# Spectral synthesis in Fourier analysis

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**Abstract**

In this paper, we state recent progress with respect to spectral synthesis in Fourier analysis whose contents are composed of three subjects such that spectral synthesis in the spaces of Fourier transforms on $\mathbf{R}^n$, spectral synthesis in the Fourier-Lebesgue spaces, and spectral synthesis in other spaces.

## §1 Spectral synthesis in the spaces of Fourier transforms on $\mathbf{R}^n$

Let $A(\mathbf{R}^n) = \mathcal{F}L^1(\mathbf{R}^n)$ be the set of all Fourier transforms on $n$-dimensional Euclidean space $\mathbf{R}^n$, i.e. the set of all $f \in C_0(\mathbf{R}^n)$ such that

$$f(x) = \int_{\mathbf{R}^n} g(y)e^{-ixy}dy (= \widehat{g}(x))$$

for some $g \in L^1(\mathbf{R}^n)$. Then, $A(\mathbf{R}^n)$ is a Banach algebra with pointwise multiplication and the norm $||f||_A = \int |g(y)|dy$(cf.[16],[20],[22]). Also for $E \subset \mathbf{R}^n$ a closed set and $I \subset A(\mathbf{R}^n)$, we define that

$$\begin{aligned}
Z(I) &= \cap_{f\in I} f^{-1}(0), \\
I(E) &= \{f \in A(\mathbf{R}^n) | f = 0 \text{ on } E\}, \\
J_0(E) &= \{f \in A(\mathbf{R}^n) | f = 0 \text{ on a neighborhood of } E\}, \text{ and} \\
J(E) &: \text{ the closed ideal generated by } J_0(E), \text{ i.e. the closure of} \\
& \quad \{\sum_{j=1}^{m} f_j g_j | f_j \in A(\mathbf{R}^n), g_j \in J_0(E), m \in \mathbf{N}\} \quad \text{in } A(\mathbf{R}^n).
\end{aligned}$$

It is known that $J(E) \subset I(E)$ and $Z(I(E)) = Z(J(E)) = E$(cf.[16],[20],[22]). $E$ is called a set of spectral synthesis (S-set) for $A(\mathbf{R}^n)$ if $I(E) = J(E)$.

Next we review known results about the spectral synthesis. Let $\Phi$ be a translation invariant $w^*$-closed subspace of $L^\infty(\mathbf{R}^n)$ and $\sigma(\Phi) = \{\lambda \in \mathbf{R}^n | e^{i\lambda x} \in \Phi\}$ (It is called the spectrum of $\Phi$). In 1945, A.Beurling[1] showed that $\Phi = \{0\}$ if $\sigma(\Phi) = \phi$(empty set), and proposed the problem :

For a closed set $E$ of $\mathbf{R}^n$, $\Phi$ such that $\sigma(\Phi) = E$ is unique ?

In this case, for any $\varphi \in \Phi$, we have

$$\varphi = \lim \sum_n c_n e^{i\lambda_n x} \quad \text{in } \sigma(L^\infty, L^1) \ (\lambda_n \in \sigma(\Phi), \ c_n \in \mathbf{C}).$$

($\varphi$ is "synthesized" from the spectrum $\sigma(\Phi)$ in Rudin[22], and Reiter[20] said that Beurling coined the term 'spectral synthesis' in this context.)

Then, we have that $\widehat{g}(\lambda) = 0$ for any $g \in L^1(\mathbf{R}^n)$ such that

$$< g, \varphi >= \int_{\mathbf{R}^n} g(\xi)\varphi(-\xi)d\xi = 0 \ (\forall \varphi \in \Phi,\ \lambda \in \sigma(\Phi)),$$

and we understand that

$$\Phi \text{ such that } \sigma(\Phi) = E \text{ is unique if and only if } I(E) = J(E).$$

Hence, Beurling' s problem is replaced by

$$\text{for any closed set } E \subset \mathbf{R}^n, \text{ we have that } I(E) = J(E)\ ?$$

Before Beurling[1], Wiener[31] proved the following :

Let $f \in A(\mathbf{R})$ with $|f| > 0$ on $\mathbf{R}$, and $I$ the closed ideal generated by $\{e^{iyx}f(x)|y \in \mathbf{R}\}$. Then $I = A(\mathbf{R})$.

This result is a special case of Beurling's result, because we have that when $\Phi = \{\varphi \in L^\infty(\mathbf{R})|\int \varphi(\xi)F(y-\xi)d\xi = 0\ (\forall\ y \in \mathbf{R})\}$ (where $f(x) = \int F(\xi)e^{-ix\xi}d\xi$), $\Phi = \{0\}$ for $\sigma(\Phi) = \phi$ by Wiener. Hence, we may have that the study of spectral synthesis was begun by Wiener[31].

In 1948, L.Schwarz[27] obtained a negative answer for the Beurling problem such that the unit sphere $S^{n-1}$ $(n \geq 3)$ is a non S-set for $A(\mathbf{R}^n)$. On the other hand, Herz[8] got an opposite result such that the unit circle $S^1$ is a S-set for $A(\mathbf{R}^2)$. After that, P.Malliavin[15] proved an excellent result such that for $n \geq 1$ there exists a non S-set for $A(\mathbf{R}^n)$, whose result is called Malliavin's Theorem.

After Malliavin[15], many papers had been done about the study of the set of spectral synthesis for various directions. Here, we state some results in two directions(study of S-set with respect to the unit sphere and study of S-set with respect to thin set).

Domar([3],[4]cf.[29]) had the study of S-set for $A(\mathbf{R}^n)$ in the $(n-1)$-dimensional manifold with non-zero Gauss curvature $(n \geq 2)$. He also investigated about S-sets of the curves with non-zero curvature in $\mathbf{R}^2$([2]cf.[4],[18],[23]). Müller[17] had the study of S-set for $A(\mathbf{R}^n)$ in the manifold with zero Gauss curvature. By these results in high dimension space, we may have that a hypersurface in the shape of flat is near S-set for $A(\mathbf{R}^n)$(cf.[24]).

Next we state some results about thin set. Let $\mathbf{T}$ be the 1-dimensional torus, and $E$ a closed subset of $\mathbf{T}$. We give some definitions :

$$\begin{aligned} A(\mathbf{T}) &= \{f \in C(\mathbf{T})|\sum_n |\hat{f}(n)| < \infty, \hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx\}. \\ S(E) &= \{f \in C(E)||f| = 1 \text{ on } E\}. \\ A(E) &: \quad \text{the restriction algebra of } A(\mathbf{T}) \text{ on } E. \end{aligned}$$

For example, a closed set $E$ is called a Kronecker set if for any $\varepsilon > 0$ and $f \in S(E)$ there exists $n \in \mathbf{Z}$ such that $||f - e^{inx}||_{C(E)} < \varepsilon$, and a closed set $E$ is called a Helson set if $A(E) = C(E)$.

It is known that a Kronecker set is a Helson set, and a Helson set has Lebesgue measure zero(cf.[22]). Varopoulos[28] showed that a Kronecker set is a S-set for $A(\mathbf{T})$. But Körner[13] and Kaufman[11] proved that there exists a Helson set $E$ which is a non S-set for $A(\mathbf{T})$. Study of thin set is important for study of harmonic analysis, for example, we have that the another proof of Malliavin's Theorem of Varopoulos by applying Kronecker sets([30]cf.[9],[16]) and the negative answer for the Wiener conjecture by Lev and Olveskii[14](which is related to Helson set), and etc.

## §2 Spectral synthesis in the Fourier-Lebesgue spaces

This part is the joint work with M.Kobayashi(Hokkaido University) in [12].

Fourier-Lebesgue spaces is a generalization of the spaces of Fourier transforms $A(\mathbf{R}^n)$. We give some definitions :

(1) for $1 \le q < \infty$, and $s \in \mathbf{R}$

$$L^q_s(\mathbf{R}^n) = \{f | ||f||_{L^q_s} = (\int_{\mathbf{R}^n} (|f(x)| < x >^s)^q dx)^{1/q} < \infty\},$$

and $||f||_{L^\infty_s} = \text{ess.sup}_{x\in\mathbf{R}^n} < x >^s |f(x)|$ with $< x >= 1 + |x|$ $(L^q(\mathbf{R}^n) = L^q_0(\mathbf{R}^n))$.

(2) $\mathcal{S}(\mathbf{R}^n)$ denotes by the Schwartz space of all complex valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}^n$.

(3) $\mathcal{S}'(\mathbf{R}^n)$ denotes by the topological dual of $\mathcal{S}(\mathbf{R}^n)$.

(4) for $1 \le q < \infty$, and $s \in \mathbf{R}$

$$\mathcal{F}L^q_s(\mathbf{R}^n) = \{f \in \mathcal{S}'(\mathbf{R}^n) | ||f||_{\mathcal{F}L^q_s} = (\int_{\mathbf{R}^n} (|\mathcal{F}[f](\xi)| < \xi >^s)^q d\xi)^{1/q} < \infty\},$$

where $\mathcal{F}[f]$ denotes by Fourier transform in $\mathcal{S}'(\mathbf{R}^n)$.

(5)

$$\widehat{g}(\xi) = \int_{\mathbf{R}^n} g(x) e^{-i\xi x} dx \quad (g \in L^1(\mathbf{R}^n)).$$

It is known that $(\mathcal{F}L^q_s(\mathbf{R}^n), ||\cdot||_{\mathcal{F}L^q_s})$ is a Banach space and its dual space is $\mathcal{F}L^{q'}_{-s}(\mathbf{R}^n)$ $(1/q + 1/q' = 1)$, i.e. $\ell \in (\mathcal{F}L^q_s(\mathbf{R}^n)'$ if and only if

$$\ell(f) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \mathcal{F}[f](\xi) \overline{\mathcal{F}[\phi](\xi)} d\xi =< f, \phi > \quad (f \in \mathcal{F}L^q_s(\mathbf{R}^n))$$

for some $\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^n)$ with $||\ell||_{op} = (2\pi)^{-n} ||\phi||_{\mathcal{F}L^{q'}_{-s}}$. Also we note that if $1 < q < \infty$ and $s > n/q'$ or $q = 1$ and $s \ge 0$, and $f \in \mathcal{F}L^q_s(\mathbf{R}^n)$, then $\mathcal{F}[f] \in L^1(\mathbf{R}^n)$.

Also $f \in C_0(\mathbf{R}^n)$ by the Riemann-Lebesgue lemma, and $\mathcal{F}L^q_s(\mathbf{R}^n)$ is a Banach algebra with respect to pointwise multiplication and the norm $||\cdot||_{\mathcal{F}L^q_s}$, when $s > n/q'$ and $1 < q < \infty$ or $q = 1$ and $s \geq 0$. In this spaces, we study spectral synthesis about $S^{n-1} \subset \mathbf{R}^n$, when $1 < q < \infty$ and $s > n/q'$ or $q = 1$ and $s \geq 0$ . For this we give the definitions such that for a closed set $E \subset \mathbf{R}^n$,

$$\begin{array}{lcl} I(E) & = & \{f \in \mathcal{F}L^q_s(\mathbf{R}^n) | f = 0 \text{ on } E\}, \\ J_0(E) & = & \{f \in \mathcal{F}L^q_s(\mathbf{R}^n) | f = 0 \text{ on a neighborhood of } E\}, \text{ and} \\ J(E) & \text{denotes} & \text{the closed ideal generated by } J_0(E) \text{ in } \mathcal{F}L^q_s(\mathbf{R}^n). \end{array}$$

Also we denotes by $Z(I) = \cap_{f \in I} f^{-1}(0)$ for $I \subset \mathcal{F}L^q_s(\mathbf{R}^n)$, and a closed set $E \subset \mathbf{R}^n$ is called a S-set for $\mathcal{F}L^q_s(\mathbf{R}^n)$ if $I(E) = J(E)$. Next we state some known results:
(1) C.S.Herz[8]

$$S^1(\subset \mathbf{R}^2) \text{ is a S}-\text{set for } \mathcal{F}L^1_0(\mathbf{R}^2) = A(\mathbf{R}^2).$$

(2) H.Reiter([20],[21])
(i) if $0 < s < \frac{1}{2}$,

$$S^1(\subset \mathbf{R}^2) \text{ is a S}-\text{set for } \mathcal{F}L^1_s(\mathbf{R}^2).$$

(ii) if $s \geq \frac{1}{2}$,

$$S^1(\subset \mathbf{R}^2) \text{ is a non S}-\text{set for } \mathcal{F}L^1_s(\mathbf{R}^2).$$

(iii) if $s > 0$ and $n \geq 3$,

$$S^{n-1}(\subset \mathbf{R}^n) \text{ is a non S}-\text{set for } \mathcal{F}L^1_s(\mathbf{R}^n).$$

In [12], we show the following results which are an extension for Herz and Reiter. We give our theorems :

**Theorem 1**

(i) if $1 < q < 2$ and $\frac{2}{q'} < s < \frac{2}{q'} + \frac{1}{2}$, $S^1$ is a S-set for $\mathcal{F}L^q_s(\mathbf{R}^2)$.

(ii) if $1 < q < \infty$ and $s > \frac{2}{q'} + \frac{1}{2}$, $S^1$ is a non S-set for $\mathcal{F}L^q_s(\mathbf{R}^2)$.

**Theorem 2**
If $1 < q < \infty$ and $s > \frac{n}{q'}$ with $n \geq 3$, $S^{n-1}$ is a non S-set for $\mathcal{F}L^q_s(\mathbf{R}^n)$.

Here, we show some key points of the proofs. Let

$$I_0 = \{g = f^2 | f \in \mathcal{F}L^q_s(\mathbf{R}^2),\ f \text{ is a radial function},\ f = 0 \text{ on } S^1\},$$

$I_1$ the closed ideal of $\mathcal{F}L^q_s(\mathbf{R}^2)$ generated by $I_0$, $I = I(S^1)$, and $J = J(S^1)$. For Theorem 1(ii), we apply the Bessel function for showing $Z(I) = Z(I_1) = S^1$ and $I = I(S^1) \neq I_1 \supset J = J(S^1)$ if $1 < q < \infty$ and $s > \frac{2}{q'} + \frac{1}{2}$. Let

$$k_0(\xi) = \int_{\mathbf{R}^2} e^{i\xi x} d\mu(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i|\xi|\cos\theta} d\theta \ (\xi \in \mathbf{R}^2),$$

$$K_0(\xi) = i\xi_2 k_0(\xi)\ (\xi = (\xi_1, \xi_2) \in \mathbf{R}^2),\ \text{and}$$

$$\ell_0(f) = \int_{\mathbf{R}^2} F(\xi)K_0(\xi)d\xi\ (\widehat{F} = f).$$

Then, we show that
(i)$\ell_0$ is a bounded linear functional on $\mathcal{F}L^q_s(\mathbf{R}^2)$, and
(ii)$\ell_0 = 0$ on $I_1$ and $\ell_0 \neq 0$ on $I(S^1)$, and $I \neq I_1 \supset J$.
For Theorem 2, we show it in the same way to the proof of Theorem 1(ii) by

$$\mathcal{F}[\sigma](\xi) = \int_{\mathbf{R}^n} e^{-i\xi x} d\sigma(x) = O(|\xi|^{-\frac{n-1}{2}})\ (|\xi| \to \infty),$$

where $d\sigma$ is the surface measure of $S^{n-1}$. When we define that
$\varphi(\xi) = \int_{\mathbf{R}^n} e^{i\xi\cdot x} d\sigma(x)\ (\xi \in \mathbf{R}^n)$,
$\Phi_n(f)(\xi) = -i\xi_n\varphi(\xi)\ (\xi = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n)$, and
$T_n(f) = \int_{\mathbf{R}^n} F(-\xi)\Phi_n(\xi)d\xi\ (\widehat{F} = f)$,
we show that
(i)$T_n$ is a bounded linear functional on $\mathcal{F}L^q_s(\mathbf{R}^n)$, and
(ii)$T_n = 0$ on $I_2$ and $T_n \neq 0$ on $I(S^{n-1})$, and $I(S^{n-1}) \neq I_2 \supset J(S^{n-1})$,
where $I_2$ is the closed ideal of $\mathcal{F}L^q_s(\mathbf{R}^n)$ generated by

$$\{f^2 | f \in \mathcal{F}L^q_s(\mathbf{R}^n),\ \text{radial function},\ f = 0 \text{ on } S^{n-1}\}.$$

Also for the proof of Theorem 1(i) we define

$$\mathcal{F}M(\mathbf{R}^2) = \{\mathcal{F}[\mu] | \mathcal{F}[\mu](x) = \int_{\mathbf{R}^2} e^{-ix\xi} d\mu(\xi),\ \mu \in M(\mathbf{R}^2)\},$$

where $M(\mathbf{R}^2)$ is the set of all bounded regular Borel measures on $\mathbf{R}^2$. Here, we give key propositions.

**Proposition 1**
Let $I'$ be a closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^2)$ and $f_0 \in \mathcal{F}L^q_s(\mathbf{R}^2)$.
If every $\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^2)$ such that

$$\mathcal{F}[\phi] \in C(\mathbf{R}^2) \text{ and } < f, \phi >= 0 \text{ for all } f \in I' \text{ satisfies } < f_0, \phi >= 0,$$

then $f_0$ is in $I'$.

**Proposition 2**
Let $\phi \in \mathcal{F}L^{q'}_{-s}$, $\mathcal{F}[\phi] \in C(\mathbf{R}^2)$ and $< f, \phi >= 0$ for all $f \in J$. Then there exists $\{\sigma_n[\phi]\}_n \subset \mathcal{F}M(\mathbf{R}^2)$ such that
(i) $\sigma_n[\phi] = \mathcal{F}[\mu_n]$ for some $\mu_n \in M(\mathbf{R}^2)$ with supp $\mu_n \subset S^1$.
(ii)

$$||\sigma_n[\phi]||_{L^{q'}_{-s}} \leq C||\phi||_{\mathcal{F}L^{q'}_{-s}}.$$

(iii) $\{\sigma_n[\phi]\}_n$ converges to $\mathcal{F}[\phi]$ uniformly on any compact set of $\mathbf{R}^2$.

Then, we show that $J \supset I$, and obtain that $S^1$ is a S-set for $\mathcal{F}L^q_s(\mathbf{R}^2)$.

In fact, if Propositions 1 and 2 are true, then we can prove Theorem 1(i) as follows :

Let $f_0 \in I$ and $\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^2)$ be such that $\mathcal{F}[\phi] \in C(\mathbf{R}^2)$ and $< f, \phi >= 0$ for all $f \in J$. Then there exists $\{\sigma_n[\phi]\}_n \subset \mathcal{F}M(\mathbf{R}^2)$ satisfying (i)-(iii) in Proposition 2. Since $\mathcal{F}[f_0] \cdot \mathcal{F}[\phi] \in L^1(\mathbf{R}^2)$ and $\mathcal{F}[f_0] \in L^q_s(\mathbf{R}^2)$, for $\varepsilon > 0$ there exists a compact set $K$ such that

$$\int_{\mathbf{R}^2\setminus K} |\mathcal{F}[f_0](\xi)\mathcal{F}[\phi](\xi)| d\xi < \frac{\varepsilon}{4}$$

and

$$(\int_{\mathbf{R}^2\setminus K} (|\mathcal{F}[f_0](\xi)| < \xi >^s)^q d\xi)^{1/q} < \frac{\varepsilon}{4C||\phi||_{\mathcal{F}L^{q'}_{-s}}},$$

where $C$ denotes the constant in Proposition 2(ii). Then we have

$$\int_{\mathbf{R}^2\setminus K} |\mathcal{F}[f_0](\xi)\sigma_n[\phi](\xi)| d\xi \leq ||\mathcal{F}[f_0]||_{L^q_s(\mathbf{R}^2\setminus K)} ||\sigma_n[\phi]||_{L^{q'}_{-s}(\mathbf{R}^2\setminus K)} < \frac{\varepsilon}{4}.$$

Therefore, we have that

$$\begin{aligned} & | < f_0, \phi > - < f_0, \mu_n > | \\ \leq\ & C'(|\int_K \mathcal{F}[f_0](\xi)(\overline{\mathcal{F}[\phi](\xi) - \sigma_n[\phi](\xi)})d\xi| + \int_{\mathbf{R}^2\setminus K} |\mathcal{F}[f_0](\xi)|(|\mathcal{F}[\phi](\xi)| + |\sigma_n[\phi](\xi)|)d\xi) \\ \leq\ & C'(\sup_{\xi\in K} |\mathcal{F}[\phi](\xi) - \sigma_n[\phi](\xi)| \int_K |\mathcal{F}[f_0](\xi)| d\xi + \frac{\varepsilon}{2}) \quad \text{for some constant } C'. \end{aligned}$$

On the other hand, since $f_{0|_{S^1}} = 0$ and supp $\mu_n \subset S^1$, we have that

$$\begin{aligned} < f_0, \mu_n > &= (2\pi)^{-2} \int_{\mathbf{R}^2} \mathcal{F}[f_0](\xi)\overline{(\int_{\mathbf{R}^2} e^{-ix\xi} d\mu_n(x))} d\xi \\ &= \overline{\int_{\mathbf{R}^2} \overline{f_0(x)} d\mu_n(x)} = 0. \end{aligned}$$

For this reason $< f_0, \phi >= 0$, and we conclude $f_0 \in J$ by Proposition 1.
After all, we have that $I = J$.

**Remark** There are some applications of spectral synthesis for partial differential equations(cf.[7],[12]).

## §3 Spectral synthesis in other two spaces

In this part, we state spectral synthesis in modulation spaces, and in the spaces of Hankel transforms.

[1] **Spectral synthesis in Modulation spaces**
This part depends on Feichtinger-Kobayashi-Sato([5],[6]).

Let $\phi$ be in $\mathcal{S}(\mathbf{R}^n)\backslash\{0\}$, $(T_x h)(t) = h(t-x)$, $(M_\xi h)(t) = e^{it\xi}h(t)$, and

$$V_\phi f(x,\xi) =< f, M_\xi T_x \phi > \quad (f \in \mathcal{S}'(\mathbf{R}^n)).$$

It is known that

$$V_\phi f(x,\xi) \in C(\mathbf{R}^n \times \mathbf{R}^n).$$

Also we define that $M_s^{p,q} = M_s^{p,q}(\mathbf{R}^n) = \{f \in \mathcal{S}'(\mathbf{R}^n) | ||f||_{M_s^{p,q}} < \infty\}$, where

$$||f||_{M_s^{p,q}} = (\int_{\mathbf{R}^n} < \xi >^{sq} (\int_{\mathbf{R}^n} |V_\phi f(x,\xi)|^p dx)^{q/p} d\xi)^{1/q}.$$

$M_s^{p,q} = M_s^{p,q}(\mathbf{R}^n)$ are called modulation spaces. Then, it is well known that $M_s^{p,q}$ is a multiplication algebra and $M_s^{p,q} \subset$ {uniformly bounded continuous functions} if $1 < q < \infty$ and $s > \frac{n}{q'}$, or $q = 1$ and $s \geq 0$. Let $1 < q < \infty$ and $s > \frac{n}{q'}$, or $q = 1$ and $s \geq 0$. For a closed set $E \subset \mathbf{R}^n$, we give some definitions :

$$\begin{aligned} I(E) &= \{f \in M_s^{p,q}(\mathbf{R}^n) | f = 0 \text{ on } E\}. \\ J_0(E) &= \{f \in M_s^{p,q}(\mathbf{R}^n) | f = 0 \text{ on a neighborhood of } E\}. \\ J(E) &: \text{ the closed ideal generated by } J_0(E) \text{ in } M_s^{p,q}(\mathbf{R}^n). \end{aligned}$$

A closed set $E \subset \mathbf{R}^n$ is called a S-set for $M_s^{p,q}(\mathbf{R}^n)$ if $I(E) = J(E)$. Then, we can show that $M_s^{p,q}(\mathbf{R}^n)$ is related to $\mathcal{F}L_s^q(\mathbf{R}^n)$ with respect to the algebraic operation, and have the followings :

**Theorem** Let $1 \leq p \leq 2$. Suppose that $q = 1$ and $s \geq 0$, or $1 < q \leq p'$ and $s > n/q'$. Then for a compact subset $K$ of $\mathbf{R}^n$, $K$ is a S-set for $M_s^{p,q}(\mathbf{R}^n)$ if and only if $K$ is a S-set for $\mathcal{F}L_s^q(\mathbf{R}^n)$.

**Corollary**

(i) Let $1 \leq p \leq 2,\ 1 < q \leq p',\ 1 < q < 2$ and $2/q' < s < 2/q' + 1/2$.
Then $S^1$ is a S-set for $M_s^{p,q}(\mathbf{R}^2)$.

(ii) For $1 \leq p \leq 2,\ 1 < q < \infty$ and $s > 2/q' + 1/2$.
Then $S^1$ is a non S-set for $M_s^{p,q}(\mathbf{R}^2)$.

(iii) Let $1 \leq p \leq 2,\ 1 < q \leq p',\ 1 < q < \infty$, and $s > n/q'$.
Then $S^{n-1}$ $(n \geq 3)$ is a non S-set for $M_s^{p,q}(\mathbf{R}^n)$.

[2] **Spectral synthesis in the spaces of Hankel transforms**
This part depends on Kanjin-Sato[10].

Let $\nu > -\frac{1}{2}$, and $f$ on $[0,\infty)$. $\mathcal{H}_\nu f(x)$ is called the Hankel transform of order $\nu$ when

$$\mathcal{H}_\nu f(x) = \int_0^\infty f(\xi)\mathcal{J}_\nu(x\xi)dm_\nu(\xi),$$

where $\mathcal{J}_\nu(0)=1,\ \mathcal{J}_\nu(\xi) = \frac{J_\nu(\xi)}{c_\nu \xi^\nu}\ (\xi \neq 0)$ , $c_\nu = \frac{1}{2^\nu \Gamma(\nu+1)}$, $dm_\nu(\xi) = c_\nu \xi^{2\nu+1}d\xi$, and $J_\nu(\xi)$ the Bessel function.

Also we define the space of Hankel transforms $A_\nu$ such that

$$A_\nu = \{\mathcal{H}_\nu f | ||\mathcal{H}_\nu f|| = \int_0^\infty |f(\xi)|dm_\nu(\xi) < \infty\}.$$

The followings are known(A.Schwartz[26]cf.[25]):

(i) $A_\nu \subset C_0([0,\infty))$.
(ii) $A_\nu$ is a commutative Banach algebra with multiplication
and the norm $||\mathcal{H}_\nu f||$.

Here, we remark about the multiplication. Let $\nu > -\frac{1}{2}$, and for $f,\ g \in L^1(dm_\nu)$

$$f * g(x) = \int_0^\infty \int_0^\infty \Phi_\nu(x,y,z)f(y)g(z)dm_\nu(y)dm_\nu(z),$$

where

$$\Phi_\nu(x,y,z) = \frac{2^{3\nu-1}\Gamma(\nu+1)^2\Delta(x,y,z)^{2\nu-1}}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}(xyz)^{2\nu}}, \quad = 0,$$

when the first value being assumed only there is a triangle of sides $x,\ y$, and $z$ with area $\Delta(x,y,z)$. Then we have that

$$\text{(i) } \Phi_\nu(x,y,z) \geq 0 \quad (0 < x,y,z < \infty), \quad \text{(ii) } \int_0^\infty \Phi_\nu(x,y,z)dm_\nu(x) = 1, \text{ and}$$

$$\text{(iii) } \int_0^\infty \mathcal{J}_\nu(xu)\Phi_\nu(x,y,z)dm_\nu(x) = \mathcal{J}_\nu(yu)\mathcal{J}_\nu(zu).$$

Hence, we have the multiplication in $A_\nu$ such that

$$\mathcal{H}_\nu(f*g) = \mathcal{H}_\nu(f)\cdot\mathcal{H}_\nu(g) \quad (f*g \in L^1([0,\infty),dm_\nu)).$$

Also we give an example of Hankel transform.

Let $\nu = \frac{n-2}{2}$ $(n \geq 2)$, and $f$ be in $L^1(\mathbf{R}^n)$ which is a radial function. i.e.

$$\begin{aligned}
f(\xi) &= \phi(|\xi|) \text{ for some } \phi \text{ on } [0,\infty). \\
\widehat{f}(x) &= \int_{\mathbf{R}^n} f(y)e^{-ixy}dy \\
&= (2\pi)^{\frac{n}{2}} \int_0^\infty \phi(s)\mathcal{J}_\nu(|x|s)s^{n-1}ds \\
&= (2\pi)^{\frac{n}{2}}\mathcal{H}_\nu(\phi).
\end{aligned}$$

For a closed set $E \subset [0, \infty)$, we define that

$$\begin{aligned} I(E) &= \{f \in A_\nu | f = 0 \text{ on } E\}, \\ J_0(E) &= \{f \in A_\nu | f = 0 \text{ on a neighborhood of } E\}, \text{ and} \\ J(E) &: \text{the closed ideal generated by } J_0(E) \text{ in } A_\nu. \end{aligned}$$

Then, a closed set $E \subset [0, \infty)$ is called a S-set for $A_\nu$ if $I(E) = J(E)$. It is known that for $y_0 > 0$,
(1) H.Reiter[19](cf.[20],[21]) when $n \geq 3$,

$$\{y_0\} \text{ is a non S} - \text{set for } A_{\frac{n-2}{2}}.$$

(2) A.Schwartz[26]: when $\nu \geq \frac{1}{2}$,

$$\{y_0\} \text{ is a non S} - \text{set for } A_\nu, \text{ and}$$

(when $n = 2$, we remark $\nu = 0$)
(3) C.S.Herz[8]

$$S^1 \text{ is a S} - \text{set for } A(\mathbf{R}^2).$$

Our results are as follows :

**Proposition**
Let $\nu > -\frac{1}{2}$ and let $I$ be a closed ideal in $A_\nu$ such that $Z(I) = \{y_0\}$ $(y_0 > 0)$. Then, we have that

$$I = \{f \in A_\nu | f^{(j)}(y_0) = 0, \ j = 0, 1, \cdots, M\} \text{ for some } M \leq \nu + \frac{1}{2}.$$

We show next theorem by the proposition.
**Theorem** When $-\frac{1}{2} < \nu < \frac{1}{2}$, and $y_0 > 0$, we have that

$$\{y_0\} \text{ is a S} - \text{set for } A_\nu.$$

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