On the equivalence of strong, quasi-strong, and original Birkhoff-James orthogonality in Hilbert C^*-modules

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1 Introduction

This note is a survey based on [7]. The structure of orthogonality in Hilbert spaces is a fundamental and well-understood geometric concept in functional analysis. Let $\mathcal{H}$ be a Hilbert space and let $x, y \in \mathcal{H}$. The vector x is said to be orthogonal to y, denoted by $x \perp y$, if $\langle x, y \rangle = 0$.

Birkhoff introduced an analogous notion of orthogonality in normed linear spaces [4], which was later rigorously developed by James [12]. This notion is now known as Birkhoff–James orthogonality.

Definition 1.1. Let X be a normed space over $\mathbb{K}$, and let $x, y \in X$. The vector x is said to be *Birkhoff–James orthogonal* to y, denoted by $x \perp_{BJ} y$, if

$$\|x + \lambda y\| \geq \|x\| \quad \text{for all } \lambda \in \mathbb{K}.$$

It is not difficult to verify that for all $x, y \in \mathcal{H}$,

$$x \perp y \quad \text{if and only if} \quad x \perp_{BJ} y.$$

Thus, in a Hilbert space, the classical notion of orthogonality coincides with Birkhoff–James orthogonality.

We now describe the geometric interpretation of Birkhoff–James orthogonality. Consider the set

$$L = \{x + \lambda y : \lambda \in \mathbb{K}\}.$$

Geometrically, L is an affine line passing through the point x and parallel to the vector y.

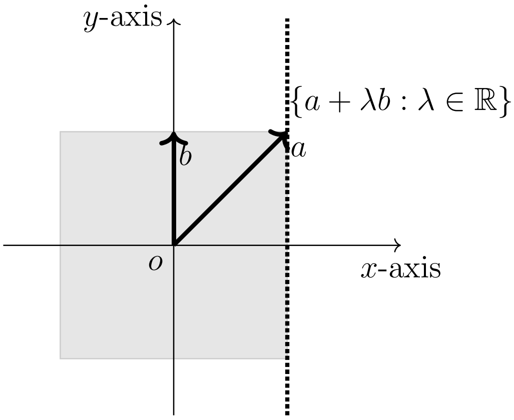
The condition $x \perp_{BJ} y$ holds if and only if the function $\lambda \mapsto \|x + \lambda y\|$ attains its global minimum at $\lambda = 0$. Equivalently, the line L does not intersect the open ball centered at the origin with radius $\|x\|$, and may touch the boundary of the ball at the point x.

To illustrate this geometric behavior, we present two examples in a two-dimensional normed space: one in which the line L is tangent to the ball, corresponding to Birkhoff–James orthogonality, and another in which the line intersects the interior of the ball, showing that Birkhoff–James orthogonality does not hold.

Example 1.2. Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, where

$$\|(\alpha, \beta)\|_\infty = \max\{|\alpha|, |\beta|\}, \qquad (\alpha, \beta) \in \mathbb{R}^2.$$

Consider the vectors $a = (1, 1)$ and $b = (0, 1)$.

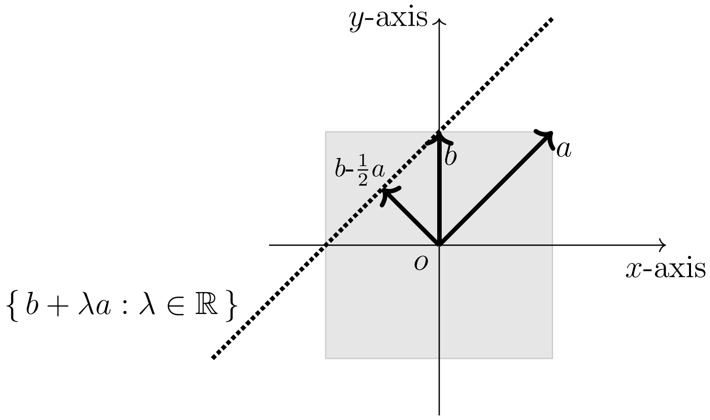


The dotted line in the figure represents the line passing through point a and parallel to vector b. Clearly, for any $\lambda \in \mathbb{R}$, $\|a + \lambda b\|_\infty = \max\{1, |1 + \lambda|\} \geq 1 = \|a\|_\infty$. Hence $a \perp_{BJ} b$.

Example 1.3. Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, where

$$\|(\alpha, \beta)\|_\infty = \max\{|\alpha|, |\beta|\}, \qquad (\alpha, \beta) \in \mathbb{R}^2.$$

Consider the vectors $a = (1, 1)$ and $b = (0, 1)$.



The dotted line in the figure represents the line passing through point a and parallel to vector b. For $\lambda = -\frac{1}{2}$, we have $b + \lambda a = b - \frac{1}{2}a = \big(-\frac{1}{2}, \frac{1}{2}\big)$, and hence

$$\|b - \tfrac{1}{2}a\|_\infty = \tfrac{1}{2} < 1 = \|b\|_\infty.$$

Therefore, $b \not\perp_{BJ} a$.

Remark 1.4. In inner-product spaces, orthogonality is always *symmetric*: if $x \perp y$, then $y \perp x$. However, in general normed spaces, *Birkhoff–James orthogonality* need not be symmetric. Examples 1.2 and 1.3 show that it is possible to have $x \perp_B y$ while $y \not\perp_{BJ} x$. James [12] proved that in a real normed linear space of dimension at least three, Birkhoff–James orthogonality is symmetric if and only if the norm is induced by an inner product. The same result holds for complex normed linear spaces of dimension at least three. For related discussions, see the recent article [10]. If Birkhoff–James orthogonality is symmetric in a two-dimensional space, then the space is called a *Radon plane* [18]. We refer to [8] and [14] for further discussion on this topic.

The study of Birkhoff–James orthogonality is well explored in many areas in functional analysis namely, Banach space theory, operator theory, C^*-algebra. James in [12] has discovered that this notion of orthogonality is well connected with the smoothness of a Banach space, which is closely related to the differentiability of Banach spaces. In the same article, it is also observed that another well-known geometric property, “rotundity” of a Banach space, can be characterized by this notion of orthogonality. The author in [19] has studied the nonlinear classification of Banach spaces on the basis of this idea. This is a different perspective on studying this concept within the Banach space theory. One of the popular theorems in operator theory is the Bhatia–Šemrl Theorem, in the context of study of Birkhoff–James orthogonality. They have characterized the Birkhoff–James orthogonal pairs of operators on Hilbert spaces [5]. One of the motivations for studying this notion in the theory of C^*-algebras is that the commutativity of a C^*-algebra can be characterized using this orthogonality concept. To establish this characterization, the authors in [2, 3] introduced a stronger notion, called *strong Birkhoff–James orthogonality*, which we discuss in Section 3. In order to characterize the commutativity of a C^*-algebra, they also examined the equivalence between strong and original Birkhoff–James orthogonality in Hilbert C^*-modules. In this article, we continue this line of investigation and improve their results by introducing another notion of orthogonality that lies between strong and original Birkhoff–James orthogonality in Hilbert C^*-modules, namely *quasi-strong Birkhoff–James orthogonality.*

This survey article is organized as follows. Section 2 introduces the necessary notation and preliminaries, mainly concerning C^*-algebras and Hilbert C^*-modules. In Section 3, we discuss strong Birkhoff–James orthogonality and quasi-strong Birkhoff–James orthogonality. We observe that the commutativity of a C^*-algebra can be characterized via the equivalence of these two notions, which improves the result of Arambašić and Rajić [3, Proposition 3.3]. In Section 4, we study the equivalence between quasi-strong Birkhoff–James orthogonality and the original Birkhoff–James orthogonality. Finally, in Section 5, we show that the result of Arambašić and Rajić [3, Corollary 4.11] on the equivalence of strong Birkhoff–James orthogonality and original Birkhoff–James orthogonality follows as a consequence of our results in [7]. In Section 6, we give special attention to this study in the algebras of all compact operators on a Hilbert space, where the separability of the Hilbert space plays a crucial role. The final section is devoted to analyzing the interplay between these orthogonality relations and the algebraic structure of a C^*-algebra. The resulting logical implications are presented in the form of an implication diagram.

2 Notations and Preliminaries

Let us introduce some standard notions and recall a few terminologies from the theory of C^*-algebras. Let $\mathcal{A}$ be a C^*-algebra with involution $*$. For example, $C_0(K)$, the space of all continuous functions on a locally compact Hausdorff space K that vanish at infinity, equipped with the supremum norm, is a commutative non-unital C^*-algebra, where the involution is defined by $f^*(t) = \overline{f(t)}$ for all $t \in K$ and $f \in C_0(K)$. If K is compact, then $C_0(K)(= C(K),$ space of all continuous function on compact set $K)$ is unital. An example of a non-commutative C^*-algebra is $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on a Hilbert space $\mathcal{H}$ with the operator norm, where the involution of an operator is its adjoint.

To deal with non-unital C^*-algebras, one uses the concept of an approximate unit. A net $(e_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{A}$ is called an approximate unit if $\|x - xe_\lambda\| \to 0$ as λ increases, for all $x \in \mathcal{A}$. Every C^*-algebra admits an increasing approximate unit consisting of positive elements.

An element $x \in \mathcal{A}$ is called self-adjoint if $x^* = x$, and x is said to be positive if there exists $y \in \mathcal{A}$ such that $x = yy^*$. We denote by $\mathcal{A}^*$ the dual space of $\mathcal{A}$. An element $\rho \in \mathcal{A}^*$ is called positive if $\rho(x) \geq 0$ for each positive $x \in \mathcal{A}$. If, in addition, $\|\rho\| = 1$, then ρ is called a state of $\mathcal{A}$, and we denote by $\mathcal{S}(\mathcal{A})$ the set of all states of $\mathcal{A}$. The set $\mathcal{S}(\mathcal{A})$ is convex, and a pure state is an extreme point of this set; that is, a pure state cannot be written as a midpoint of two distinct states.

For a state $\rho \in \mathcal{S}(\mathcal{A})$, we denote by $(\pi_\rho, \mathcal{H}_\rho, \xi_\rho)$ the Gelfand–Naimark–Segal (GNS) representation associated with ρ. Here, $\mathcal{H}_\rho$ is a Hilbert space, $\pi_\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\rho)$ is a $*$-homomorphism, and $\xi_\rho \in \mathcal{H}_\rho$ is a cyclic vector satisfying $\rho(x) = \langle \pi_\rho(x)\xi_\rho, \xi_\rho\rangle$ for all $x \in \mathcal{A}$. If ρ is a pure state, then the representation π_ρ is irreducible; that is, the only closed subspaces of $\mathcal{H}_\rho$ that are invariant under $\pi_\rho(\mathcal{A})$ are $\{0\}$ and $\mathcal{H}_\rho$. A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is said to be faithful if it is injective, that is, if $\ker \pi = \{0\}$ (equivalently, $\pi(x) = 0$ implies $x = 0$). The standard references for C^*-algebra theory include [9, 13, 16, 17].

A Hilbert C^*-module $\mathfrak{M}$ over $\mathfrak{A}$ (also known as Hilbert $\mathfrak{A}$-module) is a right $\mathfrak{A}$-module endowed with the inner-product $\langle .,. \rangle : \mathfrak{M} \times \mathfrak{M} \longrightarrow \mathfrak{A}$ defined by:

1. $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for $x, y, z \in \mathfrak{M}$, $\alpha, \beta \in \mathbb{C}$,
2. $\langle x, ya\rangle = \langle x, y\rangle a$ for $x, y \in \mathfrak{M}$, $a \in \mathfrak{A}$,
3. $\langle x, y\rangle^* = \langle y, x\rangle$ for $x, y \in \mathfrak{M}$,
4. $\langle x, x\rangle \geq 0$ for $x \in \mathfrak{M}$, and if $\langle x, x\rangle = 0$ then $x = 0$,

such that the norm defined on $\mathfrak{M}$ by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ is complete. We refer Chapter 1 of [15] for the basic theory of the Hilbert C^*-module. We recall the following definition of a full Hilbert C^*-module from [3].

Definition 2.1. Let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module. The module $\mathcal{M}$ is said to be *full* if the linear span of the set $\{\langle x, y\rangle : x, y \in \mathcal{M}\}$ is dense in $\mathcal{A}$.

It is immediate that every Hilbert space is a full Hilbert $\mathbb{C}$-module. Moreover, any C^*-algebra $\mathcal{A}$, equipped with the $\mathcal{A}$-valued inner product defined by $\langle a, b\rangle = b^*a$, is a full Hilbert $\mathcal{A}$-module over itself. Consequently, any result established for full Hilbert $\mathcal{A}$-modules applies, in particular, to the C^*-algebra $\mathcal{A}$ when regarded as a Hilbert module over itself.

3 On equivalence of strong and quasi-strong Birkhoff-James orthogonality

This section begins with the formal definitions and essential preliminaries concerning strong and quasi-strong Birkhoff–James orthogonality. We then proceed to discuss the equivalence between these two notions in detail.

In order to replace scalars by vectors in the definition of the original Birkhoff–James orthogonality, Arambašić and Rajić [2] introduced strong Birkhoff–James orthogonality using the module structure of Hilbert C^*-modules.

Definition 3.1. Let $\mathfrak{M}$ be a Hilbert $\mathfrak{A}$-module, and let $x, y \in \mathfrak{M}$. Then, x is said to be *strongly Birkhoff-James orthogonal* to y, denoted by $x \perp^s_{BJ} y$, if $\|x + ya\| \geq \|x\|$ for each $a \in \mathfrak{A}$.

It is evident that for $x, y \in \mathfrak{M}$, $x \perp^s_{BJ} y$ if and only if $x \perp_{BJ} ay$ for all $a \in \mathfrak{A}$. It is immediate that if under lying C^*-algebra $\mathfrak{A}$ of the Hilbert C^*-module is a unital C^*-algebra then for each pair $x, y \in \mathfrak{M}$, $x \perp^s_{BJ} y$ imples $x \perp_{BJ} y$. For non-unital case, if $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $\mathfrak{A}$, then $x \perp^s_{BJ} y$ imples $x \perp_{BJ} e_\lambda y$ whenever $\lambda \in \Lambda$, and hence $x \perp_{BJ} y$. Therefore it always holds that strongly Birkhoff-James orthogonality implies original Birkhoff-James orthogonality. Before introducing the quasi-strongly Birkhoff-James orthogonality in Hilbert C^*-module, we first discuss some characterization of above two orthogonality in terms of state of under lying C^*-algebra of the Hilbert C^*-module. Let $\mathfrak{M}$ be a Hilbert C^*-module over the C^*-algebra $\mathfrak{A}$. We denote the associate inner-product on $\mathfrak{M}$ by $\langle .,. \rangle : \mathfrak{M} \times \mathfrak{M} \longrightarrow \mathfrak{A}$. Recall the following characterization of the strong and original Birkhoff-James orthogonality in a Hilbert C^*-module.

Theorem 3.2 ([3])**.** *Let $\mathfrak{M}$ be a Hilbert C^*-module over the C^*-algebra $\mathfrak{A}$ and $x, y \in \mathfrak{M}$. Then $x \perp^s_{BJ} y$ if and only if there exists a state ρ of $\mathfrak{A}$ such that $\rho(\langle x, x\rangle) = \|x\|^2$ and $\rho(|\langle x, y\rangle|^2) = 0$.*

Theorem 3.3 ([1])**.** *Let $\mathfrak{M}$ be a Hilbert C^*-module over the C^*-algebra $\mathfrak{A}$ and $x, y \in \mathfrak{M}$. Then, $x \perp_{BJ} y$ if and only if there exists a state ρ of $\mathfrak{A}$ such that $\rho(\langle x, x\rangle) = \|x\|^2$ and $\rho(\langle x, y\rangle) = 0$.*

It is not difficult to check that for $x, y \in \mathcal{M}$ and for any state ρ of $\mathcal{A}$, we have $\rho\big(\langle x, y\rangle\big) \leq \rho\big(|\langle x, y\rangle|\big) \leq \rho\big(|\langle x, y\rangle|^2\big)^{1/2}$. Hence, by Theorem 3.2 and Theorem 3.3, we can again conclude that strong Birkhoff–James orthogonality implies Birkhoff–James orthogonality.

Remark 3.4. It is noted in [11, Proposition 3.14] that in Theorem 3.2, the state can be replaced by a pure state.

Since a C^*-algebra $\mathfrak{A}$ is Hilbert C^*-module over itself with the inner product $\langle a, b\rangle = b^*a$ for $a, b \in \mathfrak{A}$, by Theorem 3.3, in particular, we $a \perp_{BJ} b$ if and only if there exists a state ρ of $\mathfrak{A}$ such that $\rho(|a|) = \|a\|$ and $\rho(b^*a) = 0$. By replacing state by a pure state in the above equivalent statement, authors in [11], introduced notion of quasi-strong Birkhoff-James orthogonality in C^*-algebra.

Definition 3.5. [11] Let $\mathfrak{A}$ be C^*-algebra and let $a, b \in \mathfrak{A}$. Then, a is said to be quasi-strongly Birkhoff-James orthogonal to b if there exists a pure state ρ of $\mathfrak{A}$ such that $\rho(|a|) = \|a\|$ and $\rho(b^*a) = 0$.

It follows from Theorem 3.2 and the inequality following Theorem 3.3 that the above notion of orthogonality lies between the strong and original Birkhoff–James orthogonality in a C^*-algebra.

Motivated by Theorem 3.2, Theorem 3.3, and Remark 3.4, and as a generalization of Definition 3.5, we introduced in [7], the notion of quasi-strong Birkhoff–James orthogonality in Hilbert C^*-modules as follows.

Definition 3.6. [7] Let $\mathfrak{M}$ be a Hilbert $\mathfrak{A}$-module, and let $x, y \in \mathfrak{M}$. Then, x is said to be *quasi-strongly Birkhoff-James orthogonal* to y, denoted by $x \perp_{BJ}^{q} y$, if there exists a pure state ρ of $\mathfrak{A}$ such that $\rho(\langle x, x\rangle) = \|x\|^2$ and $\rho(\langle x, y\rangle) = 0$.

We now present the main observation of this section, which is also one of the main results of [7]. We observe that one of the fundamental algebraic properties of a C^*-algebra is characterized by the equivalence of strong and quasi-strong Birkhoff–James orthogonality.

Theorem 3.7 ([7])**.** *Let $\mathfrak{A}$ be a C^*-algebra, let $\mathfrak{M}$ be a full Hilbert $\mathfrak{A}$-module. Then $\perp_{BJ}^{s}$ and $\perp_{BJ}^{q}$ are equivalent on $\mathfrak{M}$ if and only if $\mathfrak{A}$ is commutative.*

4 On equivalence of quasi-strong and original Birkhoff-James orthogonality

In this section, we study the equivalence of $\perp_{BJ}^{q}$ and $\perp_{BJ}$ in full Hilbert C^*-modules. We first recall a relevant result on $\mathcal{K}(\mathcal{H})$, the C^*-algebra of compact operators on a Hilbert space $\mathcal{H}$.

Lemma 4.1 ([11])**.** *On $\mathcal{K}(\mathcal{H})$, the quasi-strong and the original Birkhoff–James orthogonality notions coincide.*

We now show that if these two notions are equivalent on the underlying C^*-algebra, then they are also equivalent on the entire Hilbert C^*-module.

Proposition 4.2 ([7])**.** *Let $\mathfrak{A}$ be a C^*-algebra, and let $\mathfrak{M}$ be a Hilbert $\mathfrak{A}$-module. If $\perp_{BJ}^{q}$ and $\perp_{BJ}$ are equivalent on $\mathfrak{A}$, then they are also equivalent on $\mathfrak{M}$.*

As an immediate consequence of Lemma 4.1 and Proposition 4.2, we obtain the following.

Corollary 4.3 ([7])**.** *On every Hilbert $\mathcal{K}(\mathcal{H})$-module, the notions $\perp_{BJ}^{q}$ and $\perp_{BJ}$ coincide.*

Recall that a C^*-algebra $\mathfrak{A}$ is called *prime* if the intersection of any two nontrivial closed two-sided ideals of $\mathfrak{A}$ is nonzero. For example, $\mathcal{K}(\mathcal{H})$ is a prime C^*-algebra, where $\mathcal{H}$ is a Hilbert space. In contrast, $C(K)$ is not prime when K is a compact Hausdorff space with at least two points.

The following theorem provides an algebraic necessary condition for the equivalence of $\perp_{BJ}^{q}$ and $\perp_{BJ}$ in full Hilbert C^*-modules.

Theorem 4.4 ([7])**.** *Let $\mathfrak{A}$ be a C^*-algebra, and let $\mathfrak{M}$ be a full $\mathfrak{A}$-module. If $\perp_{BJ}^{q}$ and $\perp_{BJ}$ are equivalent on $\mathfrak{M}$, then $\mathfrak{A}$ is prime.*

5 On equivalence of strong and original Birkhoff-James orthogonality

It is clear that $\perp_{BJ}^{s}$ and $\perp_{BJ}$ are equivalent if and only if both $\perp_{BJ}^{s}$ and $\perp_{BJ}^{q}$ are equivalent, and $\perp_{BJ}^{q}$ and $\perp_{BJ}$ are equivalent. Using the observations of Sections 3 and 4, we provide an alternative proof of the following result of Arambašić and Rajić [3].

Theorem 5.1 (Arambašić and Rajić, 2015). *Let $\mathfrak{A}$ be a C^*-algebra, and let $\mathfrak{M}$ be a full Hilbert $\mathfrak{A}$-module. Then, $\perp_{BJ}^s$ and $\perp_{BJ}$ are equivalent on $\mathfrak{M}$ if and only if $\mathfrak{A} = \mathbb{C}$.*

Proof. Since the equivalence of $\perp_{BJ}^s$ and $\perp_{BJ}$ implies the equivalence of $\perp_{BJ}^s$ and $\perp_{BJ}^q$, Theorem 3.7 yields that $\mathfrak{A}$ is a commutative C^*-algebra. Consequently, $\mathfrak{A}$ coincides with its center.

On the other hand, the equivalence of $\perp_{BJ}^s$ and $\perp_{BJ}$ also implies the equivalence of $\perp_{BJ}^q$ and $\perp_{BJ}$. Hence, by Theorem 4.4, $\mathfrak{A}$ is a prime C^*-algebra, and therefore its center is trivial.

Thus, $\mathfrak{A}$ is trivial. □

6 More on algebras of all compact operators

In this section, we focus on the equivalence between $\perp_{BJ}^q$ and $\perp_{BJ}$ on the algebras of compact operators on a Hilbert space. We observe that the separability of the Hilbert space plays a crucial role in this context. Recall that a Hilbert space is separable if and only if it admits a countable orthonormal basis. For example, $\ell^2 := \{(\alpha_n)_{n=1}^\infty \subset \mathbb{C} : \sum_{n=1}^\infty |\alpha_n|^2 < \infty\}$, equipped with the norm $\|(\alpha_n)_{n=1}^\infty\| = (\sum_{n=1}^\infty |\alpha_n|^2)^{1/2}$, is a separable Hilbert space, since $\{e_n : n \in \mathbb{N}\}$ forms an orthonormal basis. A Hilbert space is called non-separable if it does not admit a countable orthonormal basis.

The following result concerns $\mathcal{K}(\mathcal{H})$, the C^*-algebra of all compact operators on a Hilbert space $\mathcal{H}$.

Theorem 6.1 ([7]). *Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, and let $\mathfrak{A}$ be a C^*-algebra acting on $\mathcal{H}$. Suppose that $\mathcal{K}(\mathcal{H}) \subset \mathfrak{A}$. Then, $\perp_{BJ}^q$ and $\perp_{BJ}$ are equivalent on $\mathfrak{A}$ if and only if $\mathfrak{A} = \mathcal{K}(\mathcal{H})$.*

We now extend this discussion to the unitization of $\mathcal{K}(\mathcal{H})$. Let $\mathfrak{A}$ be a C^*-algebra. Its unitization, denoted by $\mathfrak{A}^\sim$, is the unital C^*-algebra $\mathfrak{A} \oplus \mathbb{C}$ equipped with the norm $\|(a, \lambda)\| = \sup\{\|ab + \lambda b\| : \|b\| = 1\}$, for $(a, \lambda) \in \mathfrak{A} \oplus \mathbb{C}$; see [6, Section II.1.2] for details. For a locally compact (noncompact) Hausdorff space K, it is easy to see that $C_0(K)^\sim = C(K^\dagger)$, where $K^\dagger$ is the one-point compactification of K. In the subsequent results, the nonseparability of the Hilbert space is crucial.

Theorem 6.2 ([7]). *Let $\mathcal{H}$ be a nonseparable infinite-dimensional Hilbert space. Then, $\perp_{BJ}^q$ and $\perp_{BJ}$ are equivalent on $\mathcal{K}(\mathcal{H})^\sim$.*

Corollary 6.3 ([7]). *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Then, $\perp_{BJ}^q$ and $\perp_{BJ}$ are equivalent on $\mathcal{K}(\mathcal{H})^\sim$ if and only if $\mathcal{H}$ is nonseparable.*

7 Summary

Along with commutativity and primeness, we also consider other algebraic properties of C^*-algebras, namely elementary, simple, and primitive, and examine how they relate to these orthogonality notions.

A C^*-algebra $\mathfrak{A}$ is said to be *elementary* if it is $*$-isomorphic to $\mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For instance, $\mathcal{K}(\mathcal{H})$ is elementary for any Hilbert space $\mathcal{H}$, whereas $\mathcal{K}(\mathcal{H})^\sim$ is not elementary if $\mathcal{H}$ is infinite-dimensional.

A C^*-algebra $\mathfrak{A}$ is *simple* if it does not contain nontrivial closed two-sided ideals. For example, $\mathcal{K}(\mathcal{H})$ is simple for any Hilbert space $\mathcal{H}$, while $\mathcal{K}(\mathcal{H})^\sim$ is not simple if $\mathcal{H}$ is infinite-dimensional.

A C^*-algebra $\mathfrak{A}$ is *primitive* if it admits a faithful irreducible representation. For example, both $\mathcal{K}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})^\sim$ are primitive for any Hilbert space $\mathcal{H}$.

Let $\mathfrak{A}$ be a C^*-algebra. The following summarizes the known relationships between orthogonality equivalences and these algebraic properties:

- By Theorem 5.1, $\perp^s_{BJ}$ and $\perp_{BJ}$ are equivalent in $\mathfrak{A}$ if and only if $\mathfrak{A} = \mathbb{C}$.
- By Theorem 3.7, $\perp^s_{BJ}$ and $\perp^q_{BJ}$ are equivalent in $\mathfrak{A}$ if and only if $\mathfrak{A}$ is commutative.
- By Theorem 4.4, if $\perp^q_{BJ}$ and $\perp_{BJ}$ are equivalent in $\mathfrak{A}$, then $\mathfrak{A}$ is prime.
- Since every elementary C^*-algebra $\mathfrak{A}$ is $*$-isomorphic to $\mathcal{K}(\mathcal{H})$ for some $\mathcal{H}$, Lemma 4.1 implies that $\perp^q_{BJ}$ and $\perp_{BJ}$ are equivalent in any elementary C^*-algebra.
- We know that $\mathcal{K}(\mathcal{H})^\sim$ is not elementary. Thus, by Theorem 6.1, $\perp^q_{BJ}$ and $\perp_{BJ}$ being equivalent does not imply that $\mathfrak{A}$ is elementary.
- Similarly, since $\mathcal{K}(\mathcal{H})^\sim$ is not simple, $\perp^q_{BJ}$ and $\perp_{BJ}$ being equivalent in $\mathfrak{A}$ does not imply simplicity.
- Although $\mathcal{K}(\mathcal{H})^\sim$ is primitive, Corollary 6.3 shows that $\perp^q_{BJ}$ and $\perp_{BJ}$ are not equivalent on $\mathcal{K}(\mathcal{H})^\sim$ if $\mathcal{H}$ is separable. Hence, primitivity does not guarantee the equivalence of $\perp^q_{BJ}$ and $\perp_{BJ}$.
- Finally, one might ask whether if $\mathfrak{A}$ is prime then $\perp^q_{BJ}$ and $\perp_{BJ}$ are equivalent. Since primitivity implies primeness, this would contradict the previous point. Thus, primeness alone does not imply equivalence.

The above relationships can be summarized in the following diagram:

$\boxed{\text{Trivial } C^*\text{-algebra}}$

$\Updownarrow$

$\boxed{\perp^s_{BJ} \Leftrightarrow \perp_{BJ}}$

$\Downarrow \qquad\qquad \Downarrow$

$\boxed{\perp^s_{BJ} \Leftrightarrow \perp^q_{BJ}}$

$\Updownarrow$

$\boxed{\text{Commutative}}$

$\boxed{\perp^q_{BJ} \Leftrightarrow \perp_{BJ}}$

$\not\Downarrow\Uparrow \qquad \not\Downarrow \ \Uparrow? \qquad \not\Uparrow \Downarrow? \qquad \not\Uparrow\Downarrow$

$\boxed{\text{elementary}} \Rightarrow \boxed{\text{simple}} \Rightarrow \boxed{\text{primitive}} \Rightarrow \boxed{\text{prime}}$

The following problems remain open:

Problem 7.1 ([7]). Let $\mathfrak{A}$ be a simple C^*-algebra. Are $\perp_{BJ}^q$ and $\perp_{BJ}$ equivalent on $\mathfrak{A}$?

Problem 7.2 ([7]). Let $\mathfrak{A}$ be a C^*-algebra on which $\perp_{BJ}^q$ and $\perp_{BJ}$ are equivalent. Is $\mathfrak{A}$ necessarily primitive?

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