Higher Terwilliger Algebras of Graphs

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1 Introduction

The Terwilliger algebra, introduced by Terwilliger in 1992, is an associative algebra associated with a graph together with a fixed base vertex. It refines the information provided by the adjacency matrix and has proved to be a powerful tool in algebraic graph theory and the theory of association schemes. Despite its strength, the Terwilliger algebra does not always characterize graphs up to isomorphism. In particular, there exist non-isomorphic graphs that admit isomorphic Terwilliger algebras or isomorphic standard modules. This limitation motivates the search for stronger algebraic invariants.

In this paper, we introduce a natural generalization of the Terwilliger algebra, which we call the *higher Terwilliger algebra*. Instead of fixing a single vertex, this construction is defined with respect to a sequence of vertices. By increasing the number of base vertices, we obtain a hierarchy of algebras interpolating between the ordinary Terwilliger algebra and the full matrix algebra. We investigate fundamental properties of higher Terwilliger algebras and their associated standard modules. In particular, we study the isomorphism problem and show that, in the extreme case where the number of base vertices equals the number of vertices of the graph, the corresponding higher Terwilliger algebra completely determines the graph. As an application, we describe the structure of higher Terwilliger algebras associated with hypercube graphs. This work is based on joint research with Tsuyoshi Miezaki, Takuya Saito, and Yousuke Sato.

2 Terwilliger Algebra of Graphs

In this section, we review the definition and basic properties of the Terwilliger algebra of a finite simple graph, formalize the notion of isomorphism of standard modules, prove that such an isomorphism implies cospectrality, and summarize several results for trees that motivate the higher-rank construction introduced later.

Let $G = (X, R)$ be a finite simple graph equipped with the path-length distance function ∂. Fix a vertex $x \in X$. For each nonnegative integer i, define

$$X_i(x) = \{\, y \in X \mid \partial(x, y) = i \,\}.$$

This defines a distance partition of the vertex set with respect to the base vertex x. Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of G. For each i, let $E_i^*(x)$ be the diagonal projection matrix onto the subspace spanned by the characteristic vectors of $X_i(x)$.

Definition 2.1. The $\mathbb{C}$-algebra generated by the adjacency matrix A together with the projection matrices $\{E_i^*(x)\}_i$ is called the *Terwilliger algebra* of G with respect to x, and is denoted by $T(x)$.

The full matrix algebra $\mathrm{Mat}_X(\mathbb{C})$ acts on $V = \mathbb{C}^X$ by left multiplication. By restriction, the Terwilliger algebra $T(x)$ also acts on V. We refer to V as the *standard module* of $T(x)$.

Let G and G' be finite simple graphs with base vertices x and x', respectively. Let $T(x)$ and $T'(x')$ denote their Terwilliger algebras, and let V and V' be the corresponding standard modules. We say that V and V' are *isomorphic as Terwilliger modules* if there exist an algebra isomorphism $\varphi : T(x) \to T'(x')$, satisfying $\varphi(A) = A'$ and $\varphi(E_i^*(x)) = E_i^*(x')$ for all i, and a linear isomorphism $F : V \to V'$ such that

$$F(av) = \varphi(a)\,F(v) \qquad (a \in T(x),\ v \in V).$$

Proposition 2.2. *If the standard modules for $T(x)$ and $T'(x')$ are isomorphic in the above sense, then the adjacency matrices A and A' are similar. In particular, G and G' are cospectral.*

Proof. Taking $a = A$ in the intertwining relation yields $F(Av) = \varphi(A)F(v) = A'F(v)$ for all $v \in V$, that is, $FA = A'F$. Since F is invertible, we obtain $A' = FAF^{-1}$. Thus A and A' are similar and hence have the same characteristic polynomial and eigenvalue multiplicities. □

The limitations of spectral information are particularly apparent for trees. Schwenk proved that almost all trees are cospectral, showing that the adjacency spectrum alone does not determine a tree up to isomorphism [1]. Motivated by this phenomenon, Li–Fan–Ito–Karimi–Xu studied the Terwilliger algebra of a rooted tree $\Gamma^{(x_0)}$ and its principal module, and proved that this module determines the rooted isomorphism class of the tree [2]. From a representation-theoretic viewpoint, Xu–Ito–Li compared $T(x_0)$ with the centralizer algebra of the root-fixing automorphism group, described the irreducible $T(x_0)$-modules explicitly, and characterized when $T(x_0)$ coincides with this centralizer [3]. These results demonstrate that, for trees, Terwilliger-module data captures significantly finer structure than the adjacency spectrum and provide a natural point of comparison for the higher Terwilliger algebras introduced later.

3 Higher Terwilliger Algebras

We now extend the notion of the Terwilliger algebra by fixing several base vertices simultaneously. Throughout this section, let $G = (X, R)$ be a finite simple graph equipped with the path-length distance function ∂ and adjacency matrix A.

Let $\boldsymbol{x} = (x_1, \ldots, x_g) \in X^g$ be an ordered g-tuple of vertices of G. For a vertex $y \in X$, we define the *g-way distance* from $\boldsymbol{x}$ to y by

$$\partial^{(g)}(\boldsymbol{x}, y) = (\partial(x_1, y), \ldots, \partial(x_g, y)).$$

This generalizes the usual distance by simultaneously recording distances from multiple base vertices. For each multi-index $\boldsymbol{i}$, let $E^*_{\boldsymbol{i}}(\boldsymbol{x})$ be the diagonal projection matrix onto the set of vertices whose g-way distance from $\boldsymbol{x}$ equals $\boldsymbol{i}$.

Definition 3.1. The $\mathbb{C}$-algebra generated by the adjacency matrix A together with all projection matrices $E^*_{\boldsymbol{i}}(\boldsymbol{x})$ is called the *degree-g higher Terwilliger algebra* of G with respect to $\boldsymbol{x}$. It is denoted by $T^{(g)}(\boldsymbol{x})$.

The higher Terwilliger algebra can also be described in terms of the ordinary Terwilliger algebras associated with the chosen base vertices.

Proposition 3.2. *Let $T(x_j)$ denote the ordinary Terwilliger algebra of G with respect to x_j $(1 \le j \le g)$. Then*

$$T^{(g)}(\boldsymbol{x}) = \big\langle\, T(x_1),\, T(x_2),\, \dots,\, T(x_g)\,\big\rangle.$$

Proof. For each multi-index $\boldsymbol{i} = (i_1, \dots, i_g)$ we have

$$E^*_{\boldsymbol{i}}(\boldsymbol{x}) = \prod_{j=1}^{g} E^*_{i_j}(x_j) \in \langle T(x_1), \dots, T(x_g)\rangle.$$

Hence the right-hand side contains A and all $E^*_{\boldsymbol{i}}(\boldsymbol{x})$, and therefore contains $T^{(g)}(\boldsymbol{x})$. Conversely, for fixed j and i, we recover

$$E^*_i(x_j) = \sum_{\boldsymbol{i} :\, i_j = i} E^*_{\boldsymbol{i}}(\boldsymbol{x}) \in T^{(g)}(\boldsymbol{x}),$$

which shows that $T(x_j) \subseteq T^{(g)}(\boldsymbol{x})$ for all j. This proves the equality. $\square$

Let G and G' be finite simple graphs with vertex tuples $\boldsymbol{x} \in X^g$ and $\boldsymbol{x}' \in (X')^{g'}$, respectively. Let $T^{(g)}(\boldsymbol{x})$ and $T^{(g')}(\boldsymbol{x}')$ act on $V = \mathbb{C}^X$ and $V' = \mathbb{C}^{X'}$. We say that V and V' are *isomorphic as $T^{(\cdot)}$-modules* if $g = g'$ and there exist an algebra isomorphism $\varphi : T^{(g)}(\boldsymbol{x}) \to T^{(g)}(\boldsymbol{x}')$ satisfying $\varphi(A) = A'$ and $\varphi(E^*_{\boldsymbol{i}}(\boldsymbol{x})) = E^*_{\boldsymbol{i}}(\boldsymbol{x}')$ for all $\boldsymbol{i}$, together with a linear isomorphism $F : V \to V'$ such that

$$F(av) = \varphi(a)\, F(v) \qquad (a \in T^{(g)}(\boldsymbol{x}),\ v \in V).$$

Theorem 3.3. *Let $G = (X, R)$ and let $\boldsymbol{x}$ list all vertices of X (so $g = |X|$). If $X = C_1 \sqcup \cdots \sqcup C_k$ is the decomposition of X into connected components, then*

$$T^{(g)}(\boldsymbol{x}) \cong \bigoplus_{i=1}^{k} \mathrm{Mat}_{|C_i|}(\mathbb{C}).$$

Proof. For each $y \in X$, the g-way distance $\partial^{(g)}(\boldsymbol{x}, y)$ uniquely determines y, and hence the set $\{E^*_{\boldsymbol{i}}(\boldsymbol{x})\}$ contains all diagonal matrix units $E_{y,y}$. Adjacency relations generate off-diagonal units $E_{y,z}$ with $(y, z) \in R$, and their products generate all matrix units $E_{y,z}$ whenever y and z lie in the same connected component. No matrix units arise between distinct components. $\square$

Theorem 3.4. *Let G and G' be finite simple graphs with g vertices, and let $\boldsymbol{x}$ and $\boldsymbol{x}'$ list all vertices. If the higher standard modules for $T^{(g)}(\boldsymbol{x})$ and $T^{(g)}(\boldsymbol{x}')$ are isomorphic, then G and G' are isomorphic as graphs.*

Proof. By Theorem 3.3, both algebras decompose as direct sums of full matrix algebras indexed by connected components. A module isomorphism identifies diagonal matrix units up to permutation, and the image of the adjacency matrix A preserves adjacency relations between basis vectors. This induces a graph isomorphism between G and G'. $\square$

4 Terwilliger Algebra on Hypercube Graphs

In this section, we collect some basic facts concerning the Terwilliger algebra of the n-dimensional hypercube Q_n, and present a factorization of the degree-2 higher Terwilliger algebra that reflects the Cartesian product structure of Q_n.

Let Q_n denote the n-dimensional hypercube with vertex set $\{0,1\}^n$, where two vertices are adjacent if and only if they differ in exactly one coordinate. Fix the base vertex $x_0 = (0,\dots,0)$. It is known that the Terwilliger algebra $T(x_0)$ admits an explicit block decomposition as a direct sum of full matrix algebras,

$$T(x_0) \;\cong\; \bigoplus_{r=0}^{\lfloor n/2 \rfloor} \mathrm{Mat}_{n+1-2r}(\mathbb{C}),$$

together with a complete description of its irreducible modules and the action of the adjacency and dual adjacency operators. We refer to Go [4] for a comprehensive treatment, and to Nicholson [5] for a modern perspective in the framework of generalized Terwilliger algebras on hypercubes.

We now fix two base vertices $\boldsymbol{x} = (x_0, x_i)$, where x_i denotes the vertex with ones in the first i coordinates and zeros elsewhere, for some $1 \le i \le n$. Let $A^{(m)}$ denote the adjacency matrix of Q_m, and for each m let $E_j^{*(m)}(\cdot)$ be the diagonal projection onto the distance-j sphere with respect to the indicated base vertex.

Proposition 4.1. *For $\boldsymbol{x} = (x_0, x_i)$ as above, the degree-2 higher Terwilliger algebra satisfies*

$$T^{(2)}(\boldsymbol{x}) \;\cong\; T_i(x_0) \;\otimes\; T_{n-i}(x_0) \;\subseteq\; \mathrm{End}\big(\mathbb{C}^{\{0,1\}^i} \otimes \mathbb{C}^{\{0,1\}^{n-i}}\big).$$

Proof. Identifying Q_n with the Cartesian product $Q_i \square Q_{n-i}$ by separating the first i and last $n-i$ coordinates, we obtain an identification

$$\mathbb{C}^{\{0,1\}^n} \;\cong\; \mathbb{C}^{\{0,1\}^i} \otimes \mathbb{C}^{\{0,1\}^{n-i}},$$

under which the adjacency matrix decomposes as

$$A^{(n)} \;=\; A^{(i)} \otimes I \;+\; I \otimes A^{(n-i)}. \tag{$*$}$$

Distances are additive under the Cartesian product, and hence the distance projection matrices with respect to $x_0 = (0,\dots,0)$ and $x_i = (1^i, 0^{n-i})$ factor as

$$E_j^{*(n)}(x_0) \;=\; \sum_{k+h=j} E_k^{*(i)}(x_0) \otimes E_h^{*(n-i)}(x_0),$$

$$E_j^{*(n)}(x_i) \;=\; \sum_{k+h=j} E_{i-k}^{*(i)}(x_0) \otimes E_h^{*(n-i)}(x_0). \tag{$\dagger$}$$

By definition, $T^{(2)}(\boldsymbol{x})$ is generated by $A^{(n)}$ and the two families of projection matrices $\{E_j^{*(n)}(x_0)\}_j$ and $\{E_j^{*(n)}(x_i)\}_j$. Equations $(*)$–$(\dagger)$ show that each of these generators lies in the image of the natural tensor representation of $T_i(x_0) \otimes T_{n-i}(x_0)$, and hence

$$T^{(2)}(\boldsymbol{x}) \;\subseteq\; T_i(x_0) \otimes T_{n-i}(x_0).$$

Conversely, for each $0 \le k \le i$ and $0 \le h \le n-i$ we have

$$E^{*(n)}_{k+h}(x_0)\, E^{*(n)}_{i-k+h}(x_i) \;=\; E^{*(i)}_{k}(x_0) \otimes E^{*(n-i)}_{h}(x_0),$$

since the simultaneous distance conditions select precisely the corresponding distance layers in the two factors. Thus all elementary tensor projectors $E^{*(i)}_k \otimes E^{*(n-i)}_h$ belong to $T^{(2)}(\boldsymbol{x})$.

Moreover,

$$(E^{*(i)}_k \otimes I)\, A^{(n)}\, (E^{*(i)}_{k+1} \otimes I) \;=\; (E^{*(i)}_k A^{(i)} E^{*(i)}_{k+1}) \otimes I,$$

which implies $A^{(i)} \otimes I \in T^{(2)}(\boldsymbol{x})$; by symmetry, $I \otimes A^{(n-i)} \in T^{(2)}(\boldsymbol{x})$ as well. Therefore the algebra $T_i(x_0) \otimes T_{n-i}(x_0)$ is contained in $T^{(2)}(\boldsymbol{x})$.

Combining both inclusions yields $T^{(2)}(\boldsymbol{x}) = T_i(x_0) \otimes T_{n-i}(x_0)$. □

5 Concluding Remarks

We have highlighted the limitation that the Terwilliger algebra with respect to a single base vertex does not always determine a graph up to isomorphism, and we introduced a higher variant defined by fixing several base vertices simultaneously. At maximal degree $g = |X|$, we showed that the higher Terwilliger algebra determines the graph completely and decomposes as a direct sum of full matrix algebras indexed by the connected components.

For hypercube graphs, we demonstrated that the degree-2 higher Terwilliger algebra admits a factorization reflecting the Cartesian product structure of the hypercube. This decomposition is compatible with a tensor product structure of the ordinary Terwilliger algebras associated with the factors.

Several directions for future research naturally emerge from this framework. It would be interesting to extend the theory of higher Terwilliger algebras to other families of graphs, such as Hamming graphs and group association schemes, and to investigate how small the degree can be taken while still retaining completeness as a graph invariant.

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