A note on: Characterizations of the weighted infinitesimal relative boundedness of Schrödinger operators

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1 Introduction

The purpose of this note is to give an overview of the talk "Characterizations of the weighted infinitesemial relative boundedness of Schrödinger operators" [3] at RIMS workshop "Function spaces and related topics". We focus on the class of potential $V \in \mathcal{D}'$ such that for any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$\|V\phi\|^p_{L^p(w)} \leq \epsilon\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^p_{L^p(w)} + C(\epsilon)\|\phi\|^p_{L^p(w)} \quad \forall \phi \in C_c^\infty. \tag{1.1}$$

One motivation is to extend some good properties from the fractional Laplacian $(-\Delta)^{\alpha/2}$ to the Schrödinger operator $H = (-\Delta)^{\alpha/2} + V$. A natural intuition is that if the potential V is small enough in some sense, then it does not affect the essential properties of the main part. In perturbation theory, the relative boundedness model is usually applied for this model. Let A and B be two linear operators in the Banach space X. We say the operator B is relatively A-bounded if there exist positive constant C_1 and C_2 such that for any $f \in D(A) \subseteq D(B)$,

$$\|Bf\|_X \leq C_1\|Af\|_X + C_2\|f\|_X,$$

where the infimum of such C_1 is called the *relative bound of B with respect to A*. Usually $C_1 < 1$ gives the perturbative stability of operator $A + B$. Under such restriction assumption on C_1, it is well known that the relative bounded perturbation works well for a wide range of properties, such as

- **Generator of contraction/holomorphic semigroup.** A C_0 semigroup on X is a family of operators $\{T_t\}_{t\geq 0} \in \mathcal{B}(x)$, such that $T(0) = id_X$, $T(t+s) = T(t)T(s)$ and $T(t)f \to f$ as $t \to 0^+$. A contraction C_0 semigroup satisfies $\|T(t)\|_X \leq 1$ for any $t \geq 0$, and for a holomorphic semigroup, the semigroup property and the strongly continuous could be extended to a sector $\{z \in \mathbb{C} : |\arg z| < w\}$. An operator A on X is said to be the generator of $T(t)$, if
$$x \in D(A) \iff Ax = \exists \lim_{t\to 0} \frac{T(t)x - x}{t}.$$
If A genertate a C_0 semigroup, then the semigroup is uniquely determined by A and denoted as e^{tA}. Let A generates a constraction/holomorphic semigroup, then the relative A-boundedness with small relative bound C_1 gives the same properties of $A + B$.

- **Self-adjointness.** Let A be a self-adjoint operator in the Hilbert space H, if B is symmetric and relatively A-bounded with relative bound $C_1 < 1$, then the celebrated Kato-Rellich theorem says that $A + B$ also is self-adjoint in H.

- **Spectral stability.** Let A be a self-adjoint operator in the Hilbert space H, if B is symmetric and relatively A-bounded with relative bound $C_1 < 1$. Then if A has discrete spectrum, then $A + B$ also does. Further, the distance between there spectral can be controled as $d(\sigma(A+B), \sigma(A)) \leq f(C_1, C_2)$.

However, in concrete situations, it is often difficult to characterize the relative bound for a specific constant C_1. Therefore, we strengthen the condition by requiring C_1 to be arbitrary small, which leads us to the notion of infinitesimal relative boundedness. As a result, for any potential V satisfies (1.1), the Schrödinger operator $H = (-\Delta)^{\alpha/2} + V$ can be viewed as a small perturbation of the farctional laplacian, and hence retains many of its good properties.

The characterization for the infinitesimal relative boundedness of Schrödinger operators is first developed by Kato (see for instance [7],[8]). Kato proved that the classical kato class K_n is a sufficient condition for the infinitesimal relative bound related with Δ in L^1. Later Zheng and Yao [13] generalized this result to fractional Schródinger operator $(-\Delta)^{\alpha/2} + V$. They characterized (1.1) with an extended Kato class $K_{\alpha,n}$. In the Hilbert space case $p = 2$, Maz'ya and Verbitsky [10] obtained a characterization by capacity and a dyadic Carleson condition. More recently, Cao, Deng and Jin obtained a fractional analogue for all $1 < p < \infty$ and $0 < \alpha < n$. Our result may be regarded as a weighted analogue of these characterizations.

Theorem 1.1. *Let $1 < p < \infty$, $0 < \alpha < n$, $w \in A_p$ and $0 \le V \in L^p_{loc} \cap L^p_{loc}(w) \cap L^p_{loc}(\sigma)$ with $\sigma := w^{1-p'}$. For* (i) $\sim$ (ii) *below, we have* (i) $\leftrightarrow$ (ii) *and* (iii) $\to$ (ii). *Furthermore, if $\alpha < n/p$ and $\sigma \in A_1$, then* (ii) $\to$ (iii).
(i) *For any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that for any $\phi \in C_c^\infty$,*

$$\|V\phi\|^p_{L^p(w)} \le \epsilon \|(-\Delta)^{\frac{\alpha}{2}}\phi\|^p_{L^p(w)} + C(\epsilon)\|\phi\|^p_{L^p(w)}.$$

(ii) *With $d\mu w := |V|^p w dx$,*

$$\lim_{\delta\to 0} \sup_{\mathcal{S}} \sup_{\substack{\ell(Q)\le\delta \\ Q\in\mathcal{S}}} \left\{ \frac{1}{\mu w(Q)} \sum_{\substack{Q'\in\mathcal{S} \\ Q'\subseteq Q}} |Q'|^{p'(\frac{\alpha}{n}-1)} \mu w(Q')^{p'} \sigma(Q') \right\} = 0,$$

where the first supremum is taken oven all sparse collections $\mathcal{S}$.
(iii)

$$\lim_{\delta\to 0} \sup_{diam(E)\le\delta} \left\{ \frac{\mu w(E)}{B^w_{\alpha,p}(E)} \right\} = 0,$$

where the weighted Bessel potential for a measurable set E is defined by

$$B^w_{\alpha,p}(E) := \inf\left\{ \int_{\mathbb{R}^n} f(x)^p w(x)dx : \ 0 \le f \in L^p(w), \ G_\alpha * f \ge 1 \ on \ E \right\}.$$

Actually, here the Bessel capacity $B^w_{\alpha,p}$ in (iii) can be replaced by the Riesz capacity $R^w_{\alpha,p}$

$$R^w_{\alpha,p}(E) := \inf\left\{ \int_{\mathbb{R}^n} f(x)^p w(x)dx : \ 0 \le f \in L^p(w), \ I_\alpha f \ge 1 \ on \ E \right\} \tag{1.2}$$

(this could be seen in the proof), even though the equivalence $B^w_{\alpha,p}(E) \sim R^w_{\alpha,p}(E)$ does not hold in general. However, as shown by Turesson [11], for $w \in A_p$, the Bessel capacity $B^w_{\alpha,p}(E)$ can be viewed as a localized version of the Riesz capacity in the sense that

$$B^w_{\alpha,p}(E) \sim R^w_{\alpha,p,1}(E),$$

where $R^w_{\alpha,p,1}(E)$ is defined by (1.2) with $I_{\alpha,1}f := \int_{|\cdot-y|\le 1} |f(y)||x-y|^{\alpha-n}dy$ in the place of $I_\alpha f$. This argument leads to a corollary.

Corollary 1.2. *Let $1 < p < \infty$, $0 < \alpha < n/p$, w be a weight with $\sigma := w^{1-p'} \in A_1$ and $0 \le V \in L^p_{loc} \cap L^p_{loc}(w) \cap L^p_{loc}(\sigma)$. If (1.1) holds for any $\phi \in C_c^\infty$, then*

$$\lim_{\delta\to 0} \sup \left\{ \mu w(B(x_0,\delta)) \left(\int_\delta^\infty \frac{\sigma(B(x_0,t))}{t^{(n-\alpha)p'+1}} dt \right)^{p-1} : \ x_0 \in \mathbb{R}^n \right\} = 0.$$

Corollary 1.2 follows by letting $E = B(x_0,\delta)$ in Theorem 1.1 (ii) with $B^w_{\alpha,p}$ in the place of $R^w_{\alpha,p}$, and

$$R^w_{\alpha,p}(B(x_0,\delta)) \sim \left(\int_\delta^\infty \frac{\sigma(B(x_0,t))}{t^{(n-\alpha)p'+1}} dt \right)^{1-p}$$

for $w \in A_p$, see details in Turesson [11, Theorem 3.3.10].

This paper is organized as follows. In Section 2, we recall some basic definitions and lemmas that will be applied in the proof. In Section 3, we prove our main result Theorem 1.1. Finally, In Section 4, we consider an example of both power weights and power potentials.

We conclude the introduction with some conventions on the notation. We use $a \lesssim b$ to say that there exists a constant C, which is independent of the important parameters, such that $a \le Cb$. Moreover, we write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. For any measurable set E, $|E|$ represents its Lebesgue measure. For a weight w, we set $w(E) := \int_E w dx$. Let χ_E stand for the characteristic function of E. In the paper all cubes are assumed to have edges parallel to the coordinate axes. For any open set $\Omega \subseteq \mathbb{R}^n$, $C_c^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support on Ω, and $C_c^\infty(\mathbb{R}^n) = C_c^\infty$.

2 Preliminary

In this section, we recall notations and give basic estimates for Bessel potential and an operator with sparse family. The proof of some lemmas omitted here can be found in [3].

2.1 Weight Class

We first recall the Muckenhoupt weight [9]. For $1 \le p < \infty$ we say that $w \in A_p$ if

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q w dx \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} dx \right)^{p-1} < \infty,$$

where for $p = 1$ we use the limiting interpretation $(\int_Q w^{1-p'} dx/|Q|)^{p-1} = (\operatorname{essinf}_Q w)^{-1}$. And then,

$$A_\infty := \bigcup_{p\ge 1} A_p.$$

We have althernative definition;

$$w \in A_\infty \leftrightarrow [w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q) dx < \infty,$$

where M denotes the Hardy-Littlewood maximal operator $Mf(x) = \sup_Q \frac{1}{|Q|} \int_Q |f| dy \cdot \chi_Q(x)$. This quantity $[w]_{A_\infty}$ is referred as the Fujii-Wilson A_∞ constant [4, 12].

2.2 Bessel Functions

Definition 2.1. *Let $0 < \alpha, \lambda < \infty$, the Bessel function $G_{\alpha,\lambda}$ is defined by*

$$G_{\alpha,\lambda}(x) := \mathcal{F}^{-1}\left[(2\pi)^{-\frac{n}{2}}(\lambda^2 + |\xi|^2)^{-\frac{\alpha}{2}}\right](x).$$

When $\lambda = 1$, we simply write $G_\alpha(x)$. The Bessel potential $(\lambda^2 - \Delta)^{-\alpha/2}$ is defined by

$$(\lambda^2 - \Delta)^{-\frac{\alpha}{2}} f(x) := G_{\alpha,\lambda} * f(x)$$

for $f \in C_c^\infty$.

The Bessel functions satisfy the following fundamental properties. These results can be found in standard references such as Grafakos [6].

Lemma 2.2. *Let $0 < \alpha, \lambda < \infty$, then*
(i) *For any $x \in \mathbb{R}^n$, $G_\alpha(x) > 0$ and $G_{\alpha,\lambda}(x) = \lambda^{n-\alpha} G_\alpha(\lambda x)$.*
(ii) *In the case $\alpha < n$, there exists constant $a > 0$ such that*

$$G_\alpha(x) \begin{cases} \sim |x|^{\alpha-n} & \text{if} \quad 0 < |x| < 1, \\ \lesssim e^{-a|x|} & \text{if} \quad |x| \geq 1. \end{cases}$$

Moreover, it is easy to verify that Bessel potentials are bounded on $L^p(w)$ with $w \in A_p$.

Lemma 2.3. *For $0 < \alpha, \lambda < \infty$, $p > 1$ and weight $w \in A_p$, it holds that*

$$\|(\lambda^2 - \Delta)^{-\frac{\alpha}{2}}\|_{L^p(w) \to L^p(w)} \lesssim \lambda^{-\alpha},$$

$$\|(-\Delta)^{\frac{\alpha}{2}}(\lambda^2 - \Delta)^{-\frac{\alpha}{2}}\|_{L^p(w) \to L^p(w)} \lesssim 1,$$

$$\|(\lambda^2 - \Delta)^{\frac{\alpha}{2}}(\lambda^2 + (-\Delta)^{\frac{\alpha}{2}})^{-1}\|_{L^p(w) \to L^p(w)} \lesssim 1.$$

2.3 Sparse Families

For $t \in \{0, 1, 2\}^n$, we denote the shifted dyadic cubes in $\mathbb{R}^n$ by

$$\mathcal{D}^t := \left\{2^{-k}\left([0,1)^n + m + (-1)^k \frac{t}{3}\right);\ k \in \mathbb{Z}, m \in \mathbb{Z}^n\right\}, \text{ and } \mathcal{D} := \bigcup_t \mathcal{D}^t.$$

For $Q_0 \in \mathcal{D}^t$, we denote by $\mathcal{D}^t(Q_0)$ the collection of all dyadic cubes $Q \in \mathcal{D}^t$ that satisfy $Q \subseteq Q_0$.

Definition 2.4. *For $t \in \{0, 1, 2\}^n$, a collection of cubes $\mathcal{S} \subseteq \mathcal{D}^t$ is said to be a sparse family, if there is a pairwise disjoint collection $(E_Q)_{Q \in \mathcal{S}}$, so that $E_Q \subseteq Q$ and $|E_Q| \geq |Q|/2$.*

We will use the following two lemmas later.

Lemma 2.5. (Fackler and Hytönen, [5]) *Let μ be a non-negative locally finite Borel measure, and we suppose $\alpha_1 > 0$ and $\alpha_2 \geq 0$ satisfy $\alpha_1 + \alpha_2 \geq 1$. Then for any sparse family $\mathcal{S}$ and cube $Q \in \mathcal{S}$,*

$$\sum_{\substack{Q' \in \mathcal{S} \\ Q' \subseteq Q}} |Q'|^{\alpha_1} \mu(Q')^{\alpha_2} \leq C |Q|^{\alpha_1} \mu(Q)^{\alpha_2}.$$

Lemma 2.6. *Let $1 < p < \infty$ and $\mathcal{S}$ be a sparse collection. Then for any $\{\lambda_Q\}_{Q \in \mathcal{S}} \subseteq [0, \infty)$ and $\sigma \in A_\infty$, it holds*

$$\|\sum_{Q \in \mathcal{S}} \lambda_Q \chi_Q\|_{L^p(\sigma)}^p \lesssim \sum_{Q \in \mathcal{S}} \lambda_Q^p \sigma(Q).$$

3 Sketch of the proof of the main theorem

In this section, we explain the main ideas of the proof of Theorem 1.1. We omit several technical details and focus on the structure of the argument. For the full proof, we refer the reader to [3].

3.1 Proof of (i) $\leftrightarrow$ (ii)

The first step is to reduce the infinitesimal boundedness condition to an estimate for the Bessel potential. More precisely, we prove that the following two conditions are equivalent.

Lemma 3.1. *Let $1 < p < \infty$, $0 < \alpha < n$, then for $w \in A_p$ and $0 \leq V \in L^p_{loc}(w)$, the followings are equivalent:*
(i) *For any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that for any $\phi \in C_c^\infty$,*

$$\|V\phi\|^p_{L^p(w)} \leq \epsilon \|(-\Delta)^{\frac{\alpha}{2}}\phi\|^p_{L^p(w)} + C(\epsilon)\|\phi\|^p_{L^p(w)}.$$

(ii) *It holds $\lim_{\lambda\to\infty} \|V(\lambda^2 - \Delta)^{-\alpha/2}\|_{L^p(w)\to L^p(w)} = 0$.*

This Lemma follows from the standard weighted multiplier estimates for the Bessel potential. Under the assumptions in Theorem 1.1, it is easy to see that for any $\lambda > 0$

$$C_0(\lambda) := \|(\lambda^2 - \Delta)^{-\frac{\alpha}{2}}(\cdot\mu)\|^{p'}_{L^{p'}(\mu\sigma)\to L^{p'}(\sigma)} = \|V(\lambda^2 - \Delta)^{-\frac{\alpha}{2}}\|^{p'}_{L^p(w)\to L^p(w)}$$

by duality. Thus, to prove (i) $\leftrightarrow$ (ii) in Theorem 1.1, it is sufficient to prove the following proposition.

Proposition 3.2. *With the assumption in Theorem 1.1, it holds that $C_0(\lambda) \sim C_1(\lambda)$, where*

$$C_1(\lambda) := \sup_{\mathcal{S}} \sup_{\substack{\ell(Q)\leq\frac{6}{\lambda}\\ Q\in\mathcal{S}}} \left\{ \frac{1}{\mu w(Q)} \sum_{\substack{Q'\in\mathcal{S}\\ Q'\subseteq Q}} |Q'|^{p'\left(\frac{\alpha}{n}-1\right)} \mu w(Q')^{p'} \sigma(Q') \right\}.$$

The first supremum is taken over all sparse families.

We first prove that $C_0(\lambda) \lesssim C_1(\lambda)$. The proof is divided into several steps.

Step 1. We control the resolvent of the Laplacian by the local fractional maximal operator

Lemma 3.3. *Let $0 < \alpha < n$, $1 < p < \infty$ and $\lambda > 0$, then for $w \in A_\infty$, it holds*

$$\|(\lambda^2 - \Delta)^{-\frac{\alpha}{2}} g\|_{L^p(w)} \lesssim \|M_{\alpha,\lambda} g\|_{L^p(w)},$$

where

$$M_{\alpha,\lambda} f(x) := \sup_{0<r\leq\frac{1}{\lambda}} \frac{1}{r^{n-\alpha}} \int_{B(x,r)} |f(y)|dy.$$

Sketch of the Proof. Recall that $(\lambda^2 - \Delta)^{-\alpha/2} g = G_{\alpha,\lambda} * g$, we split the integral into the local and the global parts:

$$G_{\alpha,\lambda} * g(x) = \int_{|x-y|\leq 1/\lambda} G_{\alpha,\lambda}(x-y)g(y)\,dy + \int_{|x-y|>1/\lambda} G_{\alpha,\lambda}(x-y)g(y)\,dy =: I(x) + II(x).$$

For the local part I, the pointwise estimate $G_{\alpha,\lambda}(x) \lesssim |x|^{\alpha-n}$ $(|x| \leq 1/\lambda)$ implies

$$I(x) \lesssim \int_{|x-y|\leq 1/\lambda} \frac{|g(y)|}{|x-y|^{n-\alpha}}\, dy \lesssim M_{\alpha,\lambda} g(x). \tag{3.1}$$

For the global part, the exponential decay of the Bessel kernel gives $G_{\alpha,\lambda}(x) \lesssim \lambda^{n-\alpha} e^{-a\lambda|x|}$ for $|x| > 1/\lambda$. Then By decomposing $\mathbb{R}^n$ into annuli centered at x, and using the doubling property of $w \in A_\infty$, one obtains the weighted estimate

$$\|II\|_{L^p(w)} \lesssim \|M_{\alpha,\lambda} g\|_{L^p(w)}. \tag{3.2}$$

Combining the estimates (3.1) and (3.2) yields the desired inequality. □

Step 2. Let $0 < \alpha < n$ and $\lambda > 0$. For $t \in \{0,1,2\}^n$ and a sparse family $\mathcal{S} \subseteq \mathcal{D}^t$, we define the local version of sparse operator by

$$I^{\mathcal{S}}_{\alpha,\lambda} g(x) := \sum_{\substack{Q\in\mathcal{S}\\ \ell(Q)\leq 6/\lambda}} |Q|^{\frac{\alpha}{n}-1} \int_Q |g(y)| dy \cdot \chi_Q(x).$$

We prove a pointwise bound for the local maxial operator by a local version of sparse operator.

Lemma 3.4. *Let $0 < \alpha < n$, $\lambda > 0$, there exist sparse family $\mathcal{S}^t \subseteq \mathcal{D}^t$, $t \in \{0,1,2\}^n$, that*

$$M_{\alpha,\lambda} g(x) \lesssim \sum_{t\in\{0,1,2\}^n} I^{\mathcal{S}^t}_{\alpha,\lambda} g(x).$$

The proof of Lemma 3.4 is essentially the same as that in the classical non-local case due to Curz-Uribe and Moen [2], where they proved that the fractional maximal operator is pointwise dominated by some fractional sparse operators.

Step 3. Finally, we prove that the local fractional sparse operator is bounded on L^p as follows.

Lemma 3.5. *Let $1 < p' < \infty$, $0 < \alpha < n$ and $\lambda > 0$, then for any $g \in C_c^\infty$ and $\mu w \in A_\infty$, it holds*

$$\int_{\mathbb{R}^n} |I^{\mathcal{S}}_{\alpha,\lambda}(g\mu w)(x)|^{p'} d\sigma(x) \lesssim C_1(\lambda) \int_{\mathbb{R}^n} |g|^{p'} d\mu w(x),$$

where the implicit constant depends on n, p and $[\mu w]_{A_\infty}$, and the constant $C_1(\lambda)$ is defined in Proposition 3.2.

Sketch of the Proof. Let $a_Q := |Q|^{(\frac{\alpha}{n}-1)p'} \sigma(Q) \mu w(Q)^{p'}$, $A := \{Q \in \mathcal{S} : \ell(Q) \leq 6/\lambda\}$ and

$$T_Q g(x) := \frac{1}{\mu w(Q)} \int_Q |g(x)| d\mu w(x), \quad Q \in \mathcal{S}.$$

. By Lemma 2.6 applied with $\lambda_Q = |Q|^{\alpha/n-1} \int_Q |g|\, d\mu w$, we obtain

$$\|I^{\mathcal{S}}_{\alpha,\lambda}(g\mu w)\|^{p'}_{L^{p'}(\sigma)} \lesssim \sum_{Q\in A} |Q|^{p'(\alpha/n-1)} \sigma(Q) \mu w(Q)^{p'} \left(\langle |g| \rangle_Q^{\mu w}\right)^{p'} = \sum_{Q\in A} a_Q \left(\langle |g| \rangle_Q^{\mu w}\right)^{p'}. \tag{3.3}$$

It can be simply checked that $\{T_Q\}_{Q\in\mathcal{S}}$ is bounded from $L^1(\mu w)$ to $l^{1,\infty}(\{a_Q\})$. Because the $L^\infty(\mu w) \to l^\infty(\{a_Q\})$ for $\{T_Q\}_{Q\in\mathcal{S}}$ is trivial, by interpolating them, we have that

$$\|\{T_Q\}_{Q\in\mathcal{S}}\|_{L^{p'}(\mu w)\to l^{p'}(\{a_Q\})} \lesssim C_1(\lambda)^{1/p'}.$$

Therefore,

$$\|I^{\mathcal{S}}_{\alpha,\lambda}(g\mu w)\|^{p'}_{L^{p'}(\sigma)} \lesssim C_1(\lambda) \|g\|^{p'}_{L^{p'}(\mu w)},$$

as desired. □

Now by combining Lemma 3.3, 3.4 and 3.5, we have

$$\|(\lambda^2-\Delta)^{-\frac{\alpha}{2}}(g\mu w)\|_{L^{p'}(\sigma)}^{p'} \lesssim C_1(\lambda)\|g\|_{L^{p'}(\mu w)}^{p'}. \tag{3.4}$$

By taking $g = fw^{-1}$, one gets $C_0(\lambda) \lesssim C_1(\lambda)$.

Conversely, $C_1(\lambda) \lesssim C_0(\lambda)$ is obtained by testing the operator $(\lambda^2-\Delta)^{-\alpha/2}(\cdot\,\mu w)$ on characteristic functions of dyadic cubes Q with $\ell(Q) \le 6/\lambda$. On this scale, the Bessel kernel has the lower bound

$$G_{\alpha,\lambda}(x-y) \gtrsim |x-y|^{\alpha-n}, \qquad x, y \in Q,$$

and hence the testing inequality dominates the dyadic fractional sum appearing in the definition of $C_1(\lambda)$. Therefore $C_1(\lambda) \lesssim C_0(\lambda)$.

3.2 Proof of (ii) $\leftrightarrow$ (iii)

The implication (iii) $\to$ (ii) follows from standard capacity techniques. To establish (ii) $\to$ (iii), we proceed by classifying the dyadic cubes and handling their contributions separately in the dyadic Riesz potential. We emphasize that our proof consistently employs the Riesz potential I_α to bound the Bessel potential G_α. We noted in Section 1, Theorem 1.1 remain valid when substituting the Bessel capacity $B^w_{\alpha,p}$ with the Riesz capacity $R^w_{\alpha,p}$.

Sketch of the Proof. For $\delta > 0$, set

$$k_1(\delta) := \sup_t \sup_{\substack{\ell(Q)\le\delta \\ Q\in\mathcal{D}^t}} \left\{ \frac{1}{\mu w(Q)} \sum_{Q'\in\mathcal{D}^t(Q)} |Q'|^{p'(\alpha/n-1)} \mu w(Q')^{p'} \sigma(Q') \right\}$$

and

$$k_2(\delta) := \sup_{\mathrm{diam}(E)\le\delta} \frac{\mu w(E)}{B^w_{\alpha,p}(E)}.$$

Then we have (iii) $\leftrightarrow$ $\lim_{\delta\to0} k_2(\delta) = 0$ and (ii) $\leftrightarrow$ $\lim_{\delta\to0} k_1(\delta) \to 0$.

We first indicate why the capacity condition (iii) implies the Carleson measure (ii) condition. Let $Q_0 \in \mathcal{D}^t$ with $\ell(Q_0) \le \delta$. The main point is to estimate the energy $\|G_\alpha * (\chi_{Q_0}\mu w)\|_{L^{p'}(\sigma)}^{p'}$. By duality and the definition of the weighted Bessel capacity, the condition $k_2(\delta) < \infty$ gives

$$\|G_\alpha * (\chi_{Q_0}\mu w)\|_{L^{p'}(\sigma)}^{p'} \lesssim k_2(\delta)^{1/p}\mu w(Q_0)^{1/p'}. \tag{3.5}$$

On the other hand, since $G_\alpha(x-y) \sim |x-y|^{\alpha-n}$ on small scales, we can also estimate the energy that

$$G_\alpha * (\mu w_{Q_0})(x) \ge \sum_{Q\in\mathcal{D}^t(Q_0)} |Q|^{\frac{\alpha}{n}-1}\mu w(Q)\chi_Q(x)$$

which yields that

$$\|G_\alpha * (\mu w \chi_{Q_0})\|_{L^{p'}(\sigma)}^{p'} \gtrsim \sum_{Q\in\mathcal{D}^t(Q_0)} |Q|^{\left(\frac{\alpha}{n}-1\right)p'} \mu w(Q)^{p'}\sigma(Q). \tag{3.6}$$

Combining these two estimates yields $k_1(\delta) \lesssim k_2(\delta)^{p'/p}$.

Conversely, assume in addition that $\alpha < n/p$ and $\sigma \in A_1$. Then we prove that for g supported in a cube Q_0 with $\ell(Q_0) \le \delta$, then

$$\|G_\alpha * (g\mu w)\|_{L^{p'}(\sigma)} \lesssim k_1(2\delta)^{1/p'}\|g\|_{L^{p'}(\mu w)}. \tag{3.7}$$

This is obtained by dominating $G_\alpha * (g\mu w)$ by dyadic fractional operators and applying an interpolation argument as we did in Step 3 before. The assumption $\sigma \in A_1$ is used to control the large dyadic ancestors, while $\alpha < n/p$ ensures summability. By duality, the preceding estimate implies that, for every cube Q_0 with $\ell(Q_0) \le \delta$,

$$\|(G_\alpha * f)\chi_{Q_0}\|_{L^p(\mu w)} \lesssim k_1(2\delta)^{1/p'}\|f\|_{L^p(w)}. \tag{3.8}$$

Now let $E \subset \mathbb{R}^n$ with $\operatorname{diam}(E) \le \delta$, and choose a cube Q_0 with $E \subset Q_0$ and $\ell(Q_0) \le 2\delta$. If $f \ge 0$ and $G_\alpha * f \ge 1$ on E, then

$$\mu w(E) \le \int_{Q_0} (G_\alpha * f)^p \, d\mu w \lesssim k_1(2\delta)^{p/p'} \int_{\mathbb{R}^n} f^p \, dw. \tag{3.9}$$

Taking the infimum over all such f gives $k_2(\delta) \lesssim k_1(2\delta)^{p/p'}$. □

4 An example: Power weight and Power potential

In this section, we give an example of power weights $w(x) = |x|^\theta$ and potentials $V(x) = |x|^a$, for which Theorem 1.1 holds. This is a weighted analogy results by Cao, Deng and Jin [1]

Corollary 4.1. *Under the same assumptions as in Theorem 1.1, we take power weight $w(x) = |x|^\theta$ with $0 \le \theta < (n-1)(p-1)$ and potential $V(x) = |x|^a$ satisfying $a > -n/p + (p'-1)\theta/p$. Then the infinitesimal boundedness* (i) *in Theorem 1.1 holds if and only if $-\alpha < a \le 0$.*

Sketch of the Proof. We first recall a standard estimate for power weights. Let $v = |x|^\beta$ with $\beta > -n$, and $Q(x,l)$ denote the cube centered at x with side length l. We set $Q_0 := Q(0,l)$. Then we can estimate $v(Q)$ using a standard calculation that

$$v(Q) \sim \begin{cases} |x_0|^\beta |Q|, & 2\sqrt{n}Q_0 \cap Q = \varnothing \text{ (case 1)}; \\ |Q|^{1+\beta/n}, & 2\sqrt{n}Q_0 \cap Q \ne \varnothing \text{ (case 2)}. \end{cases}$$

We prove the sufficiency first. Suppose $-\alpha < a \le 0$, denote the center of cube Q by $C(Q)$. For any $Q \in \mathcal{S}$ with $\ell(Q) \le \delta$, if $\alpha < a \le 0$, then by applying the estimate before and Lemma 2.5, we have that

$$\begin{aligned}\sum_{\substack{Q' \in \mathcal{S} \\ Q' \subseteq Q \\ Q':\ type\ I}} |Q'|^{p'\left(\frac{\alpha}{n}-1\right)} \mu w(Q')^{p'} \sigma(Q') &\lesssim \sum_{\substack{Q' \in \mathcal{S} \\ Q' \subseteq Q \\ Q':\ type\ I}} |Q'|^{\frac{\alpha p'}{n}} |C(Q')|^{ap'} \mu w(Q') \\ &\lesssim \sum_{\substack{Q' \in \mathcal{S} \\ Q' \subseteq Q}} |Q'|^{\frac{(a+\alpha)p'}{n}} \mu w(Q') \lesssim \mu w(Q) |Q|^{\frac{(a+\alpha)p'}{n}}.\end{aligned}$$

Hence, one has

$$\frac{1}{\mu w(Q)} \sum_{\substack{Q' \in \mathcal{S} \\ Q' \subseteq Q \\ Q':\ type\ I}} |Q'|^{p'\left(\frac{\alpha}{n}-1\right)} \mu w(Q')^{p'} \sigma(Q') \lesssim |Q|^{\frac{(n+\alpha)p'}{n}} \to 0 \qquad \text{as } \delta \to 0.$$

Similarly, we can also estimate that

$$\sum_{\substack{Q' \in \mathcal{S} \\ Q' \subseteq Q \\ Q':\ type\ II}} |Q'|^{p'\left(\frac{\alpha}{n}-1\right)} \mu w(Q')^{p'} \sigma(Q') \lesssim \mu w(Q) |Q|^{\frac{(n+\alpha)p'}{n}}.$$

Thus, it holds

$$\frac{1}{\mu w(Q)} \sum_{\substack{Q' \in \mathcal{S} \\ Q' \subseteq Q \\ Q' \colon \, type \, II}} |Q'|^{p'(\frac{\alpha}{n}-1)} \mu w(Q')^{p'} \sigma(Q') \lesssim |Q|^{\frac{(n+\alpha)p'}{n}} \to 0 \qquad \text{as } \delta \to 0.$$

Combining estimates for type I and type II, we checked the Carleson measure condition (ii) in Theorem 1.1. Conversely, for the necessary part, if (ii) holds, then for $a > 0$, choose a sparse family $\mathcal{S}$ and cube $Q \in \mathcal{S}$ with $2\sqrt{n}Q_0 \cap Q = \varnothing$, then we have

$$|Q|^{\frac{\alpha p'}{n} - p'} \mu w(Q)^{p'-1} \sigma(Q) \sim |Q|^{\frac{\alpha p'}{n}} |C(Q)|^{ap'},$$

which leads to a contradiction if we let $|C(Q)| \to \infty$. For $a \leq -\alpha$, we choose $Q \in \mathcal{S}$ with $2\sqrt{n}Q_0 \cap Q \neq \varnothing$, and we have that

$$|Q|^{\frac{\alpha p'}{n} - p'} \mu w(Q)^{p'-1} \sigma(Q) \sim |Q|^{\frac{p'}{n}(a+\alpha)} \to \infty \qquad \text{as } \delta \to 0.$$

This completes the proof. □

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