e}, \mathfrak{b}\}$ and $F^- \coloneqq F \setminus \mathbb{0}$. For $f \in F^-$ and $k < \omega$, let i^f_k denote the k-th element of $\{i < \omega : f(i) \neq 0\} \in [\omega]^\omega$ and put $I^f_k \coloneqq [i_{k-1}, i_k)$ ($i_{-1} \coloneqq 0$). Define an interval partition I^f by $I^f \coloneqq \langle I^f_k : k < \omega \rangle$. Since $|F^-| < \mathfrak{b}$, there is an interval partition $J = \langle J_m : m < \omega \rangle$ dominating all $\{I^f : f \in F^-\}$. Since $|F^-| < \mathfrak{e}$, there is a predictor $\pi = (D, \langle \pi_k : k \in D \rangle)$ predicting all $f \in F^-$. Inductively construct $A \coloneqq \{m_0 < m_1 < \cdots\} \in [\omega]^\omega$ and $D' \coloneqq \{d_0 < d_1 < \cdots\} \in [D]^\omega$ such that $\max J_{m_i} < d_i < \min J_{m_{i+1}}$ holds for all $i < \omega$. Define a predictor $\pi' \coloneqq (D', \langle \pi'_k : k \in D' \rangle)$ as follows: for $k = d_i \in D'$ and $\sigma \in \omega^k$,

$$\pi'_k(\sigma) \coloneqq \begin{cases} \pi_k(\sigma) & \text{if } \sigma(n) \neq 0 \text{ for some } n \in J_{m_i}, \\ 1 & \text{otherwise.} \end{cases} \tag{2.1}$$

Clearly π' does not predict any $f \in \mathbb{0}$, so it suffices to show that π' predicts all $f \in F^-$. Since J dominates I^f, for all but finitely many $i < \omega$ there is $k_i \geq 1$ such that $I^f_{k_i} \subseteq J_{m_i}$, so particularly $n \coloneqq \min I^f_{k_i} \in J_{m_i}$ satisfies $f(n) \neq 0$. Thus, by (2.1) $\pi'_{d_i}(f{\restriction}d_i) = \pi_{d_i}(f{\restriction}d_i)$ for all but finitely many i. Therefore, since π predicts f, π' also predicts f. □

Remark 2.5. A similar inequality holds for splitting game numbers: $\min\{\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^{**}}, \mathfrak{b}\} \leq \mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*} \leq \mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^{**}}$ was proved in [CGHY26].

We will see that $\mathfrak{e}_{\mathbb{0}}$ and $\mathfrak{e}$ are consistently different in the next section. In contrast, there is a case where this "$\mathbb{0}$-operation" does not change the original cardinal invariant.

Blass studied many variations of the evasion number in [Bla10, Chapter 10]. Among such variations, we treat *global (adaptive) prediction of width* k for a natural number $k \geq 1$. This concept differs from standard prediction in the following two respects: First, for a predictor $\pi = (D, \{\pi_n : n \in D\})$, we always require $D = \omega$. Second, at each stage n, the predictor guesses k possible values for $f(n)$, rather than necessarily guessing the exact value $f(n)$. This prediction is formalized (more concisely) as follows:

Definition 2.6. Let $k \geq 1$.

- A function π is a *global predictor of width k* if $\mathrm{dom}(\pi) = \omega^{<\omega}$ and $\mathrm{ran}(\pi) \subseteq [\omega]^k$.
- Pred_k denotes the set of all global predictors of width k.
- For $\pi \in \mathrm{Pred}_k$ and a function $f \in \omega^\omega$, we say π *globally predicts* f, denoted by $f \in^{\mathrm{p}} \pi$, if $f(n) \in \pi(f{\restriction}n)$ for almost all $n \in \omega$.
- Put $\mathbf{PR}^{\mathrm{G}}_k := \langle \omega^\omega, \mathrm{Pred}_k, \in^{\mathrm{p}} \rangle$ and $\mathfrak{e}^{\mathrm{G}}_k := \mathfrak{b}(\mathbf{PR}^{\mathrm{G}}_k)$.

Recall that for a natural number $k \geq 2$, a poset $\mathbb{P}$ is *σ-k-linked* if there is a countable decomposition $\mathbb{P} = \bigcup_{n\in\omega} Q_n$ such that for any n, any k conditions in Q_n have a common extension in $\mathbb{P}$. $\mathfrak{m}(\sigma\text{-}k\text{-linked})$ denotes the least cardinality κ such that Martin's axiom for σ-k-linked posets fails at κ.

Blass showed the following:

Lemma 2.7 ([Bla10])**.** *$\mathfrak{e}^{\mathrm{G}}_1 = \aleph_1$ and $\mathfrak{m}(\sigma\text{-}k\text{-linked}) \leq \mathfrak{e}^{\mathrm{G}}_k \leq \mathrm{add}(\mathcal{N})$ for any $k \geq 2$.*

We introduce $\mathbb{0}$-global prediction of width k:

Definition 2.8. Let $k \geq 1$.

- For a function $f \in \omega^\omega$ and a global predictor π, we say π *$\mathbb{0}$-globally predicts* f, denoted by $f \in^{\mathrm{p}}_{\mathbb{0}} \pi$, if either:
  - $f \notin \mathbb{0}$ and $f \in^{\mathrm{p}} \pi$, or
  - $f \in \mathbb{0}$ and $\neg(f \in^{\mathrm{p}} \pi)$.
- Put $(\mathbf{PR}^{\mathrm{G}}_k)_{\mathbb{0}} := \langle \omega^\omega, \mathrm{Pred}_k, \in^{\mathrm{p}}_{\mathbb{0}} \rangle$ and $(\mathfrak{e}^{\mathrm{G}}_k)_{\mathbb{0}} := \mathfrak{b}((\mathbf{PR}^{\mathrm{G}}_k)_{\mathbb{0}})$.

The $\mathbb{0}$-operation does not change the value of $\mathfrak{e}^{\mathrm{G}}_k$:

Theorem 2.9. *For any $k \geq 1$, $(\mathfrak{e}^{\mathrm{G}}_k)_{\mathbb{0}} = \mathfrak{e}^{\mathrm{G}}_k$.*

Proof. To see $(\mathfrak{e}^{\mathrm{G}}_k)_{\mathbb{0}} \leq \mathfrak{e}^{\mathrm{G}}_k$, take a $\mathbf{PR}^{\mathrm{G}}_k$-unbounded family F of size $\mathfrak{e}^{\mathrm{G}}_k$ in $(\omega \setminus 1)^\omega$. F is also $(\mathbf{PR}^{\mathrm{G}}_k)_{\mathbb{0}}$-unbounded, which implies $(\mathfrak{e}^{\mathrm{G}}_k)_{\mathbb{0}} \leq \mathfrak{e}^{\mathrm{G}}_k$.

To see $\mathfrak{e}^{\mathrm{G}}_k \leq (\mathfrak{e}^{\mathrm{G}}_k)_{\mathbb{0}}$, let $F \subseteq \omega^\omega$ of size $< \mathfrak{e}^{\mathrm{G}}_k$ and $F^- := F \setminus \mathbb{0}$. For $f \in F^-$ and $l < \omega$, let i^f_l denote the l-th element of $\{i < \omega : f(i) \neq 0\} \in [\omega]^\omega$, and put $I^f_l := [i_{l-1}, i_l)$ $(i_{-1} := 0)$. Define an interval partition I^f by $I^f := \langle I^f_l : l < \omega \rangle$. Since $|F^-| < \mathfrak{e}^{\mathrm{G}}_k \leq \mathrm{add}(\mathcal{N}) \leq \mathfrak{b}$, there is an interval partition $J = \langle J_m : m < \omega \rangle$ dominating all $\{I^f : f \in F^-\}$. Since $|F^-| < \mathfrak{e}^{\mathrm{G}}_k$, there is $\pi \in \mathrm{Pred}_k$ that globally predicts all $f \in F^-$. Define $\pi' \in \mathrm{Pred}_k$ as follows: for $\sigma \in \omega^n$ and $m \in \omega$ with $n \in J_m$:

$$\pi'(\sigma) := \begin{cases} \pi(\sigma) & \text{if } \sigma(i) \neq 0 \text{ for some } i \in J_{m-1}, \\ \{1, \dots, k\} & \text{otherwise.} \end{cases}$$

Then π' globally predicts every $f \in F^-$ and does not globally predict any $f \in \mathbb{0}$. Thus, F is $(\mathbf{PR}^{\mathrm{G}}_k)_{\mathbb{0}}$-dominated by π' and hence $|F| < (\mathfrak{e}^{\mathrm{G}}_k)_{\mathbb{0}}$. Therefore, we have $\mathfrak{e}^{\mathrm{G}}_k \leq (\mathfrak{e}^{\mathrm{G}}_k)_{\mathbb{0}}$. $\square$

3. Consistency of $\mathfrak{e}_{\mathbb{0}} < \mathfrak{e}$

Towards the consistency of $\mathfrak{e}_{\mathbb{0}} < \mathfrak{e}$, we introduce a poset $\mathbb{PR}$ which adds a generic predictor π_G and hence increases $\mathfrak{e}$ by iteration.

Definition 3.1. *Prediction forcing* $\mathbb{PR}$ consists of tuples (d, π, F) satisfying:

(1) $d \in 2^{<\omega}$.
(2) $\pi = \langle \pi_n : n \in d^{-1}(\{1\}) \rangle$.
(3) for each $n \in d^{-1}(\{1\})$, π_n is a finite partial function of $\omega^n \to \omega$.
(4) $F \in [\omega^\omega]^{<\omega}$.
(5) for each $f, f' \in F$, $f \restriction |d| = f' \restriction |d|$ implies $f = f'$.

$(d', \pi', F') \leq (d, \pi, F)$ if:

(i) $d' \supseteq d$.
(ii) $\forall n \in d^{-1}(\{1\}), \pi'_n \supseteq \pi_n$.
(iii) $F' \supseteq F$.
(iv) For all $n \in (d')^{-1}(\{1\}) \setminus d^{-1}(\{1\})$ and $f \in F$, we have $f \restriction n \in \mathrm{dom}(\pi'_n)$ and $\pi'_n(f \restriction n) = f(n)$.

For a generic filter G, define the generic predictor $\pi_G = (D_G, \langle \pi^G_n : n \in D_G \rangle)$ by $D_G = \bigcup \{d^{-1}(\{1\}) : (d, \pi, F) \in G\}$ and for $n \in D_G$, $\pi^G_n \coloneqq \bigcup \{\pi_n : (d, \pi, F) \in G,\ n \in d^{-1}(\{1\})\}$.

It is easy to see that $\mathbb{PR}$ is σ-centered, and hence ccc.

We can obtain a forcing which increases $\mathfrak{e}_{\mathbb{0}}$ by just restricting $\mathbb{PR}$:

Definition 3.2. $\mathbb{PR}_{\mathbb{0}} \coloneqq \{(d, \pi, F) \in \mathbb{PR} : F \subseteq \omega^\omega \setminus \mathbb{0}\}$ and the order is defined by restriction of $\mathbb{PR}$. π_G is defined in the same way.

$\mathbb{PR}_{\mathbb{0}}$ is σ-centered as well and hence it is ccc.

Lemma 3.3. *For $f \in \omega^\omega$, $\Vdash_{\mathbb{PR}_{\mathbb{0}}} f \sqsubset^{\mathrm{p}}_{\mathbb{0}} \dot{\pi}_G$.*

Proof. If $f \notin \mathbb{0}$, we obtain $\Vdash_{\mathbb{PR}_{\mathbb{0}}} f \sqsubset^{\mathrm{p}} \dot{\pi}_G$ by the same proof as the standard $\mathbb{PR}$. Assume $f \in \mathbb{0}$ and $(d, \pi, F) \in \mathbb{PR}_{\mathbb{0}}$ and $m < \omega$ are given. We shall find $n \geq m$ and $(d', \pi', F) \leq (d, \pi, F)$ forcing that $n \in D_G$ and $\pi_G(f \restriction n) \neq f(n)$. Since $f \notin F$, we may assume that $f \restriction m \neq g \restriction m$ for all $g \in F$ and $f(n) = 0$ for all $n \geq m$ by increasing m, and that $|d| \geq m$ by adding enough 0's after the binary sequence d. Let $n \coloneqq |d|$ and $d' \coloneqq d^\frown 1 \in 2^{n+1}$. Define the finite partial function $\pi'_n \colon \omega^n \to \omega$ by $\mathrm{dom}(\pi'_n) \coloneqq \{g \restriction n : g \in F \cup \{f\}\}$ and for $g \restriction n \in \mathrm{dom}(\pi'_n)$,

$$\pi'_n(g \restriction n) \coloneqq \begin{cases} g(n) & \text{if } g \in F, \\ 1 & \text{if } g = f. \end{cases} \tag{3.1}$$

This is well-defined since for any $g, g' \in F \cup \{f\}$, $g \restriction n = g' \restriction n$ implies $g = g'$. Let $\pi' \coloneqq \pi^\frown \pi'_n$ be the sequence of finite partial functions indexed by $k \in (d')^{-1}(\{1\})$. By (3.1), (d', π', F) extends (d, π, F) and forces that $n \in (d')^{-1}(\{1\}) \subseteq D_G$ and $\pi^G_n(f \restriction n) = \pi'_n(f \restriction n) = 1 \neq 0 = f(n)$. ∎

When showing the consistency of $\mathfrak{e}_{\mathbb{0}} < \mathfrak{e}$, we will use the *Fr-linkedness* property to keep $\mathfrak{e}_{\mathbb{0}}$ small in the forcing iteration. Fr-linkedness was formalized by Mejía in [Mej19] and the formalization was described in more detail in [CGHY26]. Let us review the framework of Fr-linkedness in the following.

Definition 3.4. Let $\mathbb{P}$ be a poset.

(1) For a countable sequence $\bar{p} = \langle p_m : m < \omega \rangle \in \mathbb{P}^\omega$, we define $\dot{W}(\bar{p})$ as the $\mathbb{P}$-name of an index set of the sequence $\bar{p}$ as follows:
$$\Vdash_{\mathbb{P}} \dot{W}(\bar{p}) := \{m < \omega : p_m \in \dot{G}\}$$
where $\dot{G}$ denotes the canonical $\mathbb{P}$-name of a generic filter.

(2) $Q \subseteq \mathbb{P}$ is *Fr-linked* if there exists a function $\lim\colon Q^\omega \to \mathbb{P}$ such that for any countable sequence $\bar{q} \in Q^\omega$,
$$\lim \bar{q} \Vdash |\dot{W}(\bar{q})| = \omega.$$

(3) For an infinite cardinal μ, $\mathbb{P}$ is *μ-Fr-linked* if it is a union of μ-many Fr-linked components. When $\mu = \aleph_0$, we use σ instead as usual. Define $< \mu$-Fr-linkedness in the same way (for uncountable μ).

Since any singleton $\{q\} \subseteq \mathbb{P}$ is Fr-linked, we have:

Lemma 3.5. *Every poset $\mathbb{P}$ is $|\mathbb{P}|$-Fr-linked. In particular, Cohen forcing $\mathbb{C}$ is σ-Fr-linked.*

Brendle and Shelah (essentially) proved the following:

Lemma 3.6 ([BS96])**.** *The prediction forcing $\mathbb{PR}$ is σ-Fr-linked.*

We formulate a finite support iteration (fsi) of $< \kappa$-Fr-linked forcings:

Definition 3.7. Let κ be an uncountable regular cardinal.

- A finite support iteration $\mathbb{P}_\gamma = \langle (\mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi) : \xi < \gamma \rangle$ of ccc forcings is a *$<\kappa$-Fr-iteration with witnesses $\langle \theta_\xi : \xi < \gamma \rangle$ and $\langle \dot{Q}_{\xi,\zeta} : \zeta < \theta_\xi, \xi < \gamma \rangle$* if for any $\xi < \gamma$, θ_ξ is a cardinal $< \kappa$ and $\langle \dot{Q}_{\xi,\zeta} : \zeta < \theta_\xi \rangle$ are $\mathbb{P}_\xi$-names satisfying:
$$\Vdash_{\mathbb{P}_\xi} \dot{Q}_{\xi,\zeta} \subseteq \dot{\mathbb{Q}}_\xi \text{ is Fr-linked for } \zeta < \theta_\xi \text{ and } \bigcup_{\zeta < \theta_\xi} \dot{Q}_{\xi,\zeta} = \dot{\mathbb{Q}}_\xi.$$
- A condition $p \in \mathbb{P}_\gamma$ is *determined* if for each $\xi \in \mathrm{dom}(p)$, there is $\zeta_\xi < \theta_\xi$ such that $\Vdash_{\mathbb{P}_\xi} p(\xi) \in \dot{Q}_{\xi,\zeta_\xi}$. Note that there are densely many determined conditions (proved by induction on γ).

Fr-limits for conditions of the iteration $\mathbb{P}_\gamma$ are defined for "refined" sequences:

Definition 3.8. Let $\mathbb{P}_\gamma$ be a $<\kappa$-Fr-iteration and δ be an ordinal. We say that $\bar{p} = \langle p_m : m < \delta \rangle \in (\mathbb{P}_\gamma)^\delta$ is *a uniform Δ-system* if:

(1) Each p_m is determined, witnessed by $\langle \zeta^m_\xi : \xi \in \mathrm{dom}(p_m) \rangle$,

(2) the family $\{\mathrm{dom}(p_m) : m < \delta\}$ is a Δ-system with root ∇,
(3) there is a sequence $\langle \zeta^*_\xi : \xi \in \nabla\rangle$ such that for $\xi \in \nabla$, $\zeta^*_\xi = \zeta^m_\xi$ for all $m < \delta$, i.e., all $p_m(\xi)$ are forced to be in a common Fr-linked component for $\xi \in \nabla$,
(4) all $\mathrm{dom}(p_m)$ have n' elements, and $\mathrm{dom}(p_m) = \{\alpha_{n,m} : n < n'\}$ is the increasing enumeration,
(5) there is $r' \subseteq n'$ such that $n \in r' \Leftrightarrow \alpha_{n,m} \in \nabla$ for $n < n'$,
(6) for $n \in n' \setminus r'$, $\langle \alpha_{n,m} : m < \delta\rangle$ is (strictly) increasing.

Definition 3.9. Let $\mathbb{P}_\gamma$ be a $<\kappa$-Fr-iteration and $\bar{p} = \langle p_m : m < \omega\rangle \in (\mathbb{P}_\gamma)^\omega$ be a uniform Δ-system with root ∇. We (inductively) define $p^\infty = \lim \bar{p}$ as follows:

(1) $\mathrm{dom}(p^\infty) \coloneqq \nabla$,
(2) $p^\infty{\restriction}\xi \Vdash_{\mathbb{P}_\xi} p^\infty(\xi) \coloneqq \lim\langle p_m(\xi) : m \in \dot{W}(\bar{p}{\restriction}\xi)\rangle$ for $\xi \in \nabla$, where $\bar{p}{\restriction}\xi \coloneqq \langle p_m{\restriction}\xi : m < \omega\rangle \in (\mathbb{P}_\xi)^\omega$.

To see that the second item is valid, $\dot{W}(\bar{p}{\restriction}\xi)$ has to be infinite, which is true:

Lemma 3.10 ([Mej19], [CGHY26, Lemma 3.6])**.** *Let $\mathbb{P}_\gamma$ be a $<\kappa$-Fr-iteration. Let $\bar{p} = \langle p_m : m < \omega\rangle \in (\mathbb{P}_\gamma)^\omega$ be a uniform Δ-system and $p^\infty \coloneqq \lim \bar{p}$. Then $p^\infty \Vdash_{\mathbb{P}_\gamma} |\dot{W}(\bar{p})| = \omega$.*

To show that Fr-linkedness helps to keep $\mathfrak{e}_{\mathbb{0}}$ small, we characterize $\mathfrak{e}_{\mathbb{0}}$ with the following relational system:

Definition 3.11. $\mathrm{Pred}_{\mathbb{0}} \coloneqq \{\pi \in \mathrm{Pred} : \pi \text{ does not predict any } f \in \mathbb{0}\}$. $\mathbf{PR}'_{\mathbb{0}} \coloneqq \langle \omega^\omega, \mathrm{Pred}_{\mathbb{0}}, \sqsubset^{\mathrm{p}}_{\mathbb{0}}\rangle$.

Lemma 3.12. $\mathfrak{b}(\mathbf{PR}'_{\mathbb{0}}) = \mathfrak{e}_{\mathbb{0}}$.

Proof. $\mathfrak{b}(\mathbf{PR}'_{\mathbb{0}}) \le \mathfrak{e}_{\mathbb{0}}$ is clear. To see $\mathfrak{e}_{\mathbb{0}} \le \mathfrak{b}(\mathbf{PR}'_{\mathbb{0}})$, let $F \subseteq \omega^\omega$ of size $< \mathfrak{e}_{\mathbb{0}}$. By Lemma 2.4, we particularly know that $\mathfrak{e}_{\mathbb{0}}$ is uncountable. Thus $F' \coloneqq F \cup \mathbb{0}$ has size $< \mathfrak{e}_{\mathbb{0}}$, so some $\pi \in \mathrm{Pred}$ $\mathbb{0}$-predicts all $f \in F'$. This π has to be in $\mathrm{Pred}_{\mathbb{0}}$, so π witnesses that F is $\mathbf{PR}'_{\mathbb{0}}$-bounded. □

We use the following notation.

Definition 3.13. For a predictor $\pi = (D, \{\pi_n : n \in D\})$, $f \in \omega^\omega$ and $n < \omega$, $f \sqsubset^{\mathrm{p}}_n \pi$ denotes that $f(k) = \pi_k(f{\restriction}k)$ for all $k \ge n$ in D.

The Fr-linkedness property and Cohen reals help to keep $\mathfrak{e}_{\mathbb{0}}$ small through the forcing iteration:

Lemma 3.14. *Let κ be an uncountable regular cardinal, $\gamma > \kappa$ be a limit ordinal and $\mathbb{P}_\gamma$ be a $< \kappa$-Fr-iteration whose first κ-many iterands are Cohen forcings $\mathbb{C}$. Then, $\Vdash_{\mathbb{P}_\gamma} \mathfrak{e}_{\mathbb{0}} \le \kappa$.*

Proof. We show that the first κ-many Cohen reals $\langle \dot{c}_\alpha : \alpha < \kappa\rangle$ (as members of ω^ω) witness $\Vdash_{\mathbb{P}_\gamma} \mathfrak{e}_{\mathbb{0}} \le \kappa$, by focusing on the characterization $\mathfrak{e}_{\mathbb{0}} = \mathfrak{b}(\mathbf{PR}'_{\mathbb{0}})$ in Lemma 3.12.

Assume towards contradiction that there exist a condition $p \in \mathbb{P}_\gamma$ and a $\mathbb{P}_\gamma$-name $\dot\pi = (\dot D, \langle \dot\pi_n : n \in \dot D\rangle)$ of a member of $\mathrm{Pred}_{\mathbb{0}}$ such that for all $\alpha < \kappa$, $p \Vdash \dot c_\alpha \sqsubset^{\mathrm{p}}_{\mathbb{0}} \dot\pi$. Since Cohen reals are not in $\mathbb{0}$, we actually have $p \Vdash \dot c_\alpha \sqsubset^{\mathrm{p}} \dot\pi$. Let us witness the starting point of the prediction $\sqsubset^{\mathrm{p}} = \bigcup_{n<\omega} \sqsubset^{\mathrm{p}}_n$. That is, for each $\alpha < \kappa$ we pick $p_\alpha \leq p$ and $n_\alpha < \omega$ such that $p_\alpha \Vdash \dot c_\alpha \sqsubset^{\mathrm{p}}_{n_\alpha} \dot\pi$. By extending and thinning, we may assume there is $I \in [\kappa]^\kappa$ such that:

(1) $\alpha \in \mathrm{dom}(p_\alpha)$ for $\alpha \in I$.
(2) $\{p_\alpha : \alpha \in I\}$ forms a uniform Δ-system with root ∇.
(3) All n_α are equal to n^*.
(4) All $p_\alpha(\alpha)$ are the same Cohen condition $s \in \omega^{<\omega}$.
(5) $|s| = n^*$. (By extending s or increasing n^*.)

In particular, we have that:

$$\text{For each } \alpha \in I,\ p_\alpha \text{ forces } \dot c_\alpha {\restriction} n^* = s \text{ and } \dot c_\alpha \sqsubset^{\mathrm{p}}_{n^*} \dot\pi. \tag{3.2}$$

Pick some countable $\{\alpha_0 < \alpha_1 < \cdots\} \in [I \setminus \nabla]^\omega$. For $m < \omega$, define $q_m \leq p_{\alpha_m}$ by extending the α_m-th position to $q_m(\alpha_m) \coloneqq s^\frown \mathbb{0}_m \in \omega^{n^*+m}$ where $\mathbb{0}_m$ denotes the sequence of length m whose values are all 0.

By (3.2), we obtain:

$$q_m \Vdash \text{for all } n \in [n^*, n^*+m) \cap \dot D,\ \dot\pi_n(s^\frown \mathbb{0}_{n-n^*}) = 0. \tag{3.3}$$

Since $\alpha_m \notin \nabla$, $\bar q \coloneqq \langle q_m \rangle_{m<\omega}$ also forms a Δ-system (with root ∇) and hence we can take the limit $q^\infty \coloneqq \lim \bar q$. Let $f \coloneqq s^\frown 00\cdots \in \mathbb{0}$. Since $q^\infty \Vdash \exists^\infty m < \omega\ q_m \in \dot G$ and by (3.3),

$$q^\infty \Vdash \text{for all } n \geq n^* \text{ in } \dot D,\ \dot\pi_n(f{\restriction}n) = 0 = f(n),$$

which contradicts $\dot\pi \in \mathrm{Pred}_{\mathbb{0}}$. □

Now we are ready to prove the consistency of $\mathfrak{e}_{\mathbb{0}} < \mathfrak{e}$.

Theorem 3.15. *Let $\kappa \leq \lambda$ be uncountable regular cardinals. Then, it is consistent that $\kappa = \mathfrak{e}_{\mathbb{0}} \leq \mathfrak{e} = \lambda$ holds.*

Proof. Let $\mu > \lambda$ be such that $\mu^{<\kappa} = \mu$ and $\mathrm{cf}(\mu) = \lambda$ (e.g., construct the continuous sequence $\langle \mu_\alpha : \alpha \leq \lambda\rangle$ by $\mu_0 \coloneqq \lambda$ and $\mu_{\alpha+1} \coloneqq (\mu_\alpha^{<\kappa})^+$ and put $\mu \coloneqq \mu_\lambda$). Let $\mathbb{P} = \mathbb{P}_\gamma$ be a $<\kappa$-Fr-iteration of length $\gamma = \mu$ whose first κ-many iterands are Cohen forcings and each of the rest is either $\mathbb{PR}$ or a subforcing of $\mathbb{PR}_{\mathbb{0}}$ of size $<\kappa$ (by bookkeeping), which is possible by Lemma 3.6 (and Lemma 3.5). It is easy to see that $\mathbb{P}$ forces $\mathfrak{e} \geq \lambda$ by $\mathrm{cf}(\mu) = \lambda$ and $\mathfrak{e}_{\mathbb{0}} \geq \kappa$ by bookkeeping (and $\mu^{<\kappa} = \mu$). Since we are performing a finite support iteration, λ many Cohen reals are cofinally added and they witness $\mathrm{non}(\mathcal{M}) \leq \lambda$. Since $\mathfrak{e} \leq \mathrm{non}(\mathcal{M})$ holds in ZFC, we have $\Vdash_{\mathbb{P}} \lambda \leq \mathfrak{e} \leq \mathrm{non}(\mathcal{M}) \leq \lambda$ and hence $\Vdash_{\mathbb{P}} \mathfrak{e} = \lambda$. Lemma 3.14 implies $\Vdash_{\mathbb{P}} \mathfrak{e}_{\mathbb{0}} \leq \kappa$. Therefore, $\Vdash_{\mathbb{P}} \mathfrak{e}_{\mathbb{0}} = \kappa$ and $\mathfrak{e} = \lambda$. □

4. Questions

The first question is about the exact value of $\mathfrak{e}_{\mathbb{0}}$. By Lemma 2.3, 2.4, we know $\min\{\mathfrak{e}, \mathfrak{b}\} \leq \mathfrak{e}_{\mathbb{0}} \leq \mathfrak{e}$. Then we ask:

Question 1. *Does $\mathfrak{e}_{\mathbb{0}} = \min\{\mathfrak{e}, \mathfrak{b}\}$ hold in ZFC? Equivalently, is $\mathfrak{e}_{\mathbb{0}} \leq \mathfrak{b}$ true?*

The second question is more exploratory. The invariants $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*}$ and $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^{**}}$ are quite naturally defined: The splitting* game and the splitting** game are reasonable game-theoretic formulations of the splitting property. Moreover, the need to distinguish the two games is inevitable, since if both players play finite sets, there is no fair criterion for deciding the winner.

In contrast, our definition of $\mathbb{0}$-prediction is somewhat artificial. This leads to the following question:

Question 2. *Is there another separation result based on the same mechanism as $\mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^*} < \mathfrak{s}^{\mathrm{I}}_{\mathrm{game}^{**}}$, but arising from a more natural framework than $\mathbb{0}$-prediction?*

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