Zhu Algebras of Superconformal Vertex Algebras

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1 Introduction

This note is based on a joint work with Shintarou Yanagida (Nagoya University). See [12] for more details.

1.1 Zhu algebra of a VOA

The Zhu algebra is a fundamental associative algebra attached to a vertex operator algebra (VOA), introduced by Yongchang Zhu [14] as a tool for studying its representation theory.

Let V be a VOA with vacuum vector $\mathbf{1}$, conformal vector L, and state–field correspondence

$$Y(a,z) = \sum_{n\in\mathbb{Z}} z^{-n-1} a_{(n)}, \qquad a \in V.$$

For $a, b \in V$, with a homogeneous of conformal weight $\Delta(a)$, define

$$a \underset{n}{*} b := \operatorname*{res}_{z=0}\Big[z^{-n}(1+z)^{\Delta(a)} Y(a,z)b\Big]\,dz = \sum_{j\geq 0} \binom{\Delta(a)}{j} a_{(j-n)}b, \qquad n \in \mathbb{Z}_{\geq 1}.$$

Let $O(V)$ be the subspace of V spanned by all $a \underset{n}{*} b$ with $n \geq 2$. Then

$$A(V) := V/O(V),$$

equipped with the product induced by $\underset{1}{*}$, is a unital associative algebra [14, Theorem 2.1.1], [2, Theorem 2.7(c)]. The image of $\mathbf{1}$ is the identity element of $A(V)$, and the image of L is central in $A(V)$.

Zhu's fundamental theorem gives a bijection between isomorphism classes of simple $\mathbb{Z}_{\geq 0}$-graded V-modules and simple $A(V)$-modules [14, Theorem 2.2.2]. Thus $A(V)$ is a basic invariant of the representation theory of V. It also reflects important properties of V: $A(V)$ is semisimple if V is rational, finite-dimensional if V is C_2-cofinite, and its associated graded algebra is closely related to the C_2-Poisson algebra $R(V)$. These features make Zhu algebras a natural starting point for studying more general variants and applications. See [12, §0.1] and references therein for details.

1.2 Zhu algebra of a vertex algebra

While the Zhu algebra $A(V)$ is a basic invariant of a VOA, its definition depends essentially on the conformal grading. Even the twisted Zhu algebra $\mathrm{Zhu}_H(V)$ introduced by De Sole and Kac [2] still depends on the choice of a conformal Hamiltonian H. By contrast, Yi-Zhi Huang

[9] introduced an associative algebra $\widetilde{A}(V)$ whose definition extends naturally to an arbitrary vertex algebra V.

For $a, b \in V$ and $n \in \mathbb{Z}_{\geq 1}$, define

$$a \underset{n}{\bullet} b \coloneqq \operatorname*{res}_{z=0}\big[e^z(e^z-1)^{-n}Y(a,z)b\big]\,dz.$$

Let $\widetilde{O}(V)$ be the subspace of V spanned by all $a \underset{n}{\bullet} b$ with $n \geq 2$. Then

$$\widetilde{A}(V) \coloneqq V/\widetilde{O}(V),$$

equipped with the product induced by $\underset{1}{\bullet}$, is a unital associative algebra. Moreover, if V is a VOA, then $\widetilde{A}(V) \cong A(V)$. Thus Huang's construction preserves the essential content of the Zhu algebra while providing a geometric framework that applies to arbitrary vertex algebras.

1.3 Main result 1: Zhu algebras of superconformal vertex algebras

A key advantage of Huang's formulation is that it is particularly convenient for explicit calculations. In particular, the defining relations of $\widetilde{A}(V)$ can be read off directly from the λ-brackets of the vertex algebra. We apply this idea to the standard $N = 1, 2, 3, 4$ and big $N = 4$ superconformal vertex algebras

$$V^{N=1}, \quad V^{N=2}, \quad V^{N=3}, \quad V^{N=4}, \quad V^{N=4,\mathrm{big}}.$$

These vertex algebras are closely related to affine W-algebras, so their Zhu algebras are expected to be closely connected with finite W-algebras [2, 5, 6].

Our first main result identifies the Zhu algebras $\widetilde{A}(V)$ of these examples in terms of Lie superalgebras. For $V^{N=1}, V^{N=2}$, and $V^{N=4}$, the Zhu algebra is isomorphic to the universal enveloping algebra $U(\mathfrak{g}^f)$, where $\mathfrak{g}^f$ is the centralizer of a minimal nilpotent element f in a suitable simple Lie superalgebra $\mathfrak{g}$. For $V^{N=3}$ and $V^{N=4,\mathrm{big}}$, the Zhu algebra is no longer itself an enveloping algebra, but it still contains a natural subalgebra isomorphic to $U(\mathfrak{g}^f)$, and its structure can be described explicitly as a module over this subalgebra; see § 3 for the precise statements.

These examples show that Huang's formulation gives a uniform and effective way to compute Zhu algebras of superconformal vertex algebras and produces concrete algebras closely related to finite W-algebras.

1.4 Main result 2: Zhu algebras of SUSY vertex algebras

Our second main result is a SUSY analogue of Huang's Zhu algebra. A SUSY vertex algebra may be viewed as a supersymmetric version of an ordinary vertex algebra, formulated in terms of superfields [8]. In this paper, we work with $N_K = N$ SUSY vertex algebras, a class that includes the superconformal vertex algebras considered above.

Motivated by Huang's geometric formulation, we define an associative superalgebra $\widetilde{A}(\mathbb{V})$ attached to an $N_K = N$ SUSY vertex algebra $\mathbb{V}$. Its underlying associative algebra is determined by the reduced part V_{red} of $\mathbb{V}$, namely the underlying ordinary vertex algebra: more precisely, $\widetilde{A}(\mathbb{V})$ is naturally isomorphic to $\widetilde{A}(V_{\mathrm{red}})$. Thus the SUSY Zhu algebra retains the basic algebraic properties of Huang's Zhu algebra in the ordinary setting.

The supersymmetric structure is encoded by additional odd operators. More precisely, the odd derivations $\not{D}^1, \ldots, \not{D}^N$ on $\mathbb{V}$ induce odd differentials on $\widetilde{A}(\mathbb{V})$. Moreover, in the Poisson limit, these coincide with the odd differentials on the C_2-Poisson algebra $R(\mathbb{V})$ introduced in [13, §4.1, Remark 4.1.3]. In this way, $\widetilde{A}(\mathbb{V})$ provides a natural extension of Huang's Zhu algebra to the SUSY setting.

1.5 Future directions

The results of this note suggest several natural directions for further study. First, one should pass from universal superconformal vertex algebras to their simple quotients, which are more fundamental for representation theory. It is therefore important to determine their Zhu algebras explicitly, and we expect that Huang's formulation will remain effective in this setting.

A second direction is to extend our method beyond the linear case. The approach developed here relies on the fact that the defining relations of $\widetilde{A}(V)$ can be read off from directly when the λ-brackets among generators are linear. For more general affine W-algebras, however, non-linearity is inevitable. Therefore it would be interesting to develop a corresponding theory for Huang's Zhu algebra in the non-linear setting, parallel to the theory of De Sole and Kac for the original Zhu algebra [2]; see also [6].

Finally, the SUSY Zhu algebras introduced here raise further questions in the study of SUSY vertex algebras, especially in representation theory and in SUSY analogues of Zhu bimodules and chiral/Hochschild-type theories. These problems should clarify how much genuinely supersymmetric information is captured by the SUSY Zhu algebra.

2 Preliminaries

2.1 Vertex algebras

In this note we use the word "vertex algebra" in the sense of [10, §1.3] (usually referred to as "vertex superalgebra"). Explicitly, a vertex algebra is a quadruple $(V, Y, \mathbf{1}, \partial)$ consisting of

- a linear superspace $V = V^{\bar{0}} \oplus V^{\bar{1}}$, called the *state space*,
- an even linear map $V \to (\operatorname{End} V)[\![z^{\pm 1}]\!]$, $a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} z^{-n-1} a_{(n)}$ with $a_{(n)} \in \operatorname{End} V$, where z is a formal variable,
- a non-zero even element $\mathbf{1} \in V^{\bar{0}}$, called the *vacuum*,
- an even operator $\partial \in (\operatorname{End} V)^{\bar{0}}$, called the (even) *translation operator*.

satisfying the axioms stated in [10, §1.3]. For simplicity, we sometimes denote the vertex algebra $(V, Y, \mathbf{1}, \partial)$ by V. We also use the standard abbreviation $a(z) := Y(a,z)$ for $a \in V$.

2.2 State-field correspondence vs λ-bracket

For later use in §3, we recall the λ-bracket on a vertex algebra $(V, Y, \mathbf{1}, \partial)$ and the related notions briefly. See [10, §2.3], [2, §1.5] and [8, §2] for details.

- For an even indeterminate λ, the λ-*bracket* of $a, b \in V$ is defined as

$$[a_\lambda b] := \sum_{n\geq 0} \frac{1}{n!} \lambda^n a_{(n)} b \in V[\lambda]. \tag{2.1}$$

- An even element $L \in V^{\bar{0}}$ is called *conformal* with *central charge* $c \in \mathbb{C}$ if

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3. \tag{2.2}$$

  We will use the standard notation

$$L(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n, \quad L_n = L_{(n+1)} \in \operatorname{End} V. \tag{2.3}$$

- Given a conformal element $L \in V^{\bar{0}}$, an element $a \in V$ is said to have *conformal weight* $\Delta \in \mathbb{C}$ if
$$[L_\lambda a] = (\partial + \Delta\lambda)a + O(\lambda^2). \tag{2.4}$$
An element $a \in V$ is called *primary* of conformal weight $\Delta \in \mathbb{C}$ if
$$[L_\lambda a] = (\partial + \Delta\lambda)a. \tag{2.5}$$

2.3 Vertex operator alebras

In this note we use the word "vertex operator algebra (VOA)" to mean a vertex algebra $V = V^{\bar{0}} \oplus V^{\bar{1}}$ equipped with a conformal element $L \in V^{\bar{0}}$ with central charge $c \in \mathbb{C}$ such that the following conditions hold.

1. $L_{-1} = L_{(0)}$ coincides with the translation operator ∂ on V.
2. $L_0 = L_{(1)}$ is diagonalizable on V, i.e.,
$$V = \bigoplus_{\Delta \in \mathbb{C}} V_\Delta, \quad V_\Delta := \{v \in V \mid L_0 v = \Delta v\}. \tag{2.6}$$
3. $V_\Delta = \{0\}$ holds for any $\Delta \in \mathbb{C}$ with its real part sufficiently negative.

2.4 Huang's formulation of Zhu algebra

We explain a slight variant of Huang's geometric formulation in [9, §6], allowing an arbitrary parameter γ.

Let V be a vertex algebra, and let γ be a complex number or a formal variable. For $a, b \in V$ and $n \in \mathbb{Z}_{\geq 1}$, consider the expression

$$a \underset{n}{\overset{\gamma}{\bullet}} b := \operatorname*{res}_z \big[f_n(z;\gamma)a(z)b\big]dz, \quad f_n(z;\gamma) := \begin{cases} \dfrac{\gamma^n e^{\gamma z}}{(e^{\gamma z}-1)^n} & (\gamma \neq 0 \text{ or } \gamma \text{ is a formal variable}) \\ z^{-n} & (\gamma = 0) \end{cases}. \tag{2.7}$$

Here the residue map is the even linear map

$$\operatorname*{res}_z[-]dz \colon V[\![z^{\pm 1}]\!] \longrightarrow V, \quad \operatorname*{res}_z \left[\sum_{j \in \mathbb{Z}} z^j v_j\right] dz := v_{-1}. \tag{2.8}$$

When γ is a formal variable, we regard $f_n(z;\gamma)$ as a formal series of z:

$$f_n(z;\gamma) = \sum_{j \geq 0} c(j,n)\gamma^j z^{-n+j} = z^{-n} - \frac{n-2}{2}\gamma z^{-n+1} + \cdots \in \mathbb{Q}[\gamma](\!(z)\!), \quad c(j,n) \in \mathbb{Q}. \tag{2.9}$$

Then we have

$$a \underset{n}{\overset{\gamma}{\bullet}} b = \sum_{j=0}^{n+N(a,b)} c(j,n)\gamma^j a_{(j-n)}b = a_{(-n)}b - \frac{n-2}{2}\gamma a_{(-n+1)}b + \cdots \in V[\gamma]. \tag{2.10}$$

In particular, for $n = 1$, we have

$$a \underset{1}{\overset{\gamma}{\bullet}} b = \sum_{j \geq 0} \gamma^j \frac{B_j^+}{j!} a_{(j-1)}b = a_{(-1)}b + \frac{\gamma}{2}a_{(0)}b + \frac{\gamma^2}{12}a_{(1)}b + \cdots, \tag{2.11}$$

where B_j^+ are the Bernoulli numbers.

Remark 2.1. Some remarks are in order.

1. If $\gamma = 0$, we have $a \overset{\gamma=0}{\underset{n}{\bullet}} b = a_{(-n)}b$.

2. The definition of the operation $\underset{n}{\bullet}$ in [9, §6] is given for the case $\gamma = 2\pi i$, and we modified it to accommodate any $\gamma \in \mathbb{C}$. Our $\overset{\gamma}{\underset{1}{\bullet}}$ coincides with the operation $\cdot_{(f)}\cdot$ with $f(z) \coloneqq f_1(z;c) = ce^{cz}/(e^{cz}-1)$, introduced in [3, (3.11)] (replacing c with γ).

3. The operation $\overset{\gamma}{\underset{n}{\bullet}}$ can be rewritten as
$$a \overset{\gamma}{\underset{n}{\bullet}} b \cdot \gamma^{-n+1} = \operatorname*{res}_z \left[\frac{\gamma e^{\gamma z}}{(e^{\gamma z}-1)^n} a(z)b \right] dz = \operatorname*{res}_x \left[x^{-n} a\big(\gamma^{-1}\log(1+x)\big)b \right] dx. \tag{2.12}$$
This follows from the change of variables $x = e^{\gamma z} - 1$. See [9, Remark 6.2] for details.

Definition 2.2 (cf. [9, §6])**.** Let V be a vertex algebra, and let γ be a formal variable. For $n \in \mathbb{Z}_{\geq 1}$, we define
$$\widetilde{O}_\gamma(V) \coloneqq V \overset{\gamma}{\underset{2}{\bullet}} V, \qquad \widetilde{A}_\gamma(V) \coloneqq V[\gamma]/\widetilde{O}_\gamma(V), \tag{2.13}$$
and denote the equivalence class of $a \in V$ in the quotient space by $[a] \in \widetilde{A}_\gamma(V)$.

The subspace $\widetilde{O}_\gamma(V)$ satisfies analogous properties with the defining ideal $O(V)$ of the Zhu algebra $A(V) = V/O(V)$; see [12, Theorem 1.4] and [2, Theorem 2.7].

Proposition 2.3 ([7, Proposition 6.1])**.** The definition of $\widetilde{A}_\gamma(V)$ can be specialized to $\gamma = c$ for any $c \in \mathbb{C}$, and
$$\widetilde{A}_{\gamma=c}(V) \coloneqq V/\widetilde{O}_{\gamma=c}(V)$$
is a unital associative $\mathbb{C}$-superalgebra with the product $\overset{\gamma=c}{\bullet}$ induced by $\overset{\gamma=c}{\underset{1}{\bullet}}$ on V.

When V is a VOA with a conformal element $L \in V^{\bar{0}}$, in the same way as [7, Proposition 6.3], one can construct an explicit isomorphism from $\widetilde{A}_{\gamma=c}(V)$ for $c \in \mathbb{C} \setminus \{0\}$ to $\mathrm{Zhu}_{L_0}(V)$. In particular, when V is a VOA, $\widetilde{A}_{\gamma=c}(V)$ for $c \in \mathbb{C} \setminus \{0\}$ is isomorphic to $\widetilde{A}_{\gamma=1}(V)$ (see [12, §1.3] for details). Based on this observation, we focus on the case $c = 1$.

Definition 2.4 (cf. [7, §6])**.** The Zhu algebra of a vertex algebra V is the unital associative $\mathbb{C}$-superalgebra
$$(\widetilde{A}(V), \bullet, 1) \coloneqq (\widetilde{A}_{\gamma=1}(V), \overset{\gamma=1}{\bullet}, [\mathbf{1}]).$$

Corollary 2.5 ([12, Corollary 1.7])**.** For a vertex algebra $(V, Y, \mathbf{1}, \partial)$ and $a, b \in V$, the following hold in the Zhu algebra $\widetilde{A}(V)$:
$$[\partial a] = 0, \qquad [[a],[b]] \coloneqq [a] \bullet [b] - p(a,b)[b] \bullet [a] = [a_{(0)}b].$$

3 Main Result 1

In this section we demonstrate the computation of the Zhu algebra $\widetilde{A}(V)$ for superconformal vertex algebras directly.

We will freely use the language of Lie conformal algebras and the enveloping vertex algebra of a Lie conformal algebra. See [10, §2.7, §4.7], [2, §1.5], [4, Chap. 16] and [8, §2.4] for details. We list some terminology and notation.

- A *Lie conformal algebra* $\mathcal{L}$ is a $\mathbb{C}[\partial]$-module equipped with an even bilinear map $[\cdot_\lambda\cdot]\colon \mathcal{L}\otimes\mathcal{L}\to\mathbb{C}[\lambda]\otimes\mathcal{L}$ satisfying some axioms [8, Definition 2.2].
- Given a Lie conformal algebra $\mathcal{L}$, one can associate to it a vertex algebra $V(\mathcal{L})$ called the *enveloping vertex algebra* of L [10, §4.7], [4, §16.11].
- Let $\mathcal{L}$ be a Lie conformal algebra, $C\in\mathcal{L}$ be a central element such that $\partial C=0$, and $c\in\mathbb{C}$. We denote by $V^c(\mathcal{L})$ the quotient $V(\mathcal{L})/(C-c\mathbf{1})_{(-1)}V(\mathcal{L})$. It has a natural vertex algebra structure.

In terms of λ-bracket, Corollary 2.5 implies the following.

Lemma 3.1. Let $\mathcal{L}$ be a Lie conformal algebra and V be an arbitrary quotient vertex algebra of $V(\mathcal{L})$. For $a,b\in V$, the following hold in the Zhu algebra $\widetilde{A}(V)$:

$$[\partial a]=0,\qquad [[a],[b]]=[a_\lambda b]\Big|_{\lambda=0}. \tag{3.1}$$

3.1 $N=1,2,4$ superconformal algebras

Let $V^{N=4}$ be the enveloping vertex algebra of the (small) $N=4$ superconformal algebra [10, §5.9], [11, §8.4]. It is strongly generated by eight elements

- a conformal element L of central charge $c\in\mathbb{C}$,
- three even primary elements J^0,J^+,J^- of conformal weight 1,
- and four odd primary elements $G^+,G^-,\overline{G}^+,\overline{G}^-$ of conformal weight 3/2.

The remaining non-zero λ-brackets are

$$\begin{gathered}[J^0{}_\lambda J^\pm]=\pm 2J^\pm,\quad [J^0{}_\lambda J^0]=\frac{c}{3}\lambda,\quad [J^+{}_\lambda J^-]=J^0+\frac{c}{6}\lambda,\\ [J^0{}_\lambda G^\pm]=\pm G^\pm,\quad [J^0{}_\lambda \overline{G}^\pm]=\pm\overline{G}^\pm,\quad [J^\pm{}_\lambda G^\mp]=G^\pm,\quad [J^\pm{}_\lambda \overline{G}^\mp]=-\overline{G}^\pm,\\ [G^\pm{}_\lambda \overline{G}^\pm]=(\partial+2\lambda)J^\pm,\quad [G^\pm{}_\lambda \overline{G}^\mp]=L\pm\frac{1}{2}(\partial+2\lambda)J^0+\frac{c}{6}\lambda^2.\end{gathered}$$

Now we study the Zhu algebra

$$\widetilde{A}(V^{N=4})=V^{N=4}/\widetilde{O}(V^{N=4}).$$

It is generated by the classes $[L]$, $[G^\pm]$, $[\overline{G}^\pm]$ and $[J^\alpha]$. By the λ-bracket relations and Lemma 3.1, the non-trivial commutation relations of the generators in $\widetilde{A}$ are given as follows:

$$\begin{gathered}[[J^0],[J^\pm]]=\pm2[J^\pm],\quad [[J^+],[J^-]]=[J^0],\quad [[J^0],[G^\pm]]=\pm[G^\pm],\quad [[J^0],[\overline{G}^\pm]]=\pm[\overline{G}^\pm],\\ [[J^\pm],[G^\mp]]=[G^\pm],\quad [[J^\pm],[\overline{G}^\mp]]=-[\overline{G}^\pm],\quad [[G^\pm],[\overline{G}^\pm]]=0,\quad [[G^\pm],[\overline{G}^\mp]]=[L].\end{gathered} \tag{3.2}$$

Now, let $\mathfrak{sl}(2|2)$ be the special linear Lie superalgebra, and consider the quotient $\mathfrak{psl}(2|2):=\mathfrak{sl}(2|2)/\mathbb{C}I$ (I is the identity supermatrix). Using the supermatrix notation, we take a minimal nilpotent element $f=f_{\min}\in\mathfrak{psl}(2|2)$ as

$$f:=\left[\begin{array}{c|c}O&O\\ \hline O&F\end{array}\right],$$

where $F := \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ and $O := \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then its centralizer

$$\mathfrak{psl}(2|2)^f := \{x \in \mathfrak{psl}(2|2) \mid [x, f] = 0\}$$

has a linear basis consisting of

$$f, \quad j^0 := \left[\begin{array}{c|c} H & O \\ \hline O & O \end{array}\right], \quad j^+ := \left[\begin{array}{c|c} E & O \\ \hline O & O \end{array}\right], \quad j^- := \left[\begin{array}{c|c} F & O \\ \hline O & O \end{array}\right], \tag{3.3}$$

$$g^+ := \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad g^- := \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \overline{g}^+ := \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right], \quad \overline{g}^- := \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right], \tag{3.4}$$

where $H := \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $E := \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Hence we have $\dim \mathfrak{psl}(2|2)^f = 4|4$ and

$$(\mathfrak{psl}(2|2)^f)^{\overline{0}} \cong \mathfrak{sl}(2) \oplus \mathbb{C}f, \quad (\mathfrak{psl}(2|2)^f)^{\overline{1}} \cong (\text{lowest weight component of } \mathbb{C}^2 \otimes \mathbb{C}^2 \oplus \mathbb{C}^2 \otimes \mathbb{C}^2).$$

One can check that the above basis of $\mathfrak{psl}(2|2)^f$ satisfy the same relations as (3.2), and hence we have a surjective morphism of associative $\mathbb{C}$-algebras

$$\varphi \colon U(\mathfrak{psl}(2|2)^f) \twoheadrightarrow \widetilde{A}(V^{N=4}) \tag{3.5}$$

given by

$$f \longmapsto [L], \quad j^\alpha \longmapsto [J^\alpha], \quad g^\beta \longmapsto [G^\beta], \quad \overline{g}^\beta \longmapsto [\overline{G}^\beta], \quad (\alpha \in \{0, +, -\}, \ \beta \in \{+, -\}). \tag{3.6}$$

Furthermore, we have:

Theorem 3.2 ([12, §2])**.** The algebra morphism φ is bijective, and we have

$$\varphi \colon U(\mathfrak{psl}(2|2)^f) \xrightarrow{\sim} \widetilde{A}(V^{N=4}).$$

By considering the standard $N = 2$ and $N = 1$ superconformal subalgebras of the small $N = 4$ superconformal algebra, we have the following.

Corollary 3.3 ([12, §2])**.** The algebra isomorphism φ restricts to algebra isomorphisms

$$U(\mathfrak{osp}(2|2)^f) \xrightarrow{\sim} \widetilde{A}(V^{N=2}), \qquad U(\mathfrak{osp}(1|2)^f) \xrightarrow{\sim} \widetilde{A}(V^{N=1}).$$

See [12, §2] for more details.

3.2 $N = 3$ superconformal algebra

For the $N = 3$ case, following [1, p.107] and [11, §8.5], we consider the vertex algebra $V^{N=3}$ strongly generated by

- a conformal element L of central charge $c \in \mathbb{C}$,
- three even primary elements A^i $(i = 1, 2, 3)$ of conformal weight 1,
- three odd primary elements G^i $(i = 1, 2, 3)$ of conformal weight 3/2,
- and an odd primary element Φ of conformal weight 1/2,

with the remaining non-zero λ-brackets given by

$$[A^i{}_\lambda A^j] = \varepsilon_{ijk}A^k + \frac{c}{3}\lambda\delta_{ij}, \quad [A^i{}_\lambda G^j] = \varepsilon_{ijk}G^k + \lambda\Phi\delta_{ij}, \tag{3.7}$$

$$[G^i{}_\lambda G^j] = 2L\delta_{ij} - \varepsilon_{ijk}(\partial + 2\lambda)A^k + \frac{c}{3}\lambda^2\delta_{ij}, \quad [\Phi_\lambda G^i] = A^i, \quad [\Phi_\lambda \Phi] = -\frac{c}{3} \tag{3.8}$$

for $i, j \in \{1,2,3\}$. Here ε_{ijk} is antisymmetric with $\varepsilon_{123} = 1$, and δ_{ij} denotes the Kronecker's delta. We also used Einstein's summation convention for repeated indices. According to Lemma 3.1, the OPEs (3.7) and (3.8) imply the following commutation relations in $\widetilde{A}(V^{N=3})$:

$$[L] \text{ and } [\mathbf{1}] \text{ are in the center}, \quad [[A^i],[A^j]] = \varepsilon_{ijk}[A^k], \quad [[A^i],[G^j]] = \varepsilon_{ijk}[G^k], \tag{3.9}$$

$$[[G^i],[G^j]] = 2L\delta_{ij}, \quad [[\Phi],[A^i]] = 0, \quad [[\Phi],[G^i]] = [A^i], \quad [[\Phi],[\Phi]] = -\frac{c}{3}[\mathbf{1}]. \tag{3.10}$$

To determine the Zhu algebra of $V^{N=3}$, let $\mathfrak{osp}(3|2)$ be the ortho-symplectic Lie superalgebra of superdimension $6|6$, which can be realized by supermatrices as

$$\mathfrak{osp}(3|2) := \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \,\middle|\, A \in \mathfrak{so}(3),\ D \in \mathfrak{sl}(2),\ B \in \mathrm{Mat}_{3\times 2},\ C = -J({}^tB) \right\},$$

where $J := \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Using this realization, we take a minimal nilpotent element $f = f_{\min} \in \mathfrak{osp}(3|2)$ as

$$f := \left[\begin{array}{c|c} O & O \\ \hline O & F \end{array}\right],$$

where $F := \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$. Then its centralizer

$$\mathfrak{osp}(3|2)^f := \{x \in \mathfrak{osp}(3|2) \mid [x, f] = 0\}$$

has a basis consisting of

$$f, \quad a^i := \left[\begin{array}{c|c} J^i & O \\ \hline O & O \end{array}\right], \quad g^i := E_{i4} + E_{5i} \quad (i = 1,2,3), \tag{3.11}$$

where E_{pq} is the supermatrix unit in $\mathfrak{gl}(3|2)$ and

$$J^1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J^2 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad J^3 := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence we have $\dim \mathfrak{osp}(3|2)^f = 4|3$ and

$$(\mathfrak{osp}(3|2)^f)^{\bar{0}} \cong \mathbb{C} \oplus \mathfrak{so}(3), \quad (\mathfrak{osp}(3|2)^f)^{\bar{1}} \cong \big(\text{the natural representation } \mathbb{C}^3 \text{ of } \mathfrak{so}(3)\big).$$

The universal enveloping algebra of $\mathfrak{osp}(3|2)^f$ accounts for the subalgebra generated by $[L], [A^i], [G^i]$ in (3.9) and (3.10), but it does not yet capture the odd generator $[\Phi]$. More explicitly, we have a (non-surjective) morphism of associative $\mathbb{C}$-algebras

$$\varphi \colon U(\mathfrak{osp}(3|2)^f) \longrightarrow \widetilde{A}(V^{N=3}) \tag{3.12}$$

given by

$$f \longmapsto [L], \quad a^i \longmapsto [A^i], \quad g^i \longmapsto [G^i], \quad (i \in \{1,2,3\}). \tag{3.13}$$

Furthermore, we have:

Theorem 3.4 ([12, §2]). The algebra morphism φ is injective, and $\widetilde{A}(V^{N=3})$ is a free $U(\mathfrak{osp}(3|2)^f)$-module of rank $1|1$ with basis $[\mathbf{1}]$ and $[\Phi]$.

3.3 Big $N = 4$ superconformal algebra

In this note, we omit a discussion of the relation between the big $N = 4$ superconformal algebra and the centralizer of a minimal nilpotent element in the exceptional Lie superalgebra $D(2, 1; a)$, $a \in \mathbb{C}\backslash\{0, -1\}$, since the necessary notation and data are too involved for the present exposition. See [12, §2] for details.

4 Main Result 2

4.1 $N_K = N$ SUSY vertex algebras

We recall several notations for superfields from [8, §4].

- Let $N \in \mathbb{Z}_{\geq 1}$, and denote $[N] := \{1, 2, \dots, N\}$.
- Let $Z = (z, \zeta^1, \dots, \zeta^N)$ be an $N_K = N$ *supervariable*, i.e., a tuple of an even indeterminate z and odd indeterminates $\zeta^1, \dots, \zeta^N$. We denote
$$Z^{j|J} := z^j \zeta^J = z^j \zeta^{j_1} \cdots \zeta^{j_r} \quad (j \in \mathbb{Z},\ J = \{j_1 \leq \cdots \leq j_r\} \subset [N]).$$
Given two $N_K = N$ supervariables $Z_i = (z_i, \zeta_i^1, \dots, \zeta_i^N)$ $(i = 1, 2)$, we have a group structure:
$$Z_1 - Z_2 = (z_1 - z_2 - \textstyle\sum_{k=1}^N \zeta_1^k \zeta_2^k, \zeta_1^1 - \zeta_2^1, \dots, \zeta_1^N - \zeta_2^N). \tag{4.1}$$
Hence
$$(Z_1 - Z_2)^{j|J} := (z_1 - z_2 - \textstyle\sum_{k=1}^N \zeta_1^k \zeta_2^k)^j (\zeta_1^{j_1} - \zeta_2^{j_1}) \cdots (\zeta_1^{j_r} - \zeta_2^{j_r})$$
for $j \in \mathbb{Z}$ and $J = \{j_1, \dots, j_r\} \subset [N]$.
- For an $N_K = N$ supervariable $Z = (z, \zeta^1, \dots, \zeta^N)$, we denote
$$\eth_Z^k := \partial_{\zeta^k} + \zeta^k \partial_z \quad (k \in [N]), \quad \eth_Z^{j|J} := \partial_z^j \eth_Z^{j_1} \cdots \eth_Z^{j_r} \quad (j \in \mathbb{N},\ J = \{j_1 \leq \cdots \leq j_r\} \subset [N]).$$
- For a linear superspace V, we set
$$V[\![Z^{\pm 1}]\!] := V[\![z^{\pm 1}]\!][\zeta] = \{v(Z) = \textstyle\sum_{j \in \mathbb{Z},\, J \subset [N]} Z^{j|J} v_{j|J} \mid v_{j|J} \in V\}.$$
- For a linear superspace V, we define the super residue $\operatorname{sres}_Z[\cdot]\delta Z \colon V[\![Z^{\pm 1}]\!] \to V$ by
$$\operatorname*{sres}_Z[v(Z)]\delta Z := v_{-1|[N]} \qquad (v(Z) = \textstyle\sum_{j \in \mathbb{Z},\, J \subset [N]} Z^{j|J} v_{j|J} \in V[\![Z^{\pm 1}]\!]).$$

As introduced in [8, Definition 4.13], an $N_K = N$ supersymmetric (SUSY for short) vertex algebra is a tuple $(\mathbb{V}, Y, \mathbf{1}, \eth^1, \dots, \eth^N)$ consisting of

- a linear superspace $\mathbb{V} = \mathbb{V}^{\bar{0}} \oplus \mathbb{V}^{\bar{1}}$,
- an even linear map $\mathbb{V} \to (\operatorname{End} \mathbb{V})[\![Z^{\pm 1}]\!]$,
$$a \mapsto Y(a, Z) = \sum_{j \in \mathbb{Z},\, J \subset [N]} Z^{-1-j|[N]-J} a_{(j|J)},$$
with each $Y(a, Z)$ called the *superfield*,

- a non-zero even element $\mathbf{1} \in \mathbb{V}^{\bar{0}}$, called the *vacuum*,
- odd operators $\not{\partial}^1, \ldots, \not{\partial}^N \in (\operatorname{End} \mathbb{V})^{\bar{1}}$, called the *odd translation operators*,

satisfying the following conditions for any $a, b \in \mathbb{V}$.

1. (field condition) $Y(a, Z)b \in \mathbb{V}(\!(Z)\!)$.
2. (vacuum axiom) $Y(\mathbf{1}, Z) = \mathrm{id}_{\mathbb{V}}$ and $Y(a, Z)\mathbf{1} = a + O(Z)$.
3. (translation axiom) $[\not{\partial}^k, Y(a, z)] = (\partial_{\zeta^k} - \zeta^k \partial_z)Y(a, z)$ for any $k \in [N]$.
4. (locality axiom) The superfields $Y(a, Z)$ and $Y(b, W)$ are local, i.e., there exists $M(a, b) \in \mathbb{N}$ such that $(z - w)^{M(a,b)}[Y(a, Z), Y(b, W)] = 0$.

We will use the standard notation $a(Z) \coloneqq Y(a, Z)$ for $a \in \mathbb{V}$.

All the operators $(\not{\partial}^k)^2 \in \operatorname{End} \mathbb{V}$ for $k \in [N]$ coincide, and

$$\partial \coloneqq (\not{\partial}^1)^2 = \cdots = (\not{\partial}^N)^2 \in (\operatorname{End} \mathbb{V})^{\bar{0}} \tag{4.2}$$

satisfies the translation axiom of non-SUSY vertex algebra. We call ∂ the *even translation operator* of the SUSY vertex algebra $\mathbb{V}$.

We can "reduce" a SUSY vertex algebra to a (non-SUSY) vertex algebra.

Lemma 4.1. Let $\mathbb{V}$ be an $N_K = N$ SUSY vertex algebra with vertex operators $a(Z) = a(z, \zeta^1, \ldots, \zeta^N)$ for $a \in \mathbb{V}$. Then the even linear map

$$\mathbb{V} \longrightarrow (\operatorname{End} \mathbb{V})(\!(z^{\pm 1})\!), \quad a \longmapsto a(z) \coloneqq a(z, 0, \ldots, 0) = \sum_{j \in \mathbb{Z}} z^{-j-1} a_{(j|[N])}$$

and the even translation ∂ gives a vertex algebra structure on $\mathbb{V}$. We call it the *reduced part* of the $N_K = N$ SUSY vertex algebra $\mathbb{V}$, and denote it by $\mathbb{V}_{\mathrm{red}}$.

4.2 Zhu algebra of a SUSY vertex algebra

We now define the Zhu algebra for an $N_K = N$ SUSY vertex algebra.

Let $\mathbb{V}$ be an $N_K = N$ SUSY vertex algebra (§ 4.1), and γ be a complex number or a formal variable. Also, let $a, b \in \mathbb{V}$ and $n \in \mathbb{Z}_{\geq 1}$. If $\gamma \neq 0$, then we define $a \overset{\gamma}{\underset{n}{\bullet}} b \in \mathbb{V}[\gamma]$ by

$$a \overset{\gamma}{\underset{n}{\bullet}} b \coloneqq \operatorname*{sres}_{Z} \Big[\frac{\gamma^n \zeta^{[N]} e^{\gamma z}}{(e^{\gamma z} - 1)^n} a(Z) b \Big] \delta Z = \operatorname*{res}_{z} \Big[\frac{\gamma^n e^{\gamma z}}{(e^{\gamma z} - 1)^n} a(z, 0, \ldots, 0) b \Big] dz. \tag{4.3}$$

If $\gamma = 0$, then we define

$$a \overset{\gamma=0}{\underset{n}{\bullet}} b \coloneqq \operatorname*{sres}_{Z} \big[z^{-n} \zeta^{[N]} a(Z) b \big] \delta Z = a_{(-n|[N])} b.$$

Then the operation $\overset{\gamma}{\underset{n}{\bullet}}$ coincides with the non-SUSY operation $\overset{\gamma}{\underset{n}{\bullet}}$ on the reduced part $\mathbb{V}_{\mathrm{red}}$. Hence, the corresponding result for ordinary vertex algebras implies the following.

Proposition 4.2 ([12, Proposition 3.6])**.** Let $\mathbb{V}$ be an $N_K = N$ SUSY vertex algebra, and γ be a complex number or a formal variable. Then the quotient space

$$\widetilde{A}_\gamma(\mathbb{V}) \coloneqq \mathbb{V}[\gamma] / \widetilde{O}_\gamma(\mathbb{V}), \quad \widetilde{O}_\gamma(\mathbb{V}) \coloneqq \operatorname{Span}_{\mathbb{C}[\gamma]} \{ a \overset{\gamma}{\underset{n}{\bullet}} b \mid a, b \in \mathbb{V},\ n \geq 2 \}$$

is an associative algebra whose product $\overset{\gamma}{\bullet}$ is induced by the operation $\underset{1}{\overset{\gamma}{\bullet}}$ as

$$[a] \overset{\gamma}{\bullet} [b] := [a \underset{1}{\overset{\gamma}{\bullet}} b] \quad (a, b \in \mathbb{V}),$$

and the equivalence class of $\mathbf{1} \in \mathbb{V}$ is its unit. Moreover, it is isomorphic to the Zhu algebra $\widetilde{A}_\gamma(\mathbb{V}_{\mathrm{red}})$ for the non-SUSY vertex algebra $\mathbb{V}_{\mathrm{red}}$.

Definition 4.3. The algebra $\widetilde{A}_\gamma(\mathbb{V})$ in Proposition 4.2 is called the Zhu algebra of the $N_K = N$ SUSY vertex algebra $\mathbb{V}$. We denote $\widetilde{A}(\mathbb{V}) := \widetilde{A}_{\gamma=1}(\mathbb{V})$ as in the non-SUSY case.

The algebra $\widetilde{A}_\gamma(\mathbb{V})$ for a SUSY vertex algebra $\mathbb{V}$ enjoys the same properties as in the non-SUSY case. In particular, for any $a \in \mathbb{V}$, we have

$$[\partial a] = 0. \tag{4.4}$$

We also have the specialization at $\gamma = 0$.

Lemma 4.4 ([12, Lemma 3.9])**.** Let $\mathbb{V}$ be an $N_K = N$ SUSY vertex algebra. Then, the specialization $\widetilde{A}_{\gamma=0}(\mathbb{V})$ is identical to the C_2-Poisson algebra:

$$R(\mathbb{V}) := \mathbb{V}/C_2(\mathbb{V}), \quad C_2(\mathbb{V}) := \mathrm{Span}_{\mathbb{C}}\{a_{(j|J)}b \mid a, b \in \mathbb{V},\ j \geq 2,\ J \subset [N]\}.$$

In particular, the product is supercommutative, and it is equipped with a Poisson bracket:

$$[a] \cdot [b] = [a_{(-1|N)}b] = [a] \overset{\gamma=0}{\bullet} [b], \quad \{[a], [b]\} = [a_{(0|N)}b].$$

However, the SUSY case has an additional structure. Recall that an $N_K = N$ SUSY vertex algebra $\mathbb{V}$ has the odd translation operators $\eth^k$ $(k \in [N])$. These are odd derivations with respect to the $_{(-1|N)}$-operation in $\mathbb{V}$. Then, by the relation $(\eth^k)^2 = \partial$ and (4.4), we have:

Proposition 4.5 ([12, Proposition 3.10])**.** The odd translation operators $\eth^k$ $(k \in [N])$ of $\mathbb{V}$ induce odd differentials of the superalgebra $\widetilde{A}_\gamma(\mathbb{V})$, which are denoted by the same symbols $\eth^k$.

Recall that the odd derivations $\eth^k$ on $\mathbb{V}$ induce odd differentials on the C_2-Poisson algebra $R(\mathbb{V})$. Thus, we have:

Corollary 4.6 ([12, Corollary 3.11])**.** Under the limit $\gamma \to 0$, the odd linear operators $[a] \mapsto [\eth^k a]$ $(k \in [N])$ on $\widetilde{A}_\gamma(\mathbb{V})$ converge to the differentials $\eth^k$ on $R(\mathbb{V})$.

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