The Integrality and Algebraic Connectivity of Hyper-Regular Hypergraph $C(k, l, n, m)$

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Abstract

Suppose $\mathcal{H}(\mathcal{V}, \mathcal{E})$ be a hypergraph with a vertex set $\mathcal{V}$ and an hyperedge set $\mathcal{E}$. In this paper, we investigate a class of hypergraphs called hyper-regular hypergraphs $C(k, l, n, m)$. We show that the hyper-regular $C(k, l, n, m)$ is not integral hypergraphs by its adjacency matrix. Moreover, the Laplacian spectrum of the hyper-regular hypergraphs $C(k, 1, n, m)$ shows that that hyper-regular hypergraphs $C(k, 1, n, m)$ is not Laplacian integral. In addition, several properties of the algebraic connectivity of hyper-regular hypergraphs $C(k, l, n, m)$ are studied.

Keywords: *hypergraph, integrality, hyper-regular $C(k, l, n, m)$, algebraic connectivity*

1 Introduction

A *hypergraph* of order n and size m is a structure $\mathcal{H}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, v_2, v_3, ..., v_n\}$ is a finite set and $\mathcal{E} = \{E_1, E_2, E_3, ..., E_m\}$ is a power set of $\mathcal{V}$ such that

$$\mathcal{E} \subseteq 2^{\mathcal{V}}.$$

The elements $v_1, v_2, ..., v_n$ are called *vertices* and the sets $E_1, E_2, ..., E_m$ are called *hyperedges.* [2]

A hypergraph H is called integral if all the eigenvalues of its adjacency matrix are integers. There are many integral hypergraphs that have been found, such as hyper-stars, complete $r-$uniform hypergraphs, and sunflower hypergraphs. [1]

In this paper, we study a class of hypergraphs known as hyper-regular hypergraph $C(k, l, n, m)$ introduced by Astuti (2012).

Definition 1 Let $k \geq 2$, $l \geq 1$, $m \geq 3$, $n \geq 3$, and $k, l, m, n \in \mathbb{N}$. Let Γ be a $k-$regular graph of order n and size m. A hyper-regular $C(k, l, n, m)$ is a $(2+l)-$uniform hypergraph of order $n + lm$ and size m which obtained by enlarging each edge of Γ with l additional vertex.

For example, Figure 1 is a $3 - regular$ graph (Γ) and Figure 2 is hyper-regular hypergraph $C(3, 1, 4, 6)$ which obtained by adding one vertex on each edges of Graph Γ.

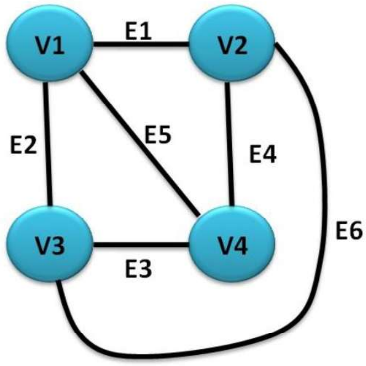

Figure 1: Graph Γ

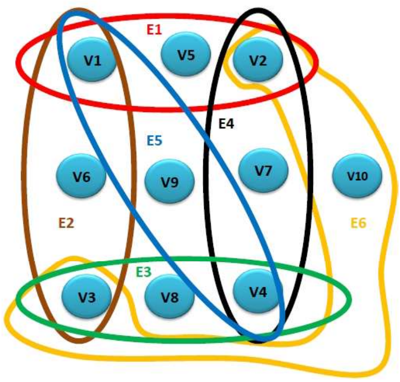

Figure 2: Hyper-regular Hypergraph $C(3, 1, 4, 6)$

Astuti (2012) showed that for $l = 1$, the hyper-regular hypergraph $C(k, l, n, m)$ is not integral. In this paper, we extend this result by proving that the hyper-regular hypergraph $C(k, l, n, m)$ is not integral for $l \geq 2$ using an induction method.

A hypergraph $\mathcal{H}$ is called Laplacian integral if all eigenvalues of its Laplacian matrix are integers [3]. Examples of Laplacian inetgral hypergraphs include sunflower hypergraphs [6] and complete tripartite hypergraphs.[8]

The Laplacian spectrum of a hypergraph $\mathcal{H}$ is the set of eigenvalues of its Laplacian matrix together with their multiplicities. [4]

The algebraic connectivity of a hypergraph defined as the second smallest eigenvalue of its Laplacian matrix.[5] One study on algebraic connectivity was conducted by Kumar [7] who showed a relationship between the algebraic connectivity of the union of two-edge disjoint hypergraph and the algebraic conectivity of each hypergraph.

In this paper, we also present the Laplacian spectrum of the hyper-regular hypergraph $C(k,l,n,m)$ for $l=1$ by determining the determinant of a block matrix. In addition, we investigate several properties of the algebraic connectivity of hyper-regular hypergraph $C(k,l,n,m)$.

2 Main Results

In this section, we present our main results on the integrality of hyper-regular hypegraphs $C(k,l,n,m)$. First, we recall the results obtained by Astuti (2012) for teh case $l=1$,

Theorem 1. Let $k>1$ and Γ is a $k-regular$ graph of order n and size m. If A be an adjacency matrix of Γ with eigenvalues λ and B be an adjacency matrix of hyper-regular $C(k,1,m,n)$ which is induced by Γ, with eigenvalues θ, then

$$\theta = \frac{\lambda \pm \sqrt{\lambda^2+4(\lambda+k)}}{2}$$

Corollary 1. The hyper-regular hypergraphs $C(k,1,n,m)$ is not integral, for $k>1$.

From the results above, we extend the non-integrality of hyper-regular hypergraphs to the case $l \geq 2$ by studying the structure of its adjacency matrix.

Definition 2. The adjacency matrix (Q) of hyper-regular hypergraph $C(k,l,n,m)$ is a $(n+lm)\times(n+lm)$ matrix that can be expressed in the form of a block matrix as follows,

$$Q = \begin{pmatrix} A_{n\times n} & B_{n\times m} & \dots & \dots & B_{n\times m} \\ B^T_{m\times n} & \mathbf{0}_{m\times m} & \mathbf{I}_{m\times m} & \dots & \mathbf{I}_{m\times m} \\ \vdots & \mathbf{I}_{m\times m} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{I}_{m\times m} \\ B^T_{m\times n} & \mathbf{I}_{m\times m} & \dots & \mathbf{I}_{m\times m} & \mathbf{0}_{m\times m} \end{pmatrix}_{(n+lm)\times(n+lm)}$$

which B is the incidence matrix of $k-regular$ graph Γ and B^T denotes its transpose. The number of blocks B and B^T is equal to l corresponding to the additional vertices added to each edge of Γ.

Theorem 2. Let M and N be the adjacency matrix of hyper-regular hypergraphs $C(k,1,n,m)$ and $C(k,2,n,m)$ respectively. If θ is the eigenvalues of M then γ is the eigenvalues of N satisfy

$$\gamma = \frac{\theta \pm \sqrt{\theta^2 + 4(\theta + k)}}{2}$$

.

Proof. By definition, $N = \begin{pmatrix} M & B^* \\ C^* & D^* \end{pmatrix}$, where
M is the adjacency matrix of hyper-regular $C(k, 1, n, m)$,
$B^* = \begin{pmatrix} B \\ I \end{pmatrix}$, B is an incidence matrix of Γ and I is an identity matrix,
$C^* = \begin{pmatrix} B^T & I \end{pmatrix}$, is a transpose of B^*, and
$D^* = \mathbf{0}$.
Let $x = \begin{pmatrix} u_{nx1} \\ v_{2mx1} \end{pmatrix}$ be an eigenvector of N corresponding to eigenvalues γ, where u is a $n \times 1$ coulumn vector and v is a $2m \times 1$ column vector then obtained,

$$Nx = \gamma x$$

then,

$$Mu + B^*v = \gamma u \ldots (1)$$
and
$$C^*u + D^*v = \gamma v \ldots (2)$$

based on one of properties of an incidence matrix we have,

$$B^*B^{*T} = M + D$$
$$\text{where } D = (\delta(v))I$$

$\delta(v)$ is degree of every vertex on hyper-regular $C(k, 1, n, m)$ with $\delta(v) \geq 2$.
Then, the equation (1) and (2) are multiplied by γ and B^* respectively, obtained

$$(\gamma + 1)Mu = (\gamma^2 - \delta(v))u.$$

If $\gamma = -1$ then $(1 - \delta(v))u = 0$ so that $\delta(v) = 1$. This is contrary to $\delta(v) \geq 2$. As a result, for $\gamma = -1$ it is undefined.
If $\gamma \neq -1$ then,

$$Mu = \frac{\gamma^2 - \delta(v)}{\gamma + 1} u.$$

From the equation above it can be seen that u is an eigenvector of M which corresponds to the eigenvalues of $\frac{\gamma^2 - \delta(v)}{\gamma + 1}$. Since θ is the eigenvalues of M then,

$$\theta\gamma + \theta = \gamma^2 - \delta(v)$$

so the eigenvalue of N is

$$\gamma = \frac{\theta \pm \sqrt{\theta^2 + 4(\theta + \delta(v))}}{2}.$$

■

Corollary 2. The eigenvalues of any hyper-regular hypergraphs $C(k,l,n,m)$ can be determined from the eigenvalues of hyper-regular hypergraphs $C(k,l-1,n,m)$.

Proof. By induction method we have,
For $l=1$, it means that the eigenvalues θ of hyper-regular hypergraphs $C(k,1,n,m)$ can be determined from the eigenvalues λ of $k-regular$ graph Γ. It can be seen from Astutis' finding in Theorem 1.
For $l=a$, $a>1$, $a\in\mathbb{N}$, we assumed that the eigenvalues γ_a of hyper-regular $C(k,a,n,m)$ can be determined from the eigenvalues γ_{a-1} of hyper-regular $C(k,a-1,n,m)$ such that,

$$\gamma_a=\frac{\gamma_{a-1}\pm\sqrt{\gamma_{a-1}^2+4(\gamma_{a-1}+\delta(v))}}{2}.$$

For $l=a+1$, we show that the eigenvalues γ_{a+1} of hyper-regular hypergraphs $C(k,a+1,n,m)$ can be determined from the eigenvalues γ_a of hyper-regular hypergraphs $C(k,a,n,m)$,

$$\gamma_{a+1}=\frac{\gamma_a\pm\sqrt{\gamma_a^2+4(\gamma_a+\delta(v))}}{2}$$

Suppose R and Q are the adjacency matrix of hyper-regular $C(k,a+1,n,m)$ and $C(k,a,n,m)$ respectively. By Definition 2 we obtained,

$$R=\begin{pmatrix} Q & B^* \\ B^{*T} & D^* \end{pmatrix}$$

Let $z=\begin{pmatrix} u_{n\times 1} \\ v_{(a+1)m\times 1}\end{pmatrix}$ be an eigenvector of R corresponding to eigenvalues γ_{a+1}, where u is a $n\times 1$ coulumn vector and v is a $(a+1)m\times 1$ column vector then obtained,

$$Rz=\gamma_{a+1}z$$

then,

$$Qu+B^*v=\gamma_{a+1}u\ldots(1)$$
and
$$B^{*T}u+D^*v=\gamma_{a+1}v\ldots(2)$$

based on the properties of incidence matrix we have,

$$B^*B^{*T}=Q+D$$
where $D=(\delta(v))\mathbf{I}$

$\delta(v)$ is degree of any vertex on hyper-regular hypergraphs $C(k,a,n,m)$ with $\delta(v)\geq 2$. The equation (1) and (2) are multiplied by γ_{a+1} and B^* respectively, obtained

$$(\gamma_{a+1}+1)Qu=(\gamma_{a+1}^2-\delta(v))u.$$

If $\gamma_{a+1}=-1$ then $(1-\delta(v))u=0$ so that $\delta(v)=1$.
This is contrary to $\delta(v)\geq 2$. As a result, for $\gamma_{a+1}=-1$ it is undefined.
If $\gamma\neq -1$ then,

$$Qu=\frac{\gamma_{a+1}^2-\delta(v)}{\gamma_{a+1}+1}u.$$

From the equation above, it showed that u is an eigenvector of Q which corresponds to the eigenvalues of $\frac{\gamma_{a+1}^2-\delta(v)}{\gamma_{a+1}+1}$. Since θ is the eigenvalues of Q then,

$$\gamma_a\gamma_{a+1}+\gamma_a=\gamma_{a+1}^2-\delta(v)$$

so the eigenvalue of R is

$$\gamma_{a+1}=\frac{\gamma_a\pm\sqrt{\gamma_a^2+4(\gamma_a+\delta(v))}}{2}.$$

■

Theorem 3. Hyper-regular hypergraphs $C(k,l,n,m)$ is not integral.

Proof.
For $l=1$, hyper-regular $C(k,1,n,m)$ hypergraphs is not an integral by Corollary 1.

For $l=p$ where $p\in\mathbb{N}$, we assume that hyper-regular hypergraphs $C(k,p,n,m)$ is not integral, hence we have

$$\gamma_p=\frac{\gamma_{p-1}\pm\sqrt{\gamma_{p-1}^2+4(\gamma_{p-1}+\delta(v))}}{2}$$

where γ_p are not all integers.

For $l=p+1$, by Corollary 2 the eigenvalues form of hyper-reguler $C(k,p+1,n,m)$ is,

$$\gamma_{p+1}=\frac{\gamma_p\pm\sqrt{\gamma_p^2+4(\gamma_p+\delta(v))}}{2}$$

Let $\gamma_p=a$ where a is integer, we only need to prove that $\gamma_p^2+4(\gamma_p+\delta(v))$ is a perfect square. By substitution, we have $a^2+4a+4\delta(v)$.
Assume that $a^2+4a+4\delta(v)$ is a perfect square then it can be written as,

$$(a+\sqrt{4\delta(v)})^2=a^2+4a\sqrt{\delta(v)}+4\delta(v)$$

It is clear that $\delta(v)=1$ which contradict with $\delta(v)\geq 2$, so this assumption is wrong. Hence, $a^2+4a+4\delta(v)$ is not perfect square. Therefore, γ_{p+1} are not all integers. ■

Theorem 4. Let $\mathcal{H}$ is hyper-regular hypergraphs $C(k,l,n,m)$ for $l=1$. The Laplacian spectrum of $\mathcal{H}$ is

$$Spec(\mathcal{H})=\begin{pmatrix} 1/2(-\sqrt{a}+2k+3) & n-1 \\ 1/2(\sqrt{a}+2k+3) & n-1 \\ 2 & m-n \\ 1/2(-\sqrt{b}+2k-n+3) & 1 \\ 1/2(\sqrt{b}+2k-n+3) & 1 \end{pmatrix}$$

where $a=4k^2-3$ dan $b=(-2k+n-3)^2-4(3k-3n+3)$.

Proof. The Laplacian matrix of hyper-regular hypergraphs $C(k,1,n,m)$ is a real block matrix $\mathcal{L}$ with size $(n+m)\times(n+m)$ and order m as follows,

$$\mathcal{L} = \begin{pmatrix} A_{n\times n} & B_{n\times m} \\ C_{m\times n} & D_{m\times m} \end{pmatrix}.$$

with characteristic equation $P(\mu) = det(\mu I - \mathcal{L})$=0, where μ is an eigenvalue of $\mathcal{L}$.

$$\mu I - \mathcal{L} = \begin{pmatrix} \mu I & 0 \\ 0 & \mu I \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mu I - A & -B \\ -C & \mu I - D \end{pmatrix}$$

Let,

$$\begin{aligned}
\mathbf{E} &= (\mu I - A) = (\mu - 2k - 1)I_{n\times n} + J_{n\times n} \\
\mathbf{F} &= -B = N_{n\times m} \\
\mathbf{G} &= -C = N^T_{m\times n} \\
\mathbf{H} &= (\mu I - D) = (\mu - 2)I_{m\times m}
\end{aligned}$$

By *Schur-Complements* we have,

$$\begin{aligned}
P(\mu) &= det \begin{pmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{pmatrix} \\
&= det \begin{pmatrix} I & \mathbf{F}\mathbf{H}^{-1} \\ 0 & I \end{pmatrix} det \begin{pmatrix} \mathbf{E} - \mathbf{F}\mathbf{H}^{-1} & 0 \\ \mathbf{G} & \mathbf{H} \end{pmatrix} \\
&= det(I)det(I)det(\mathbf{E} - \mathbf{F}\mathbf{H}^{-1}\mathbf{G})det(\mathbf{H}).
\end{aligned}$$

From the form above we will find each determinant,

1. $det(I) = 1$
2. $det(\mathbf{H}) = det((\mu - 2)I) = (\mu - 2)^m$
3. To find $det(\mathbf{E} - \mathbf{F}\mathbf{H}^{-1}\mathbf{G})$ first we find the form of $\mathbf{E} - \mathbf{F}\mathbf{H}^{-1}\mathbf{G}$, obtained

$$\begin{aligned}
\mathbf{H}^{-1} &= \left(\frac{1}{\mu - 2}\right) I \\
\mathbf{F}\mathbf{H}^{-1}\mathbf{G} &= N_{n\times m} \cdot \left(\frac{1}{\mu - 2}\right) I \cdot N^T_{m\times n} \\
&= \frac{1}{\mu - 2} \cdot N_{n\times m} \cdot N^T_{m\times n} \\
&= \frac{1}{\mu - 2} \cdot (A(\Gamma)_{n\times n} + D(\Gamma)_{n\times n}) \\
&= \frac{1}{\mu - 2} \cdot ((k+1)I_{n\times n} - J_{n\times n}) \\
&= \left(\frac{k+1}{\mu - 2}\right) I_{n\times n} - \left(\frac{1}{\mu - 2}\right) J_{n\times n}, \mu \neq 2 \\
\mathbf{E} - \mathbf{F}\mathbf{H}^{-1}\mathbf{G} &= (\mu - 2k - 1)\, I_{n\times n} + J_{n\times n} - \left(\left(\frac{k+1}{\mu - 2}\right) I_{n\times n} - \left(\frac{1}{\mu - 2}\right) J_{n\times n}\right) \\
&= \left(\mu - 2k - 1 - \frac{k+1}{\mu - 2}\right) I_{n\times n} + \left(1 + \frac{1}{\mu - 2}\right) J_{n\times n} \\
&= \left(\frac{\mu^2 - 2\mu k - 3\mu + 3k + 1}{\mu - 2}\right) I_{n\times n} + \left(\frac{\mu - 1}{\mu - 2}\right) J_{n\times n}.
\end{aligned}$$

By *Rank-One Updates*, $det(\mathbf{E} - \mathbf{F}\mathbf{H}^{-1}\mathbf{G})$
Let $A_1 = \left(\dfrac{\mu^2 - 2\mu k - 3\mu + 3k + 1}{\mu - 2}\right), pq^t = yJ$, where

$$y = \left(\frac{\mu - 1}{\mu - 2}\right), p = \begin{pmatrix} y \\ \vdots \\ y \end{pmatrix}, q = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

then, $det(\mathbf{E} - \mathbf{F}\mathbf{H}^{-1}\mathbf{G}) = det(A_1)(1 + q^t A_1^{-1} p)$.
obtained,
$det(\mathbf{E}-\mathbf{F}\mathbf{H}^{-1}\mathbf{G}) = \left(\dfrac{(\mu - 2k - 1)(\mu - 2) - (k - 1)}{\mu - 2}\right)^n \left(1 + \dfrac{n(\mu - 1)}{(\mu - 2k - 1)(\mu - 2) - (k - 1)}\right)$

$$\begin{aligned} P(\mu) &= det(\mathbf{H}) \cdot det(\mathbf{E} - \mathbf{F}\mathbf{H}^{-1}\mathbf{G}) \\ &= (\mu - 2)^m \cdot \left(\frac{(\mu - 2k - 1)(\mu - 2) - (k - 1)}{\mu - 2}\right)^n \left(1 + \frac{n(\mu - 1)}{(\mu - 2k - 1)(\mu - 2) - (k - 1)}\right) \\ &= 0 \end{aligned}$$

By simplifying $det(\mathbf{H})$ and $det(\mathbf{E} - \mathbf{F}\mathbf{H}^{-1}\mathbf{G})$, the characteristic polynomial of hyper-reguler $C(k, 1, n, m)$ is

$$P(\mu) = (\mu - 2)^{m-n}(\mu - 1/2(-\sqrt{a} + 2k + 3))^{n-1}(\mu - 1/2(\sqrt{a} + 2k + 3)^{n-1}(\mu - 1/2(-\sqrt{b} + 2k - n + 3))(\mu - 1/2(\sqrt{b} + 2k - n + 3)) = 0$$

where $a = 4k^2 - 3$ dan $b = (-2k + n - 3)^2 - 4(3k - 3n + 3)$. ∎

Corollary 3. Hyper-regular hypergraph $C(k, 1, n, m)$ is not Laplacian integral.

Proof. To see whether hyper-regular $C(k, 1, n, m)$ is Laplacian integral hypergraphs or not, we can observe from a or b in the Laplacian spectrum above whether is perfect square or not. Suppose we observe a, it is clear that a is not a perfect square because the constant in a is not square number. ∎

Theorem 5. Let $\mathcal{H}$ be a hyper-regular hypergraph $C(k, l, n, m)$ and $r - uniform$ hypergraph ($r = n + lm$) with Laplacian matrix $\mathcal{L}$ and algebraic connectivity $\mu_2(\mathcal{H})$, then for any two non-adjacent vertices s and t we have,

$$\mu_2(\mathcal{H}) \leq \tfrac{r-1}{2}(deg(s) + deg(t))$$

Proof. By Definition, we have

$$\mu_2(\mathcal{H}) \leq \frac{\mathcal{L}u \cdot u}{u \cdot u}$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n+lm} \end{pmatrix}$$

vector u defined as,

$$u_x = \begin{cases} 1, & x = s \\ -1, & x = t \\ 0, & \text{others} . \end{cases}$$

Thus we have,

$$\mathcal{L}u \cdot u = u^t \begin{pmatrix} \delta_{L1} I & 0 \\ 0 & \delta_{L2} I \end{pmatrix} u - u^t \begin{pmatrix} a_{11} & \dots & a_{(1)(n+lm)} \\ \vdots & \ddots & \vdots \\ a_{(n+lm)(1)} & \dots & a_{(n+lm)(n+lm)} \end{pmatrix} u$$

and

$$u \cdot u = \sum_{x \in V} u_x^2.$$

Let

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}, u^t = \begin{pmatrix} u_A & u_B \end{pmatrix}$$

where

$$u_A = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, u_B = \begin{pmatrix} u_{n+1} \\ u_{n+2} \\ \vdots \\ u_{n+lm} \end{pmatrix}$$

obtained

$$\mathcal{L}u \cdot u = \sum_{i=1}^{n} u_i^2 \delta_{L1} + \sum_{i=n+1}^{n+lm} u_i^2 \delta_{L2} - (\sum_{i=1}^{n+lm} u_i a_{i1} u_1 + \sum_{i=1}^{n+lm} u_i a_{i2} u_2 + \dots + \sum_{i=1}^{n+lm} u_i a_{in} u_n).$$

where δ_L is the sum of elements in each row of its adjacency matrix then,

$$\delta_{L1} = \sum_{j=1}^{n} a_{ij} \text{ dan } \delta_{L2} = \sum_{j=n+1}^{n+lm} a_{ij}$$

we have,

$$\mathcal{L}u \cdot u = \sum_{j=1}^{n+lm} a_{ij} u_j^2 - \sum_{j=1}^{n+lm} (\sum_{i=1}^{n+lm} a_{ij} u_i) u_j.$$

Thus

$$\begin{aligned} \frac{\mathcal{L}u \cdot u}{u \cdot u} &= \frac{\sum_{\{x,y\} \subset E_j} a_{xy} (u_x - u_y)^2}{\sum_{x \in V} u_x^2} \\ &= \frac{(r-1)(\sum_x a_{xs} + \sum_x a_{tx})}{2} \\ &= \frac{r-1}{2} (deg(s) + deg(t)). \end{aligned}$$

■

Definition 3. Let $\mathcal{H}$ be hyper-regular hypergraph $C(k, l, n, m)$. A $\bullet$ defined as an operation that attaches $\mathcal{H}$ to $\mathcal{H}$ at one vertex on $\mathcal{H}$. This operation $\bullet$ has condition, if $\mathcal{H} \bullet \mathcal{H} \bullet \cdots \bullet \mathcal{H}$ is operated w times, then every one vertex that attaches among two $\mathcal{H}$ is must be different.

Example. Let $\mathcal{H}$ is hyper-regular $C(3, 1, 4, 6)$, $\mathcal{H}' = \mathcal{H} \bullet \mathcal{H}$ can be represented as;

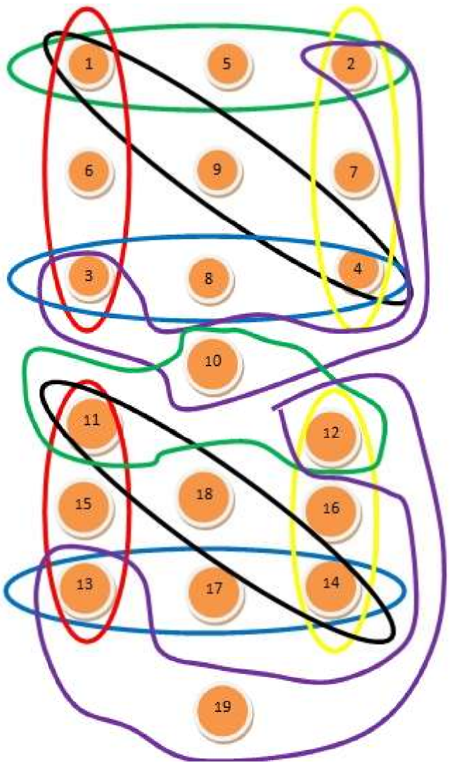

Figure 3: Hypergraph $\mathcal{H}' = \mathcal{H} \bullet \mathcal{H}$

Theorem 6. Let $\mathcal{H}$ is hyper-regular hypergraph s $C(k, l, n, m)$. If $\mathcal{H}' = \mathcal{H} \bullet \mathcal{H}$ then,

$$\mu_2(\mathcal{H}') \leq 2\mu_2(\mathcal{H})$$

Proof. By theorem 5.

$$\mu_2(\mathcal{H}) \leq \frac{r-1}{2}(deg(s) + deg(t))$$

Hyper-regular hypergraph $\mathcal{H} = C(k, l, n, m)$ consist of n main vertex and l additional vertex. For any $x \in V(\mathcal{H})$, where $x = n$, and $\delta(x) = d_1$, and for $y \in V(\mathcal{H})$, where $y = l$, and $\delta(y) = d_2$ with $d_1 > d_2$ then, for any non adjacent x and y, the sum of their degree satisfy;

$$\begin{aligned}
\delta(x) + \delta(y) &= d_2 + d_2 = 2d_2, \text{ or} \\
\delta(x) + \delta(y) &= d_1 + d_2, \text{ or} \\
\delta(x) + \delta(y) &= d_2 + d_1
\end{aligned}$$

thus,

$$\begin{aligned}
\mu_2(\mathcal{H}) &\leq \frac{r-1}{2}(2d_2), \text{ and} \\
\mu_2(\mathcal{H}) &\leq \frac{r-1}{2}(d_1 + d_2), \text{ and} \\
\mu_2(\mathcal{H}) &\leq \frac{r-1}{2}(d_2 + d_1).
\end{aligned}$$

because $d_1 > d_2$ we have,

$$\mu_2(\mathcal{H}) \leq \frac{r-1}{2}(2d_2).$$

$\mathcal{H}'$ is a $\mathcal{H} \bullet \mathcal{H}$ hypergraph. Let z is an attached vertex from $\mathcal{H}$ to $\mathcal{H}$ and d_3 is the degree of z, then d_3 satisfy,

$$\begin{aligned}
d_3 &= d_1 + d_1 = 2d_1, \text{ or} \\
d_3 &= d_1 + d_2, \text{ or} \\
d_3 &= d_2 + d_2 = 2d_2,
\end{aligned}$$

For any two non adjacent vertex in $\mathcal{H}'$ satisfy,

$$\begin{aligned}
\mu_2(\mathcal{H}') &\leq \frac{r-1}{2}(2d_1), \text{ and} \\
\mu_2(\mathcal{H}') &\leq \frac{r-1}{2}(2d_2), \text{ and} \\
\mu_2(\mathcal{H}') &\leq \frac{r-1}{2}(d_1 + d_2), \text{ and} \\
\mu_2(\mathcal{H}') &\leq \frac{r-1}{2}(d_1 + d_3), \text{ and} \\
\mu_2(\mathcal{H}') &\leq \frac{r-1}{2}(d_2 + d_3),
\end{aligned}$$

because $d_1 > d_2$ then,

$$\mu_2(\mathcal{H}') \leq \frac{r-1}{2}(2d_2)$$

it is clear that $(r-1)(d_2) < 2(r-1)(d_2)$, so we have

$$\mu_2(\mathcal{H}') \leq 2\mu_2(\mathcal{H}).$$

■

Corollary 4. Let $\mathcal{H}$ is hype-regular hypergraph $C(k, l, n, m)$. If $\mathcal{H}'$ is $\mathcal{H} \bullet \mathcal{H} \ldots \bullet \mathcal{H}$ operated w times, then the algebraic connectivity of $\mathcal{H}'$ satisfy,

$$\mu_2(\mathcal{H}') \leq 2\mu_2(\mathcal{H})$$

Proof To prove, we only see that the degree of any vertex in $\mathcal{H}'$ where the degree of $x \in V(\mathcal{H})$, the degree of $y \in V(\mathcal{H})$, and the degree of z which attached among $\mathcal{H}$ has the same composition, thus fo any two non adjacent vertex we have,

$$\mu_2(\mathcal{H}') \leq 2\mu_2(\mathcal{H}).$$

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