VERMA MODULES OF THE UNIVERSAL ASKEY–WILSON ALGEBRA

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Abstract. The universal Askey–Wilson algebra $\triangle_q$ provides an algebraic setting for encoding the bispectral property of the Askey–Wilson polynomials and the q-Racah polynomials, which sit at the top of the q-Askey scheme. In this note, we review the construction of Verma $\triangle_q$-modules and give a rigorous proof of their universal property.

1. The universal Askey–Wilson algebra $\triangle_q$

The theory of orthogonal polynomials is central to classical analysis. A characteristic feature of many significant families, such as the Askey–Wilson and q-Racah polynomials at the top of the q-Askey scheme, is the *bispectral property*: these polynomials arise as common eigenfunctions of two operators—one acting on the degree via a three-term recurrence, and the other on the argument via a differential or difference equation. To provide an algebraic setting for encoding this property, the universal Askey–Wilson algebra was introduced in [3]. This formulation offers a unified description by treating the structural parameters as central elements. The formal definition is given below.

Let $\mathbb{F}$ be a field and fix a nonzero scalar $q \in \mathbb{F}$ such that $q^4 \neq 1$.

Definition 1.1 ([3], Definition 1.2)**.** The *universal Askey–Wilson algebra* $\triangle_q$ is the unital associative algebra over $\mathbb{F}$ generated by three elements A, B, C subject to the condition that each of the following elements is central:

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \qquad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \qquad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}. \tag{1}$$

For convenience, we let α, β, γ denote the central elements of $\triangle_q$ obtained by multiplying the expressions in (1) by $q + q^{-1}$. That is,

$$\frac{\alpha}{q+q^{-1}} = A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \tag{2}$$

$$\frac{\beta}{q+q^{-1}} = B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \tag{3}$$

$$\frac{\gamma}{q+q^{-1}} = C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}. \tag{4}$$

By using (4) to eliminate C from the relations (2) and (3), we obtain the following identities, which resemble the original Askey–Wilson relations.

Lemma 1.2. *The following relations hold in* $\triangle_q$*:*

$$\alpha = \frac{B^2A - (q^2+q^{-2})BAB + AB^2 + (q^2-q^{-2})^2 A + (q-q^{-1})^2 B\gamma}{(q-q^{-1})(q^2-q^{-2})}, \tag{5}$$

$$\beta = \frac{A^2B - (q^2+q^{-2})ABA + BA^2 + (q^2-q^{-2})^2 B + (q-q^{-1})^2 A\gamma}{(q-q^{-1})(q^2-q^{-2})}. \tag{6}$$

2. Verma $\triangle_q$-modules and their universal property

Throughout this note, we fix four nonzero scalars $a, b, c, \lambda \in \mathbb{F}$. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all nonnegative integers. For each $i \in \mathbb{N}$, we define the following parameters:

$$\begin{aligned}
\theta_i &= a\lambda^{-1}q^{2i} + a^{-1}\lambda q^{-2i},\\
\theta_i^* &= b\lambda^{-1}q^{2i} + b^{-1}\lambda q^{-2i},\\
\varphi_i &= a^{-1}b^{-1}\lambda q(q^i - q^{-i})(\lambda^{-1}q^{i-1} - \lambda q^{1-i})\\
&\qquad \times\ (q^{-i} - abc\lambda^{-1}q^{i-1})(q^{-i} - abc^{-1}\lambda^{-1}q^{i-1}).
\end{aligned}$$

Furthermore we set

$$\begin{aligned}
\omega &= (b + b^{-1})(c + c^{-1}) + (a + a^{-1})(\lambda q + \lambda^{-1}q^{-1}),\\
\omega^* &= (c + c^{-1})(a + a^{-1}) + (b + b^{-1})(\lambda q + \lambda^{-1}q^{-1}),\\
\omega^\varepsilon &= (a + a^{-1})(b + b^{-1}) + (c + c^{-1})(\lambda q + \lambda^{-1}q^{-1}).
\end{aligned}$$

By [1, Section 3], there exists an infinite-dimensional $\triangle_q$-module $M_\lambda(a, b, c)$ satisfying the following conditions:

(M1): There exists a basis $\{m_i\}_{i\in\mathbb{N}}$ for the $\triangle_q$-module $M_\lambda(a, b, c)$ such that

$$\begin{aligned}
(A - \theta_i)m_i &= m_{i+1} \qquad \text{for all } i \in \mathbb{N},\\
(B - \theta_i^*)m_i &= \varphi_i m_{i-1} \qquad \text{for all } i \in \mathbb{N},
\end{aligned}$$

where m_{-1} is understood as an arbitrary vector in $M_\lambda(a, b, c)$.

(M2): The central elements α, β, γ act on $M_\lambda(a, b, c)$ as the scalars $\omega, \omega^*, \omega^\varepsilon$, respectively.

The $\triangle_q$-module $M_\lambda(a, b, c)$ is referred to as a *Verma $\triangle_q$-module*. The Verma $\triangle_q$-module $M_\lambda(a, b, c)$ possesses the following universal property:

Proposition 2.1 ([1], Section 3.1)**.** *For any $\triangle_q$-module V and $v \in V$, the following conditions are equivalent:*

(i) *There exists a $\triangle_q$-module homomorphism $M_\lambda(a, b, c) \to V$ that sends m_0 to v.*
(ii) *The following equations hold on V:*

$$\begin{gathered}
Bv = \theta_0^* v,\\
(B - \theta_1^*)(A - \theta_0)v = \varphi_1 v,\\
\alpha v = \omega v, \qquad \beta v = \omega^* v, \qquad \gamma v = \omega^\varepsilon v.
\end{gathered}$$

3. A rigorous proof for Proposition 2.1

Theorem 3.1 ([3], Theorem 4.1)**.** *The elements*

$$A^i B^j C^k \alpha^r \beta^s \gamma^t \qquad \text{for all } i, j, k, r, s, t \in \mathbb{N}$$

form a basis for $\triangle_q$.

Let $I_\lambda(a, b, c)$ denote the left ideal of $\triangle_q$ generated by the elements

$$B - \theta_0^*, \tag{7}$$

$$(B - \theta_1^*)(A - \theta_0) - \varphi_1, \tag{8}$$

$$\alpha - \omega, \qquad \beta - \omega^*, \qquad \gamma - \omega^\varepsilon. \tag{9}$$

We now consider the quotient space $\triangle_q / I_\lambda(a, b, c)$.

Lemma 3.2. *For each $n \in \mathbb{N}$ the following statements hold:*

(i) $BA^n + I_\lambda(a,b,c)$ *is a linear combination of* $A^i + I_\lambda(a,b,c)$ *for all* $0 \leq i \leq n$.
(ii) $CA^n + I_\lambda(a,b,c)$ *is a linear combination of* $A^i + I_\lambda(a,b,c)$ *for all* $0 \leq i \leq n+1$.
(iii) $C^n + I_\lambda(a,b,c)$ *is a linear combination of* $A^i + I_\lambda(a,b,c)$ *for all* $0 \leq i \leq n$.

Proof. (i): We proceed by induction on n. By (7) the statement holds for $n = 0$. By (7) and (8) the statement holds for $n = 1$. Suppose that $n \geq 2$. Multiplying both sides of (6) by A^{n-2} on the right, we see that BA^n is equal to

$$\begin{aligned}&(q-q^{-1})(q^2-q^{-2})A^{n-2}\beta - (q-q^{-1})^2A^{n-1}\gamma \\ &\quad - A^2BA^{n-2} + (q^2+q^{-2})ABA^{n-1} - (q^2-q^{-2})^2BA^{n-2}.\end{aligned}$$

Using (9) and the induction hypothesis, it follows that $BA^n + I_\lambda(a,b,c)$ is a linear combination of $A^i + I_\lambda(a,b,c)$ for all $0 \leq i \leq n$. Statement (i) follows.

(ii): By (4) the element CA^n can be expressed as

$$\frac{\gamma}{q+q^{-1}}A^n - \frac{q}{q^2-q^{-2}}ABA^n + \frac{q^{-1}}{q^2-q^{-2}}BA^{n+1}.$$

By (9) and (i), the above element is congruent to a linear combination of $A^i + I_\lambda(a,b,c)$ for all $0 \leq i \leq n+1$ modulo $I_\lambda(a,b,c)$. Statement (ii) follows.

(iii): We proceed by induction on n. The case $n = 0$ is trivial. Suppose that $n \geq 1$. By the induction hypothesis, the element $C^n = CC^{n-1}$ is congruent to a linear combination of CA^i for all $0 \leq i \leq n-1$ modulo $I_\lambda(a,b,c)$. Applying (ii) to each CA^i yields statement (iii). $\square$

Lemma 3.3. *The quotient space* $\triangle_q/I_\lambda(a,b,c)$ *is spanned by*

$$A^i + I_\lambda(a,b,c) \qquad \text{for all } i \in \mathbb{N}. \tag{10}$$

Proof. According to Theorem 3.1, the vector space $\triangle_q/I_\lambda(a,b,c)$ is spanned by the cosets

$$A^iB^jC^k\alpha^r\beta^s\gamma^t + I_\lambda(a,b,c) \qquad \text{for all } i,j,k,r,s,t \in \mathbb{N}. \tag{11}$$

By (9) each coset in (11) is a linear combination of

$$A^iB^jC^k + I_\lambda(a,b,c) \qquad \text{for all } i,j,k \in \mathbb{N}. \tag{12}$$

By Lemma 3.2(iii), each coset in (12) is a linear combination of

$$A^iB^jA^k + I_\lambda(a,b,c) \qquad \text{for all } i,j,k \in \mathbb{N}. \tag{13}$$

Applying Lemma 3.2(i) repeatedly, each coset in (13) can be further reduced to a linear combination of (10). The lemma follows. $\square$

Theorem 3.4. *There exists a unique* $\triangle_q$*-module homomorphism*

$$\Phi : \triangle_q/I_\lambda(a,b,c) \to M_\lambda(a,b,c)$$

that sends $1 + I_\lambda(a,b,c)$ *to* m_0*. Moreover* Φ *is an isomorphism.*

Proof. We view $\triangle_q$ as a left $\triangle_q$-module. There is a unique $\triangle_q$-module homomorphism $\Psi : \triangle_q \to M_\lambda(a,b,c)$ defined by

$$u \quad \mapsto \quad u \cdot m_0 \qquad \text{for all } u \in \triangle_q.$$

By conditions **(M1)** and **(M2)**, the kernel of Ψ contains the elements in (7)–(9). Consequently, the ideal $I_\lambda(a,b,c)$ is contained in $\ker\Psi$. Since $I_\lambda(a,b,c)$ is the left ideal of $\triangle_q$ generated by these elements, this induces the $\triangle_q$-module homomorphism Φ as stated in the theorem.

For each $i \in \mathbb{N}$, we define the elements in $\triangle_q/I_\lambda(a,b,c)$ as follows:

$$\bar{m}_i = \prod_{h=0}^{i-1}(A - \theta_h) + I_\lambda(a,b,c). \tag{14}$$

By **(M1)**, the image of $\bar{m}_i$ under Φ is precisely the basis vector m_i of $M_\lambda(a,b,c)$ for all $i \in \mathbb{N}$. Since Lemma 3.3 implies that the cosets $A^i + I_\lambda(a,b,c)$ for all $i \in \mathbb{N}$ span $\triangle_q/I_\lambda(a,b,c)$, it follows that $\bar{m}_i$ for all $i \in \mathbb{N}$ also span $\triangle_q/I_\lambda(a,b,c)$. Therefore Φ must be an isomorphism. The result follows. $\square$

Proof of Proposition 2.1. This is an immediate consequence of Theorem 3.4. $\square$

A simplified description for Proposition 2.1 was recently established as follows:

Proposition 3.5 ([2], Proposition 3.1)**.** *For any $\triangle_q$-module V and $v \in V$, the following conditions are equivalent:*

(i) *There exists a $\triangle_q$-module homomorphism $M_\lambda(a,b,c) \to V$ that sends m_0 to v.*
(ii) *The following equations hold on V:*

$$\begin{gathered} Bv = \theta_0^* v, \\ (B - \theta_1^*)Av = (q - q^{-1})\varphi v, \\ \beta v = \omega^* v, \qquad \gamma v = \omega^\varepsilon v, \end{gathered}$$

where

$$\varphi = (c + c^{-1})(\lambda - \lambda^{-1}) - (a + a^{-1})(bq - b^{-1}q^{-1}).$$

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