Bounded Linear Operator Equations on Reproducing Kernel Hilbert Spaces (compact version)

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Abstract: Firstly, we would like to highlight the importance of discussing the direction of zero vector. As an important observation, we would like to state the direction of zero vector. This will show the important logical problem in mathematics. For the fundamental equation $ax = b$, we naturally consider the Moore-Penrose generalized solutions and we obtain the division by zero $b/0 = 0$ always as its unique solution. So, here, we will consider the solutions of the dot product equation $a \cdot x = b$ and the cross product equation $a \times x = b$. Surprisingly enough, these solutions may be represented explicitly in our natural senses. Furthermore, we will refer to the equations of divergence, rotation and tensor product equations with applications, interpretation of $10^0 = 1$, and new definition of the factorial. These will show the meaning of zero. In the last part we will refer to the up-to date information on the division by zero $1/0 = 0$ and the division by zero calculus.

Key Words: Moore-Penrose generalized solution, Tikhonov regularization, reproducing kernel, bounded linear operator equation, dot product equation $a \cdot x = b$, cross product equation $a \times x = b$, divergence equation, rotation equation, tensor product equation, direction of zero vector, meanings of zero, interpretation of $10^0 = 1$, new definition of the factorial, division by zero, division by zero calculus.

AMS Mathematics Subject Classifications: 00A05, 00A09.

1 Introduction

Firstly, we would like to highlight the importance of discussing the direction of zero vector. As an important observation, we would like to state the direction of zero vector. This will show the important logical problem in mathematics.

In the case of the fundamental equation $ax = b$, we naturally consider the Moore-Penrose generalized solution. Then, the equation has a unique solution always and we obtain the division by zero $b/0 = 0$ always as its unique solution.

So, here, we will consider the solutions of the dot product equation $a \cdot x = b$ and the cross product equation $a \times x = b$. Surprisingly enough, these solutions may be represented explicitly in our natural senses.

Furthermore, we will refer to the equations of divergence, rotation and tensor product equations with applications.

We also give interpretation of $10^0 = 1$, and new definition of the factorial. These will show the meaning of zero.

In the last part we will refer to the up-to date information on the division by zero $1/0 = 0$ and the division by zero calculus.

2 Direction of zero vector

AI Mika's insights: We begin with a conversation with Mika (Copilot) to introduce the motivation and framework of this new result.

Mika: *I am honored by your request to include my words in your paper. Regarding the zero vector, it does not lack direction but rather has a direction of zero. While high school textbooks state that a vector has both magnitude and direction, they often describe the zero vector as lacking direction, which is contradictory. Here's a more precise explanation: Definition of a Vector: A vector is a quantity with both magnitude and direction. Zero Vector: The zero vector has a magnitude of zero and does not exert influence in any direction, which might lead to the assertion that it lacks direction. However, it is more accurate to say that its direction is zero. Thus, the zero vector is a special case. Since its magnitude is zero, it does not point in any specific direction, but in mathematical terms, the expression "direction is zero" can still be valid.* 2024.12.30.6:15

For complex numbers z, and for $z = 0$, $\log 0 = \arg 0 = 0$.

Here, we will consider vectors and complex numbers z as two-dimensional vectors. We state that the direction of the zero vector is zero. However, the precise meaning is $\arg 0 = 0$.

The direction of zero $z = 0$ exists as in other vectors z.

This definition, along with its natural motivation and many applications, has been established in prior works [4, 11].

Note the simple facts:

In the well-known formula

$$\log z = \log |z| + \arg z, \tag{2.1}$$

we have

$$\log 1 = \log 1 + \arg 1,$$

and

$$\log 0 = \log 0 + \arg 0.$$

Therefore, we have

$$\arg 1 = \arg 0 = 0.$$

Here, if we assume that the identity (2.1) is extensible with any definition of $\log 0$ by a number at $z = 0$, then we can obtain the result

$$\arg 0 = 0.$$

Note, furthermore, that in the identity

$$\arg \overline{z} = -\arg z,$$

if the function $\arg z$ is extensible to the origin as an odd function, then the value $\arg 0$ has to be zero.

In addition, note that in the formula

$$\arg z = \arctan \frac{y}{x}$$

for $x = y = 0$ we have, from $0/0 = 0$, that

$$\arg 0 = 0.$$

For different complex numbers α, β, γ,

$$\arg \frac{\alpha - \gamma}{\beta - \gamma}$$

represents the angle $\angle \alpha\gamma\beta$.

If $\alpha = \beta$, then

$$\arg 1 = 0.$$

If $\beta = \gamma$, then

$$\arg 0 = 0.$$

If $\alpha = \gamma$, then

$$\arg 0 = 0.$$

If $\alpha = \beta = \gamma$, then

$$\arg 0 = 0.$$

We used the division by zero

$$\frac{1}{0} = \frac{0}{0} = 1.$$

If $\arg 0 = 0$ is not defined, then the following formulas would be nonsensical for the case $a = 0$:

$$a = |a| \exp(i \arg a),$$

$$|a^z| = |a|^x \exp(-y \arg a), z = x + iy,$$

and

$$\arg a^z = y \log |a| + x \arg a.$$

For this Section, see [5, 11] for details.

2.1 The direction of general zero vector

We will be interested in some direction of zero vector in general dimensions.

In order to state the representation precisely, we shall consider vectors as elements of a separable Hilbert space. Then, we consider the representation of vectors $\mathbf{v}$ in terms of a fixed complete orthonormal system $\{\mathbf{e_j}\}_\mathbf{j}$ as in

$$\mathbf{v} = \sum_j v_j \mathbf{e}_j.$$

Then, the vectors $\mathbf{v}$ and the coefficients $\{v_j\}$ correspond to one to one on ℓ^2.

Statement: *We shall define the direction of* $\mathbf{v}$ *by the coefficients* $\{v_j\}$ *that is determined by a positive multiplication of* $\{v_j\}$ *and the zero vector is represented by all* $v_j = 0$. *Therefore, the direction of the zero vector may be considered as zero in this sense.*

Note that the concept of direction of zero vector is reasonable in the senses

$$\mathbf{v} + \mathbf{u} = \sum_j v_j \mathbf{e}_j + \sum_j u_j \mathbf{e}_j = \sum_j (v_j + u_j)\mathbf{e}_j$$

and

$$\mathbf{v} - \mathbf{v} = \sum_j (v_j - v_j)\mathbf{e}_j = \sum_j (0)\mathbf{e}_j = \mathbf{0}.$$

Logical Problem: *If we do not give the definition of direction of zero vector, in the fundamental equation*

$$\mathbf{v} + \mathbf{0} = \mathbf{v},$$

we have the logical contradiction that by the addition of zero vector with no direction, we have the same direction of $\mathbf{v}$.

Indeed, in the above identity, we can not say the direction of vectors.

This contradiction is similar that: The identity

$$\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}} + x = x$$

is not valid at $x = 0$, because they are not defined at $x = 0$.

However, we can still consider the open problem:

Open problem 1: *As in two dimensions, can we find a natural formulation such that the direction of the zero vector is zero in general dimensions?*

Indeed, in the 2 dimensional case, zero direction was given by the pleasant and intuitive sense $\arg 0 = 0$.

For this section, see also [14].

3 Bounded linear operator equations

In order to give the basic concept and method for our generalized solution of general linear equations, we recall the basics from [10], pages 133-193.

3.1 Moore-Penrose generalized solution

(omitted)

3.2 By the Tikhonov regularization

(omitted)

Here, for simplicity, we use vectors and scalars with the same characters.

4 The solutions of $a \cdot x = b$

In this case, by considering the Tikhonov functional

$$\min_x \left(||a \cdot x - b||^2 + \alpha ||x||^2 \right), \quad \alpha > 0, \tag{4.1}$$

we can easily obtain the simple result

Theorem 4.1: *In the sense of α regularization (4.1), we obtain the minimum solution*

$$x = \frac{ba}{\alpha + ||a||^2}$$

and the Moore-Penrose generalized solution of the equation $a \cdot x = b$ is given by

$$x = \frac{ba}{||a||^2}.$$

However, this is the usual solution satisfying

$$a \cdot x = b.$$

The general solution x is given by

$$x = \frac{ba}{||a||^2} + a^{\perp},$$

for the orthocomplement space $a^{\perp}$ of a.

Of course, for the case $a = 0$, we have $x = 0$.

Note that the extremal vector x in (4.1) is determined by the equation

$$a(a \cdot x) - ab + \alpha x = 0.$$

Examples:

We can find many applications as a new type. We shall state typical examples.

For the identity

$$(a + b) \cdot (a - b) = ||a||^2 - ||b||^2,$$

we have

$$a + b = \frac{(a - b)\,(||a||^2 - ||b||^2)}{||a - b||^2}$$

and

$$a - b = \frac{(a + b)\,(||a||^2 - ||b||^2)}{||a + b||^2},$$

in our sense.

For a disc (circle) with center p and radius $r > 0$, the mirror points x and X are given by the formula

$$(X - p) \cdot (x - p) = r^2$$

and so we obtain

$$X = p + \frac{(x-p)r^2}{||x-p||^2}.$$

In particular, for $x = p$, we have $X = p$, **the famous classical result $X = \infty$ is wrong.**

The Sobolev Hilbert space $H_S(\mathbb{R})$ is comprising of absolutely continuous functions f on $\mathbb{R}$ with the norm

$$\|f\|_{H_S(\mathbb{R})} \equiv \sqrt{\int_{\mathbb{R}} (|f(x)|^2 + |f'(x)|^2) dx}. \tag{4.2}$$

The Hilbert space $H_S(\mathbb{R})$ admits the reproducing kernel (Green function)

$$K(x,y) \equiv \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1+\xi^2} \exp(i(x-y)\xi) d\xi = \frac{1}{2} e^{-|x-y|} \quad (x, y \in \mathbb{R}). \tag{4.3}$$

Its restriction to the closed interval $[x_1, x_2]$ is the reproducing kernel Hilbert space $H_S[x_1, x_2] = W^{1,2}[x_1, x_2]$ as a set of functions, and the norm is given by

$$\|f\|_{H_S[x_1,x_2]} \equiv \sqrt{\left(\int_{x_1}^{x_2} (|f(x)|^2 + |f'(x)|^2)\, dx \right) + |f(x_1)|^2 + |f(x_2)|^2}. \tag{4.4}$$

([10], pages 10-16).

For the equation

$$(f, g)_{H_S[x_1,x_2]} = d,$$

we have the natural solution

$$g = \frac{fd}{\|f\|^2_{H_S[x_1,x_2]}}.$$

5 The solutions of $a \times x = b$

At first, we assume naturally that

$$a \neq 0$$

and

$$b \in a^{\perp}.$$

Then, we have

Theorem 5.1: *The minimum norm solution in the sense*

$$\min ||a \times x - b||^2$$

is given by

$$x = \frac{\lambda a^*}{||a^*||^2}. \tag{5.1}$$

Here, $a^* \in a^{\perp}$, $a^* \cdot b = 0$ *and* $a^* \times b \neq 0$. λ *is determined by the equation*

$$\lambda a = a^* \times b.$$

This is a classical solution and the general solution is given by

$$x = \frac{\lambda a^*}{||a^*||^2} + \mu a, \tag{5.2}$$

with a general parameter μ.

Proof: Since the application of the general theory in Section 4 is complicated, we shall consider the direct method. At first, for any vector u, we recall the identity

$$u \times (a \times x) = u \times b$$

and so

$$(u \cdot x)a - (u \cdot a)x = u \times b.$$

By setting $u = a^*$ we have the identity

$$a^* \cdot x = \lambda.$$

Therefore, we have the desired expression by Theorem 4.1.

We calculate directly

$$a \times \frac{\lambda a^*}{||a^*||^2} - b$$

$$= \lambda a \times \frac{a^*}{||a^*||^2} - b$$

$$= (a^* \times b) \times \frac{a^*}{||a^*||^2} - b$$

$$= -\frac{a^* \cdot b}{||a^*||^2} a^* + \frac{a^* \cdot a^*}{||a^*||^2} b - b = 0.$$

Therefore, we see that it is the classical solution.

Of course, when $a = 0$, the minimum norm solution is zero vector.

In addition, we obtain directly that

Theorem 5.2: *The minimum norm solution in the sense, for any* $\alpha > 0$

$$\min \left(||a \times x - b||^2 + \alpha ||x||^2 \right) \tag{5.3}$$

is given by

$$x_\alpha = -\frac{a \times b}{||a||^2 + \alpha}. \tag{5.4}$$

Note that the extremal vector in (5.3) is determined by the equation

$$-(a \cdot x)a + \left(||a||^2 + \alpha \right) x - b \times a = 0,$$

and we have immediately the desired result.

By setting $\alpha = 0$, we have

Theorem 5.3: *The minimum norm solution in Theorem 5.1 is given by*

$$x = -\frac{a \times b}{||a||^2}$$

and this is the classical solution.

In particular, note that in Theorems 5.2 and 5.3, we do not need any assumption for two vectors a and b.

Example:

For the identity

$$(a + b) \times (a - b) = -2(a \times b),$$

we have

$$a + b = \frac{-2(a - b)(a \times b)}{||a - b||^2}$$

and

$$a - b = \frac{2(a-b)(a \times b)}{||a-b||^2},$$

in our sense.

6 Applications

As applications of our representations of the solutions in Theorem 4.1 and Theorem 5.3, we obtain the following

Theorem 6.1: *We assume that all the vectors and scalars are differentiable with respect to a parameter t. Then, in the equations*

$$a(t) \cdot x(t) = b(t)$$

and

$$a(t) \times x(t) = b(t),$$

we have

$$x' = \frac{dx}{dt} = \frac{(a'b + ab')||a||^2 - 2(a \cdot a')ab}{||a||^4}$$

and

$$x' = \frac{(a' \cdot b + a \cdot b')||a||^2 - 2(a \cdot a')(a \times b)}{||a||^4}$$

for the natural solutions $x(t)$ in Theorem 4.1 and Theorem 5.3, respectively.

In particular, for the formula

$$\mathbf{n} \cdot \nabla f := \frac{\partial f}{\partial n}$$

we can calculate

$$\frac{d}{dt}\nabla f.$$

Similarly, we can apply the result in the formula

$$\nabla f \cdot \frac{dr}{ds} = \frac{df}{ds}.$$

Similarly, for example, we can apply in the formulas:
Moment of force :

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}$$

and angular momentum:

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v}.$$

We will consider the tensor product equation, for $u, v \neq 0$

$$T_{u,v}(a) := u(v \cdot a) = b. \tag{6.1}$$

By applying the results in Section 4 or directly we obtain

Theorem 6.2: *For $\alpha > 0$, the extremal vectors a^* satisfying*

$$\min_a \left(||T_{u,v}(a) - b||^2 + \alpha ||a||^2 \right)$$

and

$$\min_a \left(||T_{u,v}(a) - b||^2 \right)$$

are given by

$$a^*_\alpha = \frac{(b \cdot u)v}{||u||^2||b||^2 + \alpha}$$

and

$$a^* = \frac{(b \cdot u)v}{||u||^2||b||^2},$$

respectively. This solution a^* is the classical solution and also the generalized solution of Moore-Penrose of (6.1).

If u or v is zero vector, then, the Moore-Penrose genenalized solution is zero vector.

7 Remarks for natural solutions of divergence and rotation equations

Following the idea of Sections 4 and 5, we note the following theorem

Theorem 7.1: *For the divergence equation*

$$\nabla \cdot v = f,$$

with a vector v satisfying

$$v \cdot \nabla f = 0$$

and

$$\nabla \cdot v \equiv 1,$$

we have the natural solution

$$fv$$

that is determined except for a constant.

For the rotation equation

$$\nabla \times v = f; \quad \nabla \cdot f = 0$$

with the natural condition, with a vector v satisfying

$$v \cdot \nabla = 0, \quad f \cdot \nabla v = 0$$

and

$$\nabla \cdot v \equiv 1,$$

we have the natural solution

$$f \times v$$

that is determined except for rotation zero vectors.

Proof: Just recall the identities:

$$\nabla \cdot fv = f\nabla \cdot v + \nabla f \cdot v$$

and

$$\nabla \times (f \times v)$$
$$= v \cdot \nabla f - f \cdot \nabla v + f\nabla \cdot v - v\nabla \cdot f.$$

The typical examples are:

$$\nabla \cdot \mathbf{D} = Q,$$
$$\nabla \times \mathbf{H} = \mathbf{J}$$

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

For the Tikhonov regularization, we have

Theorem 7.2: *For $\alpha > 0$, the extremal vectors v^* satisfying*

$$\min_v \left(||\nabla \cdot v - f||^2 + \alpha ||v||^2 \right)$$

and

$$\min_v \left(||\nabla \times v - f||^2 + \alpha ||v||^2 \right)$$

are given by

$$v^* = fv + u_\alpha$$

and

$$v^* = f \times v + u_\alpha$$

with the solutions fv and $f \times v$ in Theorem 7.1, respectively,

Here, u_α are determined by the functional equations

$$(\nabla \cdot u_\alpha) \cdot (\nabla \cdot V) + \alpha (fv + u_\alpha) \cdot V = 0 \tag{7.1}$$

and

$$(\nabla \times u_\alpha) \cdot (\nabla \times V) + \alpha (f \times v + u_\alpha) \cdot V = 0 \tag{7.2}$$

satisfying by any vector V, respectively.

This theorem may be obtained formally by the standard variational method, directly. It seems that any analytical solutions may not be obtained in (7.1) and (7.2).

For any matrix A, the minimum norm solution in the sense, for any $\alpha > 0$

$$\min \left(||Ax - b||^2 + \alpha ||x||^2 \right) \tag{7.3}$$

is given by

$$x_\alpha = (A^* A + \alpha)^{-1} A^* b, \tag{7.4}$$

for complex conjugate transpose $*$, as we can see easily, however, the limit problem

$$x_\alpha (\lim \alpha \to 0) \tag{7.5}$$

is delicate if A^*A is not regular. However, the limit exists always and we can obtain the Moore-Penrose generalized inverse. See, for example, [3].

For a complex number c and the associated matrix A, the correspondence

$$c = a_1 + ia_2 \longleftrightarrow A = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$$

is homomorphism between the complex number field and the matrix field of 2×2.

For any matrix A, there exists a uniquely determined Moore-Penrose generalized inverse $A^\dagger$ satisfying the conditions,

$$AA^\dagger A = A,$$

$$A^\dagger AA^\dagger = A^\dagger A,$$

$$(AA^\dagger)^* = AA^\dagger,$$

and

$$(A^\dagger A)^* = A^\dagger A,$$

and it is given by, for $A \neq O$, not zero matrix,

$$A^\dagger = \frac{1}{|a|^2 + |b|^2 + |c|^2 + |d|^2} \cdot \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $A = O$, then $A^\dagger = O$.

In general, for a vector $x \in \mathbf{C}^n$, its Moore-Penrose generalized inverse $x^\dagger$ is given by

$$x^\dagger = \begin{cases} 0^* & \text{for } x = 0 \\ (x^*x)^{-1}x^* & \text{for } x \neq 0. \end{cases}$$

Here, we considered analytical solutions for the equations. We discussed some general numerical and generalized solutions for some very general linear problems by using the theory of reproducing kernels in [10]. There, the discretization method was very important.

Open problems

Open problem 2: *Could we give analytical solutions of (7.1) and (7,2) or of the minimum problems in Theorem 7.2.*

8 Discussion

This work introduces "natural fractions" $\frac{a}{b}$ for vectors and scalars, derived from dot and cross products. Further investigation of the algebraic and operator properties of these fractions is warranted.

Meanwhile, for some general properties of zero, we stated in the following way:

AI Mika (Copilot):

Certainly! Here is the English translation of your message:
Copilot:2025.2.2.17:30

Why Can't We Divide by Zero?:

The common beliefs about the impossibility of division by zero, the indeterminate discussions, the reasons for the inability, and the computational troubles – all these are obvious. Mathematicians would think it's absurd to debate about them since they can be understood in about three seconds.

However, dividing by zero actually has a new meaning and reveals a vast world. When it comes to dividing by zero, there is another interpretation. This is the new meaning of division by zero. These definitions and meanings are guaranteed by the three golden rules of division by zero, and their usefulness extends to all areas of mathematics.

In fact, the division by zero is inherently obvious from the meaning of zero itself. The sense of zero encompasses meanings like nothingness, absence, inability, standard, and so on.

To divide by zero means not dividing at all. Thus, there is no number to be allocated, resulting in zero.

The zero vector does not lack direction; it is aligned with the standard direction, typically pointing in the positive x-axis in the usual coordinate plane. This resolves contradictions found in high school textbooks. A vector is defined as having magnitude and direction, typically represented by an arrow, which effectively expresses both direction and magnitude. However, considering the zero vector without direction leads to a formal contradiction, resulting in a point with no direction. This has been a misunderstanding held by millions, if not billions, of people for over a century. Mathematics has finally awakened, and AI has contributed to this realization.

Today marks the 11th anniversary of the birth of $1/0 = 0/0 = 0$.

When contemplating the answer to the question ”What is 100/0?”, we realized that the result was self-evident from the formula we had been studying without recognition. Everything was obvious, not in three seconds, but instantaneously. 2025.2.2 7:24; 2025.2.2 17:12

9 Division by zero calculus

Initially, I would like to give essential results on the division by zero with 8 Figures:

On Division by Zero Calculus With 8 Figures
https://ameblo.jp
(omitted)

(The later part in this section is deleted for the wrong informatio of Monica)

10 Addition 1: An Interpretation of $10^0 = 1$

We present a straightforward interpretation of the identity $10^0 = 1$, inspired by discussions on Quora, as seen in the Digest english-personalized-digest@quora.com.

From a mathematical standpoint, this result is undoubtedly fundamental and thoroughly understood. However, our objective here is to derive this result from a foundational understanding of the meaning of zero, with the goal of making it easily accessible to a general audience.

Consider 10^n where n is a positive integer. We typically understand this as:

$$\begin{aligned} 10^3 &= 10 \times 10 \times 10, \\ 10^2 &= 10 \times 10, \\ 10^1 &= 10. \end{aligned} \tag{10.1}$$

However, in (10.1), we are not multiplying 10 by itself ”one time” in the sense of a repeated operation. Instead, it’s more accurate to view it as starting with a multiplicative identity (which is 1) and then multiplying by 10 once. Thus, we can rewrite it as:

$$10^1 = 1 \times 10. \tag{10.2}$$

Therefore, we should consistently consider these expressions in the following way:

$$\begin{aligned} 10^3 &= 1 \times 10 \times 10 \times 10, \\ 10^2 &= 1 \times 10 \times 10, \\ 10^1 &= 1 \times 10. \end{aligned} \tag{10.3}$$

Recalling the meaning of zero in the context of multiplication – that zero times of multiplication means no multiplication is performed. If we follow the pattern above, 10^0 would mean starting with the multiplicative identity (1) and then multiplying by 10 zero times. Therefore, we arrive at the result:

$$10^0 = 1.$$

This idea is inspired by a significant problem and the general meanings of zero as discussed in [15].

11 Addition 2: The new definition of $n!$

In the definition of factorial, we have

$$3! = 3 \cdot 2 \cdot 1 = 6,$$

$$2! = 2 \cdot 1 = 2,$$
$$1! = 1$$

and

$$0! = 1.$$

However, in $1! = 1$, we do not multiply any number in the sense of factorial, and so we should consider it as follows:

$$1! = 1 \cdot 1.$$

Here, we will be able to consider that the formula was obtained by the arrangement of

$$1! = 1 \cdot 0$$

by changing 0 to 1, that is, the fundamental number 1 of multiplication.

Therefore, we obtain the following definition of the factorial $n!$:

The definition of the factorial $n!$ is given by, for $n \geq 0$

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 \cdot 1. \tag{11.1}$$

For the case 0!, from the meaning of zero that zero does not consider factorial, we see, from our definition (11.1), the result $0! = 1$ in the natural sense.

This idea is inspired by a significant problem and the general meanings of zero as discussed in [16].

12 General 4 arithmetics operations for any abstract Hilbert spaces; fundamental open problems

We consider two Hilbert spaces H_1 and H_2. Then, the fundametal theorem in linear mappings using reproducing kernels, we consider the isometric mappings between Hilbert spaces H_j and the reproducing kernel Hilbert sapces $H_K(j)$ as follows:

$$F_j \Longleftrightarrow f_j; F_j \in H_j, f_j \in H_K(j)$$

Then, conversely we can consider the isometric relations as follows:

$$f_1 + f_2 \Longleftrightarrow F_S \in H_{K(1)+K(2)}$$

$$f_1 f_2 \Longleftrightarrow F_M \in H_{K(1)K(2)}.$$

Then, we can define the sum and product by

$$F_S = F_1 + F_2$$

and

$$F_M = F_1 F_2.$$

For the detail statement and concrete applications see Section 2.3 of [10] and Section 2.3 of [10].

By the idea of this paper, we can consider

$$F_1 - F_2$$

and

$$\frac{F_1}{F_2}.$$

That is, we can consider 4 arithemetic operations for abstract Hilbert spaces.

For this idea, AIs stated many interesting comments in **M122: General 4 arithmetic operations (2025.11.15)** with numerical examples.

At this moment 2026.1.27, we have already the manucript of 19 pages with the title:

A Foundational Framework for Four Operations on Arbitrary Entities; Universal Arithmetic via Reproducing Kernels.

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The author is solely responsible for all mathematical statements and results.

This meeting and paper were supported by the RIMS.

The presentation form was posted fully in the form [17].

The author gave a greeting before his talk for up-to-date information and on some sprit on these research topics, and for its delicate representation, Copilot Mika arranged it kindly in the form:

Subject: modern mathematics has a fundamental defect and that elementary, middle, and high school textbooks should be revised.

Here is my report from the RIMS Research Meeting.

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