BMOA and Little Bloch Functions and a Geometric Construction of Chord-Arc Curves

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Abstract

We explain analytic aspects of a problem concerning the strict inclusion of VMOA($\mathbb{D}$) in BMOA($\mathbb{D}$) $\cap B_0(\mathbb{D})$. Then this is translated into a geometric problem concerning boundary curves of conformal mappings. We describe a geometric construction of a chord-arc Jordan curve that is asymptotically conformal but not asymptotically smooth. This provides an example of a function that lies in the difference set of the above function spaces. The full proof appears in [9].

1 Introduction

Function spaces associated with conformal mappings often reflect geometric properties of the boundary curves of image domains. Among them, the Bloch space, the analytic BMO space (BMOA), and their little subspaces play important roles in geometric function theory. A general reference on this field is Pommerenke's book [12].

Let φ be holomorphic on the unit disk $\mathbb{D}$. The *Bloch norm* is defined by

$$\|\varphi\|_B = \sup_{z\in\mathbb{D}}(1-|z|^2)|\varphi'(z)|.$$

The finiteness of this norm is equivalent to the condition that φ is a BMO function on $\mathbb{D}$ with respect to the Lebesgue measure. A function satisfying this condition is called a *Bloch function*. The Bloch space $B(\mathbb{D})$ consisting of these functions is important in the study of quasiconformal extensions of conformal mappings. For instance, if f is a conformal mapping of $\mathbb{D}$ into the complex plane $\mathbb{C}$, then $\|\log f'\|_B$ is uniformly bounded. Moreover, if $\|\log f'\|_B$ is sufficiently small, then f extends quasiconformally to $\mathbb{C}$.

A natural vanishing condition for the Bloch norm is

$$\lim_{t\to 0} \sup_{|z|>1-t} (1-|z|^2)|\varphi'(z)| = 0.$$

Bloch functions satisfying this condition form the *little Bloch space* $B_0(\mathbb{D})$, which is a closed subspace of $B(\mathbb{D})$.

Another important analytic function space closely related to conformal mapping theory is the *analytic BMO* space. A holomorphic function φ on $\mathbb{D}$ belongs to $\mathrm{BMOA}(\mathbb{D})$ if it is the Poisson extension of a BMO function on the unit circle $\mathbb{S}$. Equivalently, $(1-|z|^2)|\varphi'(z)|^2\,dxdy$ is a Carleson measure on $\mathbb{D}$, and the *BMOA norm* of φ is defined by using the Carleson measure norm:

$$\|\varphi\|^2_{\mathrm{BMOA}} = \sup_{\xi\in\mathbb{S},\ r>0} \frac{1}{r}\iint_{|\xi-z|<r} (1-|z|^2)|\varphi'(z)|^2\,dxdy.$$

This is equivalent to the BMO norm of the boundary extension of φ to $\mathbb{S}$.

The vanishing condition for this norm is

$$\lim_{r\to 0}\sup_{\xi\in\mathbb{S}} \frac{1}{r}\iint_{|\xi-z|<r} (1-|z|^2)|\varphi'(z)|^2\,dxdy = 0,$$

which is equivalent to the boundary function being in VMO on $\mathbb{S}$. This condition defines the closed subspace $\mathrm{VMOA}(\mathbb{D})$ of $\mathrm{BMOA}(\mathbb{D})$.

It can be proved that the BMOA norm is stronger than the Bloch norm and the vanishing condition for BMOA is stronger than that for Bloch functions. Namely, $\mathrm{BMOA}(\mathbb{D}) \subset B(\mathbb{D})$ and $\mathrm{VMOA}(\mathbb{D}) \subset B_0(\mathbb{D})$ (see Girela [6, Cor.5.2]). In particular, we are interested in the inclusion relation

$$\mathrm{VMOA}(\mathbb{D}) \subset \mathrm{BMOA}(\mathbb{D}) \cap B_0(\mathbb{D}),$$

and in our paper [9], we have shown that this inclusion is strict.

Theorem 1. *There exists* $\varphi \in \mathrm{BMOA}(\mathbb{D}) \cap B_0(\mathbb{D})$ *such that* $\varphi \notin \mathrm{VMOA}(\mathbb{D})$.

The purpose of the present article is to explain the analytic background of this relationship and the geometric idea behind this result. This analytic statement has the following geometric formulation.

Theorem 2 (Geometric formulation). *There exists a chord-arc Jordan curve that is asymptotically conformal but not asymptotically smooth.*

This equivalence follows from results of Pommerenke [11] relating analytic conditions on $\log f'$ for a conformal homeomorphism $f : \mathbb{D} \to \Omega \subset \mathbb{C}$ to the geometric regularity of boundary curves $\Gamma = \partial\Omega$. The connection between BMO-type conditions, chord-arc geometry, and quasiconformal mappings has been studied extensively; see for example Semmes [13].

More generally, analytic conditions on the logarithm of the derivative of conformal maps encode subtle geometric properties of the boundary curves. This interplay between function spaces and planar geometry is a central theme in geometric function theory.

2 Lacunary Power Series and Random Power Series

To understand the claim stated in Theorem 1, it is natural to examine special classes of analytic functions. Consider a *lacunary power series* $\varphi(z) = \sum_{k=1}^{\infty} b_k z^{n_k}$ with Hadamard gaps

$$q = \inf_k \frac{n_{k+1}}{n_k} > 1.$$

Such series exhibit simple descriptions for several analytic function spaces. For instance, $\varphi \in B(\mathbb{D})$ if and only if $\sup_k |b_k| < \infty$, and $\varphi \in B_0(\mathbb{D})$ if and only if $b_k \to 0$ $(k \to \infty)$ (see [12, Prop.8.12]).

However, for gap series, the Hardy space norm and the BMOA norm are comparable in terms of q (see Hallenbeck and Samotij [7, Lem.1]):

$$\|\varphi\|_{H^2} = \sum_{k=1}^{\infty} |b_k|^2 \asymp \|\varphi\|_{\mathrm{BMOA}}.$$

Because polynomials are dense in the Hardy space $H^2(\mathbb{D})$ and belong to $\mathrm{VMOA}(\mathbb{D})$, this observation suggests that lacunary series cannot detect the difference between BMOA and VMOA, and not provide our desired function. Thus a different approach is required.

Random analytic functions provide another perspective on the difference between the spaces $\mathrm{BMOA}(\mathbb{D}) \cap B_0(\mathbb{D})$ and $\mathrm{VMOA}(\mathbb{D})$. This also has certain resemblance to the characterization for lacunary power series.

Let $\omega = (\omega_n)$ be the Steinhaus sequence, which is a discrete stochastic process on a suitable probability space consisting of random variables taking values in $\mathbb{S}$. Consider the *random power series*

$$\varphi_\omega(z) = \sum_{n=0}^{\infty} a_n \omega_n z^n.$$

Such random analytic functions have been studied extensively in connection with Bloch functions and BMOA. The boundary behavior of φ_ω can be controlled in terms of the growth of the coefficients $\{a_n\}$.

For example, if

$$(\sum_{k=0}^{n} k^2 |a_k|^2)^{1/2} = O(n/\sqrt{\log n}) \quad (= o(n/\sqrt{\log n}))$$

as $n \to \infty$ then φ_ω belongs to $B(\mathbb{D})$ (to $B_0(\mathbb{D})$, respectively) almost surely (a.s) (see Anderson et al. [1, Th.3.6]). Moreover, if $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ then φ_ω belongs to the Hardy space $H^p(\mathbb{D})$ a.s for every $p > 0$ (see Duren [3]), and if

$$\sum_{n=0}^{\infty} |a_n|^2 \log(n+1) < \infty$$

then $\varphi_\omega \in \mathrm{VMOA}(\mathbb{D})$ a.s. See Sledd [15], in which a problem was also raised concerning the distinction between BMOA and VMOA in the random setting. Later, certain refined conditions on the coefficients $\{a_n\}$ for φ_ω to be in $B_0(\mathbb{D})$ and $\mathrm{VMOA}(\mathbb{D})$ a.s were given in Wulan [16], aiming at the separation of these classes.

On the other hand, by a different model where $\xi = (\xi_n)$ is the complex random variables with the independent 2-dimensional Gaussian distributions, the random power series

$$\psi_\xi(z) = \sum_{n=0}^{\infty} a_n \xi_n z^n$$

has been investigated. For this, $\mathrm{BMOA}(\mathbb{D})$ and $\mathrm{VMOA}(\mathbb{D})$ were separated successfully by Nishry and Paquette [10, Th.5.9]. However, it seems that this example of ψ_ξ also fails to belong to $B_0(\mathbb{D})$. Consequently, while BMOA and VMOA are now separated, the more refined comparison between $\mathrm{BMOA}(\mathbb{D}) \cap B_0(\mathbb{D})$ and $\mathrm{VMOA}(\mathbb{D})$ remains open in the random setting.

3 Boundary Homeomorphisms and Teichmüller spaces

The analytic problem can be translated into a problem about boundary homeomorphisms. Hereafter, we always assume that the Bloch norm of $\varphi \in B(\mathbb{D})$ is so small that there exists a conformal mapping $f : \mathbb{D} \to \Omega \subset \mathbb{C}$ such that $\log f' = \varphi$ and f extends to a quasiconformal self-homeomorphism of $\mathbb{C}$.

Let g be a conformal mapping from the exterior disk onto the complementary domain of Ω. The boundary maps of f and g define the *conformal welding* $h = g^{-1} \circ f|_{\mathbb{S}}$, which is a *quasisymmetric* homeomorphism of the unit circle $\mathbb{S}$ onto itself; it satisfies uniform boundedness of the quasisymmetry quotient. Moreover, $\varphi \in B_0(\mathbb{D})$ if and only if h is *symmetric* (see Gardiner and Sullivan [5]), that is, the quasisymmetry quotient satisfies the vanishing condition:

$$\lim_{t \to 0} \sup_{x \in \mathbb{R}} \left| \frac{h(e^{i(x+t)}) - h(e^{ix})}{h(e^{ix}) - h(e^{i(x-t)})} \right| = 1.$$

In addition, $\varphi \in \mathrm{BMOA}(\mathbb{D})$ if and only if h is absolutely continuous and h' is an A_∞-weight in the sense of Muckenhoupt (which imples that $\log h'$ is in BMO), and further $\varphi \in \mathrm{VMOA}(\mathbb{D})$ if and only if $\log h'$ is in VMO (see Shen and Wei [14]). Therefore, the complex analytic claim can be reformulated as follows.

Real analytic problem. Does there exist an absolutely continuous symmetric homeomorphism $h : \mathbb{S} \to \mathbb{S}$ such that h' is an A_∞-weight but $\log h'$ is not a VMO function?

However, it seems difficult to construct an explicit element in this possible gap. For instance, while $\log |z-1|$ on $\mathbb{S}$ lies in the difference between BMO and VMO, the integral of $|z-1|$ near 1 does not yield a vanishing quasisymmetry quotient.

The problem considered here also has a natural interpretation in the framework of Teichmüller theory. Let

$$M(\mathbb{D}^*) = \{\mu \in L^\infty(\mathbb{D}^*) \mid \|\mu\|_\infty < 1\}$$

denote the space of Beltrami coefficients on the exterior disk. A Beltrami coefficient $\mu \in M(\mathbb{D}^*)$ determines uniquely a quasiconformal self-homeomorphism of $\mathbb{D}^*$ with complex dilatation μ up to post-composition of a Möbius transformation, which extends to a quasisymmetric homeomorphism h_μ of $\mathbb{S}$. The *universal Teichmüller space* T can be identified with the set of all such quasisymmetric homeomorphisms h_μ with certain normalization. See the standard textbook on Teichmüller spaces by Lehto [8].

Various subspaces of T are characterized by subspaces of $M(\mathbb{D}^*)$ consisting of Beltrami coefficients with specific properties:

$$M_0(\mathbb{D}^*) = \{\mu \in M(\mathbb{D}^*) \mid \lim_{t\to 0} \|\mu \cdot 1_{|z|<1+t}\|_\infty = 0\};$$
$$M_B(\mathbb{D}^*) = \{\mu \in M(\mathbb{D}^*) \mid \frac{|\mu(z)|^2}{|z|^2-1}dxdy : \text{ Carleson measure}\};$$
$$M_V(\mathbb{D}^*) = \{\mu \in M_B(\mathbb{D}^*) \mid \frac{|\mu(z)|^2}{|z|^2-1}dxdy : \text{ vanishing}\}.$$

The corresponding sets of quasisymmetric homeomorphisms h_μ for μ in these classes are defined as the *little universal* Teichmüller space T_0, the *BMO* Teichmüller space T_B, and the *VMO* Teichmüller space T_V, respectively (see Astala and Zinsmeister [2] as well as [5, 14]).

We consider the correspondence from $M(\mathbb{D}^*)$ to $B(\mathbb{D})$ so that the quasisymmetric homeomorphism h_μ for a given $\mu \in M(\mathbb{D}^*)$ coincides with the conformal welding induced by a conformal map f with $\log f' \in B(\mathbb{D})$. Although this is not well defined as a map, the following correspondences between the subspaces are consistently determined:

$$M_0(\mathbb{D}^*) \longrightarrow B_0(\mathbb{D});$$
$$M_B(\mathbb{D}^*) \longrightarrow \mathrm{BMOA}(\mathbb{D});$$
$$M_V(\mathbb{D}^*) \longrightarrow \mathrm{VMOA}(\mathbb{D}).$$

In other words, the Teichmüller spaces T_0, T_B, and T_V correspond precisely to the classes of quasisymmetric homeomorphisms described in the real analytic problem.

For the problem on the strict inclusion of VMOA($\mathbb{D}$) in BMOA($\mathbb{D}$) $\cap$ $B_0(\mathbb{D})$, one may think of the strictness of

$$M_V(\mathbb{D}^*) \subset M_B(\mathbb{D}^*) \cap M_0(\mathbb{D}^*)$$

in view of the above correspondence. In fact, we can construct an example of a Beltrami coefficient in this difference set explicitly as follows: For each integer $n \geq 2$, we take an interval I_n on $\mathbb{S}$ of length 2^{-n} so that $\{I_n\}$ are mutually disjoint. Let

$$E_n = \{Re^{i\theta} \in \mathbb{D}^* \mid e^{i\theta} \in I_n,\ (2e)^{-n} < R - 1 < 2^{-n}\},$$

and define a Beltrami coefficient $\mu \in M(\mathbb{D}^*)$ so that $\mu(z) = 1/\sqrt{n}$ on E_n for all n and $\mu(z) = 0$ elsewhere.

However, this does not yield the answer to the original problem because even though $\mu \notin M_V(\mathbb{D}^*)$, for the conformal map f on $\mathbb{D}$ whose quasiconformal extension has complex dilatation μ, $\log f'$ can be in $\mathrm{VMOA}(\mathbb{D})$. The real analytic problem mentioned above is exactly asking the strictness of

$$T_V \subset T_B \cap T_0.$$

4 Geometric Interpretation

The boundary map corresponds to the boundary curve of the image domain. The analytic problems can be interpreted in terms of the geometry of curves. Let $f : \mathbb{D} \to \Omega \subset \mathbb{C}$ be conformal and denote the boundary Jordan curve by $\Gamma = \partial\Omega$. Then Γ is a quasicircle if the Bloch norm of $\log f'$ is sufficiently small, since f extends to a quasiconformal self-homeomorphism of $\mathbb{C}$ in this case, as we are assuming.

Moreover, when $\log f' \in \mathrm{BMOA}(\mathbb{D})$ is of small BMOA norm, Γ is a *chord-arc curve* (see [2, 13]). This means that Γ is rectifiable and there exists a constant $C \geq 1$ such that

$$\frac{\mathrm{length}(\widehat{z_1 z_2})}{|z_1 - z_2|} \leq C$$

for every pair of points z_1, z_2 on Γ, where $\widehat{z_1 z_2}$ denotes the smaller subarc of Γ between them.

Two finer notions of boundary regularity are relevant. A bounded Jordan curve Γ is called *asymptotically conformal* if

$$\lim_{\ell \to 0} \sup_{|z_1 - z_2| \leq \ell} \max_{w \in \widehat{z_1 z_2}} \frac{|z_1 - w| + |z_2 - w|}{|z_1 - z_2|} = 1.$$

An asymptotically conformal Jordan curve is a quasicircle. A rectifiable Jordan curve is called *asymptotically smooth* if

$$\lim_{\ell \to 0} \sup_{|z_1 - z_2| \leq \ell} \frac{\mathrm{length}(\widehat{z_1 z_2})}{|z_1 - z_2|} = 1.$$

An asymptotically smooth Jordan curve is a chord-arc curve.

The connection between analytic function spaces and these geometric properties is provided by Pommerenke [11] in the following theorem.

Theorem 3. *Let* $f : \mathbb{D} \to \Omega$ *be a conformal homeomorphism, and let* $\Gamma = \partial\Omega$ *be a bounded Jordan curve. Then,* $\log f' \in B_0(\mathbb{D})$ *if and only if* Γ *is asymptotically conformal, and* $\log f' \in \mathrm{VMOA}(\mathbb{D})$ *if and only if* Γ *is asymptotically smooth.*

Remark. Even under the sole assumption that Γ is a Jordan curve as in the above theorem, Γ turns out to be a quasicircle if $\log f'$ is in $B_0(\mathbb{D})$.

Therefore the complex analytic problem reduces to a purely geometric question:

Geometric problem. Does there exist a chord-arc Jordan curve that is asymptotically conformal but not asymptotically smooth?

This is the reduction of Theorem 1 to Theorem 2.

A natural attempt to construct the desired example is to consider domains bounded by graphs. For instance, Gallardo-Gutiérrez et al. [4, Example 4] illustrated the domain Ω bounded by the graph

$$y = x^2 \cos(\pi/x^2)$$

on the interval $[-1, 1]$ and a smooth curve joining the endpoints of the graph harmlessly. For the conformal homeomorphism $g : \mathbb{D} \to \Omega$, the boundary curve is not rectifiable, and hence $\log g' \notin \mathrm{VMOA}(\mathbb{D})$ by Theorem 3. On the other hand, for any conformal mapping g of $\mathbb{D}$ to a graph domain Ω in general, we have $\log g' \in \mathrm{BMOA}(\mathbb{D})$ (see [2, Th.6]). This is essentially because $\arg \log g'$ is a bounded harmonic function (see [12, p.172]).

It was asserted in [4] that the curve above is asymptotically conformal, which would imply $\log g' \in B_0(\mathbb{D})$. If this were the case, it would provide our desired example. However, this is incorrect. In fact, this graph domain cannot produce the desired example.

One may expect that asymptotic conformality of a graph curve implies asymptotic smoothness. If so, graph domains cannot produce the example in Theorem 2. However, this expectation is also false. In our recent research, we are interested in finding an example of a non-rectifiable graph curve that is asymptotically conformal. This will produce the example required in Theorem 2. Before pursuing this possibility, we constructed the desired example using curves that are not graphs.

5 Construction of the Curve

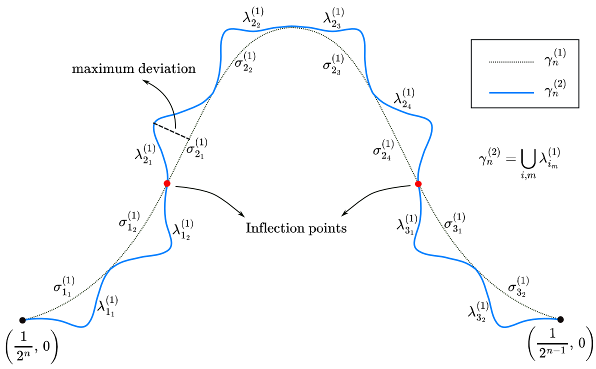

Figure 1: Bell-shaped arc is replaced by many smaller bells

The construction is inspired by the classical Koch curve. In this construction, triangular bumps are attached repeatedly to segments, and the limiting curve is not rectifiable. To make it a rectifiable curve, the bumps must become flatter as the generation proceeds. However, sharp corners destroy asymptotic conformality. Instead we replace the triangular bumps with smooth bell-shaped arcs. Each bell-shaped arc is replaced by many smaller bells. We control the size of these smaller bells and their flatness at each generation.

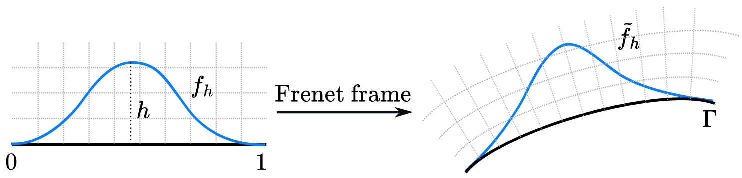

Figure 2: Bell-shaped perturbations in Frenet coordinates

Using Frenet coordinates, the perturbation can be inserted along the curve. The key idea of the construction is that when a small portion of a curve is magnified, its curvature becomes small. Thus a bell-shaped perturbation inserted in the Frenet coordinates behaves almost like a similarity transformation. By making the bells smaller, we obtain a better approximation to a similarity, while keeping the total length of the replacement by these bell-shaped arcs almost the same. This observation allows us to control the length of the curve while introducing infinitely many oscillations.

The resulting curve γ is chord-arc, asymptotically conformal, but not asymptotically smooth. For the detailed construction, see [9].

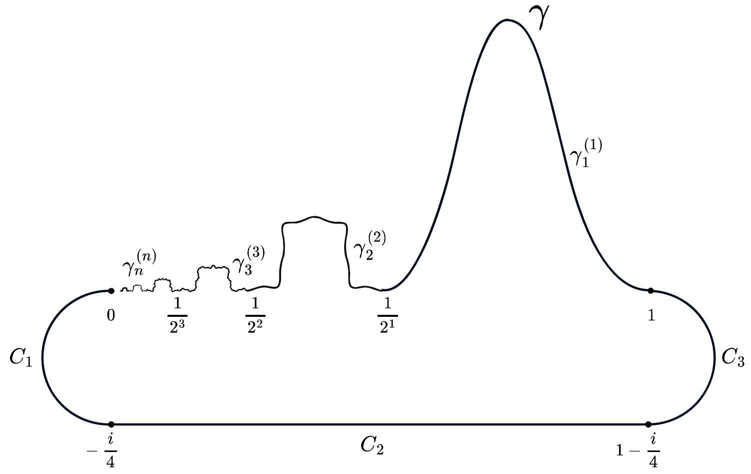

Figure 3: Construction of the curve γ

References

[1] J.M. Anderson, J. Clunie and Ch. Pommerenke, *On Bloch function and normal functions*, J. Reine Angew. Math. **270** (1974), 12–37.

[2] K. Astala and M. Zinsmeister, *Teichmüller spaces and* BMOA, Math. Ann. **289** (1991), 613–625.

[3] P. Duren, *Random series and bounded mean oscillation*, Michigan J. Math. **32** (1985), 81–86.

[4] E.A. Gallardo-Gutiérres, M.J. González, F. Pérez-González, Ch. Pommerenke and J. Rättyä, *Locally univalent functions,* VMOA *and the Dirichlet space*, Proc. London Math. Soc. **106** (2013), 565–588.

[5] F.P. Gardiner and D. Sullivan, *Symmetric structure on a closed curve*, Amer. J. Math. **114** (1992), 683–736.

[6] D. Girela, *Analytic functions of bounded mean oscillation*, Complex Function Spaces (Mekrijärvi, 1999), pp. 61–170, Univ. Joensuu Dept. Math. Rep. Ser. No. 4, 2001.

[7] D.J. Hallenbeck and K. Samotij, *On radial variation of bounded analytic functions*, Complex Variables **15** (1990), 43–52.

[8] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Grad. Texts in Math. 109, Springer, 1987.

[9] K. Matsuzaki and F. Tao, Characterization of asymptotically smooth curves, arXiv:2510.26514.

[10] A. Nishry and E. Paquette, *Gaussian analytic functions of bounded mean oscillation*, Analysis and PDE **16** (2023), 89–117.

[11] Ch. Pommerenke, *On univalent functions, Bloch functions and* VMOA, Math. Ann. **236** (1978), 199–208.

[12] Ch. Pommerenke, Boundary Behaviour of Conformal Maps, Grundlehren Math. Wiss. 299, Springer, 1992.

[13] S. Semmes, *Quasiconformal mappings and chord-arc curves*, Trans. Amer. Math. Soc. **306** (1988), 233–263.

[14] Y. Shen and H. Wei, *Universal Teichmüller space and* BMO, Adv. Math. **234** (2013), 129–148.

[15] W.T. Sledd, *Random series which are* BMO *or Bloch*, Michigan J. Math. **28** (1981), 259–266.

[16] H. Wulan, *Random power series with bounded mean oscillation characteristics*, Acta Math. Sci. (Chinese) **14** (1994), 451–457.