DEEP AND SKELETON ENUMERATION OF YOUNG TABLEAUX

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ABSTRACT. We introduce a new family of polynomials, *crystal skeleton polynomials*, to better understand enumeration of standard Young tableaux, quasi-Yamanouchi tableaux and interactions with Gessel's expansion of a Schur function, quasi-crystals and crystal skeletons as Maas-Gariépy studied in 2023.

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1. Introduction

Enumeration of *standard Young tableaux* (SYT) is a classical topic in algebraic combinatorics such as hook-length formulas, Specht modules, reduced words of permutations, to name a few. The goal of this extended abstract is to introduce a new family of polynomials, *crystal skeleton polynomials*, to better understand such enumeration. First, we explain our motivation briefly.

Let $X = (x_1, x_2, \dots)$ denote a set of infinite variables and $g(X) \in \mathbf{C}[[X]]$. Suppose $g(X)$ is homogeneous of degree n. Say g is *F-positive* if

$$g(X) = \sum_{\alpha \models n} c_\alpha F_\alpha(X), \quad c_\alpha \in \mathbf{Z}_{\geq 0}$$

where α is a strong composition of n and F_α a fundamental quasi-symmetric function. It is a basic question to ask when an F-positive function is Schur-positive.

A simpler approach.
When is an F-positive function a *single* Schur function?

Gessel found the following expansion [7]

$$s_\lambda(X) = \sum_{T \in \mathrm{ST}(\lambda)} F_{\operatorname{des} T}(X)$$

where $\mathrm{ST}(\lambda)$ denotes the set of all SYTs of shape λ and $\operatorname{des} T$ is the descent composition of T (not descent set). Assaf–Searles [2] improved it a bit in terms of *quasi-Yamanouchi tableaux* under polynomial setting. Moreover, Assaf [1] gave axioms of a *dual equivalence graph* which characterizes such a sum, roughly speaking. It originally goes back to Haiman [9]. See also [6, 8, 11, 12] on F- and Schur-positivity.

Let us now call a multiset $\mathrm{Sch}(\lambda) := \{\operatorname{des} T \mid T \in \mathrm{ST}(\lambda)\}$ the *Schur family* for λ. We aim to understand a larger collection $\mathbf{Sch} = \{\mathrm{Sch}(\lambda) \mid \lambda \in \mathrm{Par}\}$ rather globally. Our framework fits in with the brand new theory of a *crystal skeleton* which Maas-Gariépy [10] introduced only in 2023. Afterward, Cain–Malheiro–Rodrigues–Rodrigues [5] and Brauner–Corteel–Daugherty–Schilling [4] (both 2025) follow her project.

Remark 1.1. We assume familiarity with basic ideas on Young tableaux; for undefined terms, see [13]. Throughout, a fundamental quasi-symmetric function and a descent *composition* are key concepts [4]. Below, λ is always a partition, $n = |\lambda|$ and α is a strong composition of size n. We sometimes use shorthand notation $\alpha = \alpha_1 \dots \alpha_l$ to mean $\alpha = (\alpha_1, \dots, \alpha_l)$.

2. Skeleton polynomials

2.1. **Quasi-crystal, crystal skeleton.** Our discussion proceeds with $\mathbf{B}(\lambda) = (\mathrm{SSYT}(\lambda) \sqcup \{0\}, \boldsymbol{e}_i, \mathbf{f}_i, \to)$, a crystal graph of type A. By T^{st} for $T \in \mathbf{B}(\lambda)$, we mean its standardization.

Definition 2.1. Say $T, U \in \mathbf{B}(\lambda)$ are *standard equivalent* ($T \sim U$) if $T^{st} = U^{st}$. A *quasi-crystal* is an equivalent class of this equivalent relation on $\mathbf{B}(\lambda)$. A *crystal skeleton* $\mathrm{CS}(\lambda)$ is the quotient set $\mathbf{B}(\lambda)/{\sim}$.

Remark 2.2. For here, treat $\mathrm{CS}(\lambda)$ just as a set. We can however impose a graph structure with certain vertex- or edge-labeling and direction of edges [4, 5, 10].

Fact 2.3 ([10])**.** Each quasi-crystal in $\mathbf{B}(\lambda)$ forms a connected subgraph. Its generating function is single $F_\alpha(X)$ for some $\alpha \models |\lambda|$.

Let us now clarify a deeper interpretation of Gessel expansion [10] with little more new words. If $T \sim U$ in $\mathbf{B}(\lambda)$, then $\operatorname{des} T = \operatorname{des} U$ as we can check. Therefore, the descent composition for a quasi-crystal is well-defined. Say quasi-crystals $\mathbf{Q}, \mathbf{R} \subseteq \mathbf{B}(\lambda)$ are *descent equivalent* if $\operatorname{des} \mathbf{Q} = \operatorname{des} \mathbf{R}$. In fact, such quasi-crystals have the identical generating function.

Definition 2.4. A *fundamental system* for α in $\mathbf{B}(\lambda)$ is

$$\mathbf{F}(\lambda, \alpha) = \bigsqcup \mathbf{Q}$$

where $\mathbf{Q}$ runs all quasi-crystals in $\mathbf{B}(\lambda)$ such that $\operatorname{des} \mathbf{Q} = \alpha$.

As a consequence, $\mathbf{B}(\lambda)$ decomposes into a union of such $\mathbf{F}(\lambda, \alpha)$'s.

2.2. **Kostka and quasi-Kostka numbers.**

Definition 2.5 (Kostka numbers)**.** For λ and a weak composition a of size $|\lambda|$, let

$$\mathrm{SSYT}(\lambda, a) = \{T \in \mathrm{SSYT}(\lambda) \mid \operatorname{wt} T = a\} \quad \text{and} \quad K_{\lambda a} = |\mathrm{SSYT}(\lambda, a)|.$$

Definition 2.6 (quasi-Kostka numbers)**.** For λ and a strong composition α of size $|\lambda|$, let

$$\mathrm{ST}(\lambda, \alpha) = \{T \in \mathrm{ST}(\lambda) \mid \operatorname{des} T = \alpha\} \quad \text{and} \quad f_{\lambda\alpha} = |\mathrm{ST}(\lambda, \alpha)|.$$

Remark 2.7.

(1) Recall the traditional notation $f_\lambda = |\mathrm{ST}(\lambda)|$ and so $f_\lambda = \sum_\alpha f_{\lambda\alpha}$.
(2) The definition of $f_{\lambda\alpha}$ implies $\sum_{T \in \mathbf{F}(\lambda,\alpha)} x^{\operatorname{wt} T} = f_{\lambda\alpha} F_\alpha(X)$.
(3) Egge–Loehr–Warrington [6] and Orellana–Saliola–Schilling–Zabrocki [11] discussed such numbers as entries of certain transition matrices.

Definition 2.8. Say $T \in \mathrm{SSYT}(\lambda)$ is *quasi-Yamanouchi* if $\operatorname{des} T = \operatorname{wt} T$. Denote the set of all such tableaux by $\mathrm{QY}(\lambda)$.

There is the canonical identification $\mathrm{QY}(\lambda) \cong \mathrm{ST}(\lambda)$: standardization $T \mapsto T^{st}$, and destandardization $T \mapsto T^{dst}$ are des-preserving bijections and the inverse map of each other. Hence

$$f_{\lambda\alpha} = |\{T \in \mathrm{QY}(\lambda) \mid \operatorname{des} T = \alpha\}| = |\{T \in \mathrm{QY}(\lambda) \mid \operatorname{wt} T = \alpha\}| \leq K_{\lambda\alpha}.$$

Observe that $0 \le f_{\lambda\lambda} \le K_{\lambda\lambda} = 1$ and the superstandard tableau $T(\lambda)$ ($\operatorname{shape} T(\lambda) = \operatorname{wt} T(\lambda) = \lambda$) satisfies $\operatorname{des} T(\lambda) = \operatorname{wt} T(\lambda) = \lambda$. We thus showed that $f_{\lambda\lambda} = 1$ for all λ.

Example 2.9. SYTs of shape $\lambda = 332$ of descent $\alpha = 12221$ are only these:

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 7 \\ \hline 2 & 5 & 8 \\ \hline 4 & 6 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 & 7 \\ \hline 4 & 8 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 7 \\ \hline 6 & 8 \\ \cline{1-2}\end{array}$$

Thus, $f_{\lambda\alpha} = 3$.

2.3. **Skeleton polynomials.** For a composition α, denote its length by $\ell(\alpha)$. In particular, for a partition λ, $\ell(\lambda)$ is the number of rows of λ. Let $m(\lambda) = \max\{\ell(\operatorname{des} T) \mid T \in \mathrm{ST}(\lambda)\}$.

Lemma 2.10 ([14, Lemma 3.1])**.** $m(\lambda) = n - (\lambda_1 - 1)$ for $\lambda \neq \emptyset$.

By x^α we mean a monomial $x_1^{\alpha_1} \cdots x_{\ell(\alpha)}^{\alpha_{\ell(\alpha)}}$.

Definition 2.11. The *skeleton polynomial* for λ is

$$\mathrm{Sk}_\lambda(x_1, \dots, x_{m(\lambda)}) = \sum_{T \in \mathrm{CS}(\lambda)} x^{\operatorname{des} T}.$$

This is a homogeneous polynomial of degree $|\lambda|$ with nonnegative integer coefficients. It has $m(\lambda)$ variables. However, if no confusion arises, we often abbreviate them just as x. In particular, $\mathrm{Sk}_\emptyset = 1$ and $\mathrm{Sk}_\lambda(1, \dots, 1) = f_\lambda$.

Remark 2.12. We may regard $\mathrm{Sk}_\lambda(x)$ as the generating function of $\mathrm{QY}(\lambda)$ ($\subseteq \mathrm{SSYT}(\lambda)$):

$$\mathrm{Sk}_\lambda(x) = \sum_{\alpha \models n} f_{\lambda\alpha} x^\alpha. \qquad \left(\text{cf.} \quad s_\lambda(X) = \sum_{\alpha \models n} f_{\lambda\alpha} F_\alpha(X)\right)$$

For this reason, we call $(f_{\lambda\alpha})$ *skeleton coefficients*.

Example 2.13. Table 1 shows $\mathrm{Sk}_\lambda(x)$ for $|\lambda| \le 4$.

The cases for one-row and one-column shapes are $\mathrm{Sk}_{(c)}(x) = x_1^c$, $\mathrm{Sk}_{(1^r)}(x) = x_1 \cdots x_r$. Note that $s_\lambda(X)$ is symmetric while $\mathrm{Sk}_\lambda(x)$ is not necessarily. However, as we see, it has subtle symmetry on polynomials with *distinct* number of variables.

For $\ell(\lambda) \le i \le m(\lambda)$, define *$i$-skeleton polynomial* for λ

$$\mathrm{Sk}_{\lambda,i}(x) = \sum_{T \in \mathrm{ST}(\lambda,\alpha), \ell(\alpha) = i} x^\alpha$$

as an i-variable one. For all other i, set $\mathrm{Sk}_{\lambda,i}(x) = 0$ for convenience.

Proposition 2.14. If $\ell(\lambda) \le i \le m(\lambda)$, then $\mathrm{Sk}_{\lambda,i}(x) \neq 0$.

Sketch of a proof. An undirected crystal skeleton graph $\mathrm{CS}(\lambda)$ is connected and if T, U are adjacent vertices, then $\ell(\operatorname{des} T) - \ell(\operatorname{des} U) \in \{-1, 0, 1\}$ [4, Theorem 4.26]. $\square$

Example 2.15. $\mathrm{QY}(3,2)$ consists of the following five tableaux

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&2\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2&3\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2&3\\\cline{1-2}\end{array}$$

so that $\mathrm{Sk}_{32,2}(x) = x^{32} + x^{23}$ and $\mathrm{Sk}_{32,3}(x) = x^{221} + x^{131} + x^{122}$.

Definition 2.16. For an i-variable polynomial $g(x_1, \ldots, x_i)$, define its reversal as

$$g^*(x_1, x_2, \ldots, x_i) = g(x_i, x_{i-1}, \ldots, x_1).$$

Theorem 2.17. If $\ell(\lambda) \leq i \leq m(\lambda)$, then $\mathrm{Sk}^*_{\lambda,i}(x) = \mathrm{Sk}_{\lambda,i}(x)$.

We will prove this with Theorem 2.21 together.

2.4. **Evacuation.**

Definition 2.18 ([4, Remark 2.5])**.** For $T \in \mathbf{B}(\lambda)$, let $w = \operatorname{row} T = w_1 \cdots w_n$ ($n = |\lambda|$) be its row word (reading from bottom to top, left to right). Let $w^* = (n+1-w_n)\cdots(n+1-w_1)$ and define the *evacuation* (Lusztig/Schützenberger involution) of T as $T^* = P(w^*)$.

Example 2.19. Observe that

$$T = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 3&4\\\cline{1-2} 4\\\cline{1-1}\end{array}, w = \operatorname{row} T = 4341112, w^* = 6777454, T^* = P(w^*) = \begin{array}{|c|c|c|c|}\hline 4&4&7&7\\\hline 5&7\\\cline{1-2} 6\\\cline{1-1}\end{array}$$

with $\operatorname{des} T = 412$, $\operatorname{des}(T^*) = 214$.

For $\alpha = (\alpha_1, \ldots, \alpha_l) \models n$, let $\alpha^* = (\alpha_l, \ldots, \alpha_1)$ denote its reversal.

Lemma 2.20 (See [4, Section 4.4])**.** Let $T \in \mathbf{B}(\lambda)$.

(1) $\operatorname{des}(T^*) = (\operatorname{des} T)^*$. In particular, $\ell(\operatorname{des}(T^*)) = \ell(\operatorname{des} T)$.

(2) Let $\mathbf{B}_N(\lambda)$ (for $N \geq |\lambda|$) be a bounded crystal graph with all SSYTs of entries at most N. Then $* : \mathbf{B}_n(\lambda) \to \mathbf{B}_n(\lambda)$ ($n = |\lambda|$) is an involutive crystal anti-isomorphism, that is,

$$(T^*)^* = T \quad \text{and} \quad T \underset{i}{\to} U \iff U^* \underset{n-i}{\longrightarrow} T^* \quad (1 \leq i \leq n-1)$$

where $T \underset{i}{\to} U$ means $U = \mathbf{f}_i(T)$, $T, U \neq 0$. In particular, it restricts to an involutive bijection on $\mathrm{ST}(\lambda)$.

Theorem 2.21. For all λ, α, we have $f_{\lambda\alpha} = f_{\lambda\alpha^*}$. As a consequence, Theorem 2.17 holds.

2.5. **Inner crystal.** Note that $\ell(\lambda) = \min\{\ell(\operatorname{des} T) \mid T \in \mathrm{ST}(\lambda)\}$ because the superstandard tableau $T(\lambda)$ satisfies $\ell(\operatorname{des} T(\lambda)) = \ell(\operatorname{wt} T(\lambda)) = \ell(\lambda)$.

Definition 2.22. Define the *inner crystal* of $\mathbf{B}(\lambda)$ as

$$\mathbf{A}(\lambda) = \{T \in \mathrm{CS}(\lambda) \mid \ell(\operatorname{des} T) = \ell(\lambda)\}.$$

There is a specific reason to call this a "crystal".

Theorem 2.23 (See [4, Theorem 4.37])**.** After arranging vertices and edges appropriately, $\mathbf{A}(\lambda)$ is isomorphic to $\mathbf{B}_{\ell(\lambda)}(\lambda)$ as a crystal graph.

Remark 2.24. We can say that the inner crystal of $\mathbf{B}_N(\lambda)$ is also $\mathbf{B}_{\ell(\lambda)}(\lambda)$. Inclusions

$$\mathbf{A}(\lambda) = \mathbf{B}_{\ell(\lambda)}(\lambda) \subseteq \mathrm{CS}(\lambda) \subseteq \mathbf{B}_N(\lambda) \subseteq \mathbf{B}(\lambda)$$

indicate "saturated sub-crystal property" among ambient crystals of shape λ.

Definition 2.25. The *inner crystal polynomial* for λ is $A_\lambda(x) = \mathrm{Sk}_{\lambda,\ell(\lambda)}(x)$.

Theorem 2.26. $A_\lambda(x) = s_\lambda(x_1, \ldots, x_{\ell(\lambda)})$. In particular, this is symmetric.

Example 2.27. $\mathbf{B}_3\left(\yng(2,1)\right)$ consists of two quasi-crystals.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{2} \\ \hline \mathbf{3} & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{3} \\ \hline \mathbf{2} & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$$

Two standard tableaux forms its skeleton and the inner crystal. It is easier to see the edge between them after destandardization:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \ \underset{1}{\longrightarrow}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \quad \left(\cong \mathbf{B}_2\left(\yng(2,1)\right)\right)$$

3. Calculus of Skeleton polynomials

Rather than studying a solo skeleton polynomial, it is desirable to investigate interactions over a family of those. We confirm several basic facts in this section.

3.1. **Linear indepenency.** For $\lambda, \mu \in \mathrm{Par}(n)$, define the *dominance order* $\lambda \geq \mu$ if

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$$

for all i. Extend this order onto $\mathrm{Comp}(n)$, the set of all strong compositions of size n.

Lemma 3.1. If $\lambda \not\geq \alpha$, then $f_{\lambda\alpha} = 0$. As a consequence, $\mathrm{Sk}_\lambda(x) = x^\lambda + \sum_{\alpha<\lambda} f_{\lambda\alpha} x^\alpha$.

Proof. This follows from $f_{\lambda\lambda} = 1$, $0 \leq f_{\lambda\alpha} \leq K_{\lambda\alpha}$ and $K_{\lambda\alpha} = 0$ if $\lambda \not\geq \alpha$. □

Theorem 3.2. $(\mathrm{Sk}_\lambda(x) \mid \lambda \vdash n)$ is linearly independent in $\mathbf{C}[x_1, \ldots, x_n]$. In particular, $\mathrm{Sk}_\lambda(x) = \mathrm{Sk}_\mu(x) \Longrightarrow \lambda = \mu$.

Proof. Let $N = |\mathrm{Par}(n)|$ and choose a sequence $\lambda^{(1)}, \ldots, \lambda^{(N)}$ from $\mathrm{Par}(n)$ such that $\lambda^{(i)}$ is maximal in the subposet $\{\lambda^{(i)}, \ldots, \lambda^{(N)}\}$. To prove the linear independency, suppose $\sum_{i=1}^{N} c_i \mathrm{Sk}_{\lambda^{(i)}}(x) = 0, \quad c_i \in \mathbf{C}$. Thanks to Lemma 3.1, the coefficient of $x^{\lambda^{(1)}}$ comes from only $\mathrm{Sk}_{\lambda^{(1)}}(x)$, that is,

$$c_1 x^{\lambda^{(1)}} + \Big(\text{ all other terms without } x^{\lambda^{(1)}}\Big) = 0.$$

Thus, c_1 must be 0. Similarly, we can show $c_2 = \cdots = c_N = 0$. □

3.2. **Skeleton RSK.** Recall the classical *RSK correspondence*

$$\mathbf{K}_n \to \bigsqcup_{\lambda \vdash n} \mathrm{SSYT}(\lambda) \times \mathrm{SSYT}(\lambda), \quad w \mapsto (P(w), Q(w))$$

where $\mathbf{K}_n$ is the set of all Knuth arrays (lexicographic biwords) of length n. According to Stanley [13, p.419], we have

$$\sum_{\lambda \vdash n} s_\lambda(X) s_\lambda(Y) = \sum_{w \in S_n} F_{\operatorname{des} w^{-1}}(X) F_{\operatorname{des} w}(Y)$$

as $\operatorname{des} w^{-1} = \operatorname{des} P(w), \operatorname{des} w = \operatorname{des} Q(w)$. Moreover, it restricts to each of the following bijections

$$S_n^{\mathrm{invol}} \to \bigsqcup_{\lambda \vdash n} \mathrm{ST}(\lambda), \quad w \mapsto P(w),$$

$$S_n \to \bigsqcup_{\lambda \vdash n} \mathrm{ST}(\lambda) \times \mathrm{ST}(\lambda), \quad w \mapsto (P(w), Q(w)),$$

$$\mathbf{N}^n \to \bigsqcup_{\lambda \vdash n} \mathrm{SSYT}(\lambda) \times \mathrm{ST}(\lambda), \quad w \mapsto (P(w), Q(w))$$

where S_n^{invol} is the set of all involutions in S_n and $\mathbf{N}^n$ the set of all words of length n.

Theorem 3.3 (Skeleton R, RS correspondences)**.**

$$\sum_{\lambda \vdash n} \mathrm{Sk}_\lambda(x) = \sum_{w \in S_n^{\mathrm{invol}}} x^{\operatorname{des} w}.$$

$$\sum_{\lambda \vdash n} \mathrm{Sk}_\lambda(x) \mathrm{Sk}_\lambda(y) = \sum_{w \in S_n} x^{\operatorname{des}(w^{-1})} y^{\operatorname{des} w}.$$

$$\sum_{\lambda \vdash n} s_\lambda(X) \mathrm{Sk}_\lambda(y) = \sum_{w \in S_n} F_{\operatorname{des}(w^{-1})}(X) y^{\operatorname{des} w}.$$

Proof. Each proof of these is quite similar. Here we prove only the second identity.

$$\mathrm{RHS} = \sum_{\lambda \vdash n} \sum_{(S,T) \in \mathrm{ST}(\lambda) \times \mathrm{ST}(\lambda)} x^{\operatorname{des} S} y^{\operatorname{des} T} = \sum_{\lambda \vdash n} \sum_{(S^{\mathrm{dst}}, T^{\mathrm{dst}}) \in \mathrm{QY}(\lambda) \times \mathrm{QY}(\lambda)} x^{\operatorname{des} S^{\mathrm{dst}}} y^{\operatorname{des} T^{dst}}$$

FIGURE 1. (Comp(4), $\leq$)

$$
\begin{array}{ccccccc}
\text{depth} & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

$$
\begin{tikzcd}[arrows=-]
 & & & 211 \arrow[dr] & & & \\
4 \arrow[r] & 31 \arrow[r] & 22 \arrow[ur] \arrow[dr] & & 121 \arrow[r] & 112 \arrow[r] & 1111 \\
 & & & 13 \arrow[ur] & & &
\end{tikzcd}
$$

$$= \sum_{\lambda \vdash n} \left(\sum_{U \in \mathrm{QY}(\lambda)} x^{\operatorname{des} U} \right) \left(\sum_{V \in \mathrm{QY}(\lambda)} y^{\operatorname{des} V} \right) = \mathrm{LHS}.$$

$\square$

Example 3.4. Table 1 shows that

$$\sum_{\lambda \vdash 3} \mathrm{Sk}_\lambda(x)\mathrm{Sk}_\lambda(y) = x_1^3 y_1^3 + (x_1^2 x_2 + x_1 x_2^2)(y_1^2 y_2 + y_1 y_2^2) + x_1 x_2 x_3 y_1 y_2 y_3.$$

4. DEEP CALCULUS

4.1. **Depth.** Finally, we develop "deep calculus" of skeleton/Schur polynomials. For $\alpha \models n$, its *depth* is $\operatorname{dep}(\alpha) = \sum_{i=1}^{\ell(\alpha)} (i-1)\alpha_i$. This is meaningful in our context with $(f_{\lambda\alpha} | \lambda \geq \alpha)$ in the following sense.

Proposition 4.1. The poset $(\mathrm{Comp}(n), \leq, \operatorname{dep})$ is graded with lowest depth 0 and highest $n(n-1)/2$.

Sketch of Proof. Clearly, $\operatorname{dep}(n) = 0$ and $\operatorname{dep}(1^n) = n(n-1)/2$. Any $\alpha \in \mathrm{Comp}(n)$ can be obtained from (n) by a sequence of the following raising/refinement operations:

$$R_i \alpha = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1} + 1, \alpha_{i+2}, \ldots, \alpha_{\ell(\alpha)}) \quad \text{if} \quad \alpha_i \geq 2, \ i \leq \ell(\alpha) - 1,$$
$$R_{\ell(\alpha)} \alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)-1}, \alpha_{\ell(\alpha)} - 1, 1) \quad \text{if} \quad \alpha_{\ell(\alpha)} \geq 2.$$

Check that if $\alpha \neq (1^n)$, there exists at least one i such that $R_i(\alpha)$ is defined. Moreover, $\alpha \rhd R_i \alpha$ (a covering relation in dominance order) and $\operatorname{dep}(R_i\alpha) - \operatorname{dep}(\alpha) = 1$. $\square$

Figure 1 shows an example of such posets. Now for $T \in \mathrm{SSYT}(\lambda)$, define its *depth* as $\operatorname{dep}(T) = \operatorname{dep}(\operatorname{des} T)$. For example, $\operatorname{dep}\left(\begin{array}{|c|c|c|c|} \hline 1 & 1 & 4 & 4 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \right) = \operatorname{dep}(214) = 9.$

It is then natural to think of *deep Schur function* $s_\lambda(X, q) = \sum_{T \in \mathrm{SSYT}(\lambda)} q^{\operatorname{dep}(T)} x^{\operatorname{wt}(T)}$.

Indeed, $\mathrm{dep}(T)$ depends on only $\operatorname{des} T$ so that it really makes sense to talk about deep Gessel expansion and skeleton polynomials

$$s_\lambda(X,q) = \sum_\alpha f_{\lambda\alpha} q^{\mathrm{dep}(\alpha)} F_\alpha(X), \quad \mathrm{Sk}_\lambda(x,q) = \sum_\alpha f_{\lambda\alpha} q^{\mathrm{dep}(\alpha)} x^\alpha \left(= \sum_{T\in\mathrm{ST}(\lambda)} q^{\mathrm{dep} T} x^{\operatorname{des} T} \right).$$

Observe that Example 2.15 extends to

$$\mathrm{Sk}_{\yng(3,2)}(x,q) = q^2x^{32} + q^3x^{23} + q^4x^{221} + q^5x^{131} + q^6x^{122}.$$

We can show q- or (p,q)-analogs of all identities in Theorem 3.3.

4.2. **Major index.** Now is time to explain details on technical twists among our arguments and known results such as *descent sets* and *major index*. Consider two sets $\mathrm{Comp}(n)$ and $2^{[n-1]}$ of both cardinality 2^{n-1}. There is the traditional identification. For $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathrm{Comp}(n)$, define $A(\alpha) = \{a_1, \dots, a_{l-1}\}$ with $a_i = \alpha_1 + \cdots + \alpha_i$. On the other hand, given $A = \{a_1, \dots, a_k\} \in 2^{[n-1]}$, say $a_1 < \cdots < a_k$ $(0 \le k \le n-1)$, define $\alpha(A) = (\alpha_1, \dots, \alpha_{k+1})$ with $\alpha_1 = a_1, \alpha_i = a_i - a_{i-1}$ $(2 \le i \le k)$ and $\alpha_{k+1} = n - a_k$. In fact, $\alpha \mapsto A(\alpha), A \mapsto \alpha(A)$ are inverse maps of each other. To go into more details, introduce a superboolean structure onto $2^{[n-1]}$: for $A, B \in 2^{[n-1]}$, declare $A \lhd B$ if either

- $A \subset B, 1 \notin A, 1 \in B$ and $B \setminus \{1\} = A$ or
- there exists some $a \in A$ such that $a \notin B$, $a+1 \notin A$, $a+1 \in B$ and $B \setminus \{a+1\} = A \setminus \{a\}$.

Define $A \le B$ by the transitive closure of this relation so that $2^{[n-1]}$ forms a poset graded by $\mathrm{maj}(A) = \sum_{a\in A} a$. It is isomorphic to $(\mathrm{Comp}(n), \le, \mathrm{dep})$. However, maps $\alpha \mapsto A(\alpha), A \mapsto \alpha(A)$ are *not* order-preserving nor order-reversing. For example, $\mathrm{dep}(214) = 9$ $(n = 7, \frac{n(n-1)}{2} = 21)$ while $\mathrm{maj}\{2,3\} = 5$. Let us clarify the precise correspondence between depth and major index. For $T \in \mathrm{ST}(\lambda)$, let

$$\mathrm{Des}T = \{i \in [n-1] \mid i+1 \text{ is strictly lower than } i \text{ in } T\}.$$

More generally, for $T \in \mathrm{SSYT}(\lambda)$, define $\mathrm{Des}T = \mathrm{Des}(T^{st})$ and $\mathrm{maj}(T) = \sum_{i\in\mathrm{Des}T} i$. Observe that $\operatorname{des} T = (\alpha_1, \dots, \alpha_l) \iff \mathrm{Des}T = \{\alpha_1, \alpha_1+\alpha_2, \dots, \alpha_1 + \cdots + \alpha_{l-1}\}$.

Lemma 4.2. For each $T \in \mathrm{SSYT}(\lambda)$, $\mathrm{dep}(T) = \mathrm{maj}(T^*)$. (We omit a proof)

It follows from bijectivity of $T \mapsto T^*$ on $\mathrm{ST}(\lambda)$ that

$$\mathrm{Sk}_\lambda(1, \dots, 1, q) = \sum_{T\in\mathrm{ST}(\lambda)} q^{\mathrm{dep}(T)} = \sum_{T\in\mathrm{ST}(\lambda)} q^{\mathrm{maj}(T)} \quad (= f_\lambda(q))$$

known as fake degree polynomials. For example, $f_{22}(q) = q^2 + q^4$. Such a polynomial so rarely (but possibly) has an internal zero in its coefficient sequence as Billey et al. proved [3].

There is another application of Lemma 4.2 to symmetric groups. For $w \in S_n$, define $\operatorname{des} w = \operatorname{des} Q(w)$ and $\operatorname{maj} w = \operatorname{maj} Q(w)$ (Mahonian). Due to the (p,q)-skeleton RSK, we have

$$\sum_{\lambda \vdash n} \mathrm{Sk}(x,p)\mathrm{Sk}_\lambda(y,q) = \sum_{w \in S_n} x^{\operatorname{des}(w^{-1})} p^{\operatorname{dep}(w^{-1})} y^{\operatorname{des}(w)} q^{\operatorname{dep} w}.$$

Letting $x_i = p = y_i = 1$ for all i and Lemma 4.2, it specializes to $[n]_q!$; that is, dep gives another Mahonican statistic. Similarly, $\sum_{\lambda \vdash n} s_\lambda(X,p)\mathrm{Sk}_\lambda(y,q)$ with $p = y_i = 1$ recovers the graded Frobenius character of S_n.

Example 4.3. $\sum_{\lambda \vdash 4} f_\lambda \mathrm{Sk}_\lambda(y,q)$ is

$$\begin{aligned} &y^4 + 3(qy^{31} + q^2y^{22} + q^3y^{13}) + 2(q^2y^{22} + q^4y^{121}) + 3(q^3y^{211} + q^4y^{121} + q^5y^{112}) + q^6y^{1111} \\ &= y^4 + 3y^{31}q + 5y^{22}q^2 + 3(y^{13} + y^{211})q^3 + 5y^{121}q^4 + 3y^{112}q^5 + y^{1111}q^6. \end{aligned}$$

Indeed, $1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6$ is $[4]_q!$.

5. Summary

We introduced a new family of polynomials, crystal skeleton polynomials, to better understand Gessel F-expansion of s_λ, quasi-crystals and crystal skeletons. We also described interactions with q-analogs of our polynomials on depth and major index (we further proved that $\mathrm{charge}(w) = \mathrm{dep}(w^{-1})$). Actually, a part of it was a reformulation of previous results in the related literature. Nonetheless, our arguments connected several ideas among little different topics. One advantage of using descent compositions is that we can deal with partitions and compositions together in the *graded* dominance order.

Our next task is to start a theory of the following algebra

$$\mathbf{S}_n[x,y] = \left\{ \sum_{w \in S_n} a_w \, x^{\operatorname{des} w^{-1}} y^{\operatorname{des} w} \,\middle|\, a_w \in \mathbf{C} \right\}$$

and the subalgebra generated by $\mathrm{Sk}_\lambda(x)\mathrm{Sk}_\lambda(y), \lambda \vdash n$. Note that we can always recover gradings by $x_i \mapsto p^{i-1}x_i, y_i \mapsto q^{i-1}y_i$. A computer experiment suggests that $\dim \mathbf{S}_4[x,y] = 22 < 4!$ because each of pairs $(1324, 3412), (2143, 4231)$ rarely gives identical monomials as $x^{22}y^{22}$ and $x^{121}y^{121}$, respectively. We wish to bring Poincaré polynomials of *Bruhat intervals* in S_n into the discussion.

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TABLE 1. Skeleton polynomials

λ	$\mathrm{QY}(\lambda)$	$\mathrm{Sk}_\lambda(x)$
$\emptyset$	$\emptyset$	1
$\ydiagram{1}$	$\begin{ytableau}1\end{ytableau}$	x_1
$\ydiagram{2}$	$\begin{ytableau}1 & 1\end{ytableau}$	x_1^2
$\ydiagram{1,1}$	$\begin{ytableau}1\\2\end{ytableau}$	x_1x_2
$\ydiagram{3}$	$\begin{ytableau}1 & 1 & 1\end{ytableau}$	x_1^3
$\ydiagram{2,1}$	$\begin{ytableau}1 & 1\\2\end{ytableau}$ $\begin{ytableau}1 & 2\\2\end{ytableau}$	$x_1^2x_1 + x_1x_2^2$
$\ydiagram{1,1,1}$	$\begin{ytableau}1\\2\\3\end{ytableau}$	$x_1x_2x_3$
$\ydiagram{4}$	$\begin{ytableau}1 & 1 & 1 & 1\end{ytableau}$	x_1^4
$\ydiagram{3,1}$	$\begin{ytableau}1 & 1 & 1\\2\end{ytableau}$ $\begin{ytableau}1 & 1 & 2\\2\end{ytableau}$ $\begin{ytableau}1 & 2 & 2\\2\end{ytableau}$	$x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3$
$\ydiagram{2,2}$	$\begin{ytableau}1 & 1\\2 & 2\end{ytableau}$ $\begin{ytableau}1 & 2\\2 & 3\end{ytableau}$	$x_1^2x_2^2 + x_1x_2^2x_3$
$\ydiagram{2,1,1}$	$\begin{ytableau}1 & 1\\2\\3\end{ytableau}$ $\begin{ytableau}1 & 2\\2\\3\end{ytableau}$ $\begin{ytableau}1 & 3\\2\\3\end{ytableau}$	$x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2$
$\ydiagram{4}$	$\begin{ytableau}1\\2\\3\\4\end{ytableau}$	$x_1x_2x_3x_4$