日合 –Petz 不等式の補完について

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1. Introduction and Motivation

This study is based on [18].

Let $\mathbb{M}_n(\mathbb{C}) = \mathbb{M}_n$ be the algebra of $n \times n$ complex matrices, $\mathbb{P}_n$ the set of positive definite matrices in $\mathbb{M}_n$, $\mathbb{S}_n$ the set of density matrices in $\mathbb{M}_n$ and Tr the usual trace.

The Hiai-Petz inequality in [11] refers to the following trace inequality: For positive definite matrices A and B in $\mathbb{P}_n$,

$$\text{(HP)} \qquad \frac{1}{p}\mathrm{Tr}[A(\log B^{-\frac{p}{2}}A^pB^{-\frac{p}{2}})] \le S_U(A|B) \le \frac{1}{p}\mathrm{Tr}[A(\log A^{\frac{p}{2}}B^{-p}A^{\frac{p}{2}})] \quad \text{for all } p > 0.$$

Currently, the second inequality of (HP) is being deeply studied as a cornerstone of the mathematical framework in quantum information theory. See, for example, [4, 7, 9, 17, 20]. In this context, we would like to focus on the first inequality in (HP). It serves as an expression for the lower bound of Umegaki relative entropy and may offer a new perspective for understanding the properties of Umegaki relative entropy. Therefore, we will first introduce the mathematical background and recent results of the second inequality of (HP), thereby clarifying the significance of the research objective for the first inequality of (HP). Let us begin by reviewing the fundamental concepts of quantum relative entropy.

For two density matrices A and B in $\mathbb{P}_n$, the Umegaki relative entropy in [19] is defined by

$$S_U(A|B) = \mathrm{Tr}[A(\log A - \log B)],$$

which is a quantum generalization of the relative entropy due to Kullback-Leibler [13]: Suppose that two random variables X and Y on a discrete alphabet χ have the probability distribution function $P(x)$ and $Q(x)$, respectively. The divergence between the two random variables X and Y is a quantity that expresses the degree of separation between the two probability distributions:

$$D(X|Y) = D(P|Q) = \sum_{x\in\chi} P(x)\log\frac{P(x)}{Q(x)}.$$

The Umegaki relative entropy has the following fundamental properties, see also [4]:

Theorem 1.1. *Let A and B be positive definite matrices in $\mathbb{P}_n$.*

(i) (Non-negativity): $S_U(A|B) \ge 0$, *and* $S_U(A|B) = 0$ *if and only if* $A = B$.
(ii) (Additivity): $S_U(A_1 \otimes A_2|B_1 \otimes B_2) = S_U(A_1|B_1) + S_U(A_2|B_2)$.

(iii) (Joint convexity): *For any* $\lambda \in [0,1]$

$$S_U(A_\lambda|B_\lambda) \le (1-\lambda)S_I(A_1|B_1) + \lambda S_U(A_2|B_2)$$

where $A_\lambda = (1-\lambda)A_1 + \lambda A_2$ *and* $B_\lambda = (1-\lambda)B_1 + \lambda B_2$.

(iv) (Data Processing Inequality): *For any trace-preserving completely positive linear map* Φ

$$S_U(\Phi(A)|\Phi(B)) \le S_U(A|B).$$

Let A and B be two density matrices. In [1], the Tsallis relative entropy of A to B is defined by

$$D_\alpha(A|B) = \frac{1}{\alpha}\mathrm{Tr}[A - A^{1-\alpha}B^\alpha]$$

for any $\alpha \in (0,1]$. Then it is known in Ruskai-Stillinger [16] that

$$D_{-\alpha}(A|B) \ge S_U(A|B) \ge D_\alpha(A|B) \qquad \text{for all } \alpha \in (0,1]$$

and if $\alpha \to 0$, then the Tsallis relative entropy $D_\alpha(A|B)$ converges to the Umegaki relative entropy $S_U(A|B)$:

$$\lim_{\alpha\to\pm 0} D_\alpha(A|B) = S_U(A|B).$$

The quantum Tsallis relative entropy $D_\alpha(A|B)$ has the following properties:

Theorem 1.2. *Let A and B be density matrices in $\mathbb{S}_n$.*

(i) (Non-negativity): *For any* $\alpha \in [-1,1]\backslash\{0\}$, $D_\alpha(A|B) \ge 0$, *and* $D_\alpha(A|B) = 0$ *if and only if* $A = B$.

(ii) (Pseudoadditivity): *For any* $\alpha \in [-1,1]\backslash\{0\}$,

$$D_\alpha(A_1\otimes A_2|B_1\otimes B_2) = D_\alpha(A_1|B_1) + D_\alpha(A_2|B_2) - \alpha D_\alpha(A_1|B_1)D_\alpha(A_2|B_2).$$

(iii) (Joint convexity): *For any* $\alpha \in [0,1]$

$$D_\alpha(A_\lambda|B_\lambda) \le (1-\lambda)D_\alpha(A_1|B_1) + \lambda D_\alpha(A_2|B_2)$$

where $A_\lambda = (1-\lambda)A_1 + \lambda A_2$ *and* $B_\lambda = (1-\lambda)B_1 + \lambda B_2$.

(iv) (Data processing inequality): *For any trace-preserving completely positive linear map* Φ *and* $\alpha \in [0,1]$

$$D_\alpha(\Phi(A)|\Phi(B)) \le D_\alpha(A|B).$$

On the other hands, the Fujii-Kamei relative entropy is defined by

$$S_{FK}(A|B) = -\mathrm{Tr}[S(A|B)] = -\mathrm{Tr}[A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}],$$

where the relative operator entropy $S(A|B)$ due to Fujii-Kamei [6] is defined by

$$S(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}},$$

which is a relative version of the operator entropy defined by Nakamura-Umegaki [15].

We list the fundamental properties of the relative operator entropy $S(A|B)$, see [10]:

Proposition 1.3. *Let A, B, C and D be positive definite matrices in $\mathbb{P}_n$.*

(i) (Right monotonicity): *If* $B \le C$, *then* $S(A|B) \le S(A|C)$.

(ii) (Transformer inequality): $T^*S(A|B)T \le S(T^*AT|T^*BT)$ *for all* T.

(iii) (Information monotonicity): $\Phi(S(A|B)) \le S(\Phi(A)|\Phi(B))$ *for normal positive linear map* Φ.

(iv) (Sub-additivity): $S(A|B) + S(C|D) \le S(A+C|B+D)$.

(v) (Bound): $A - AB^{-1}A \leq S(A|B) \leq B - A$.
(vi) (Kernel inclusion): $\ker S(A|B) \supset \ker A$.

If A and B commute, the the Umegaki relative entropy $S_U(A|B)$ coincides with the Fujii-Kamei relative entropy $S_{FK}(A|B)$:

$$S_U(A|B) = S_{FK}(A|B).$$

Hence, the Fujii-Kamei relative entropy $S_{FK}(A|B)$ is another quantum generalization of the Kulbac-Leiber relative entropy.

Yanagi, Kuriyama and Furuichi in [20] defined the Tsallis relative operator entropy as an operator generalization of the Tsallis relative entropy:

$$T_\alpha(A|B) = \frac{A \natural_\alpha B - A}{\alpha} \qquad \text{for } \alpha \in \mathbb{R},$$

where the matrix α-geometric mean is defined by

$$A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \qquad \text{for } \alpha \in (0, 1],$$

see also [12]. We use the notation $\natural_\alpha$ for the binary operation

$$A \natural_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \qquad \text{for } \alpha \notin [0, 1],$$

that have formula in common with $\sharp_\alpha$.

For convenience, we denote another quantum Tsallis relative entropy of order $\alpha \in \mathbb{R}$ by

$$NT_\alpha(A|B) = -\mathrm{Tr}[T_\alpha(A|B)],$$

see also [7, 17]. If A and B commute, then the two quantum Tsallis relative entropy $D_\alpha(A|B)$ and $NT_\alpha(A|B)$ coincide:

$$D_\alpha(A|B) = NT_\alpha(A|B) \qquad \text{if } AB = BA.$$

Since

$$T_{-\alpha}(A|B) \leq S(A|B) \leq T_\alpha(A|B),$$

it follows that

$$NT_{-\alpha}(A|B) \geq S_{FK}(A|B) \geq NT_\alpha(A|B) \qquad \text{for all } \alpha \in (0, 1].$$

Since $\lim_{\alpha \to \pm 0} T_\alpha(A|B) = S(A|B)$, we have

$$\lim_{\alpha \to \pm 0} NT_\alpha(A|B) = S_{FK}(A|B).$$

The Umegaki relative entropy $S_U(A|B)$ and the Fujii-Kamei relative entropy $S_{FK}(A|B)$ have the 1-parameter extensions $D_\alpha(A|B)$ and $NT_\alpha(A|B)$ respectively. It is natural to ask what is the relation between $S_U(A|B)$ and $S_{FK}(A|B)$:

$$D_{-\alpha}(A|B) \geq S_U(A|B) \geq D_\alpha(A|B) \qquad \text{for all } \alpha \in (0, 1]$$

and

$$NT_{-\alpha}(A|B) \geq S_{FK}(A|B) \geq NT_\alpha(A|B) \qquad \text{for all } \alpha \in (0, 1].$$

Hiai-Petz in [11] showed that

$$S_U(A|B) \leq S_{FK}(A|B).$$

Moreover, it follows from [9, 17] that

$$D_\alpha(A|B) \leq NT_\alpha(A|B) \qquad \text{for all } \alpha \in [-1, 1] \setminus \{0\}. \tag{1.1}$$

Hiai-Petz in [11] showed a 1-parameter extension of the Umegaki relative entropy as follows:

$$\frac{1}{p}\mathrm{Tr}[A(\log B^{-\frac{p}{2}}A^pB^{-\frac{p}{2}})] \le S_U(A|B) \le \frac{1}{p}\mathrm{Tr}[A(\log A^{\frac{p}{2}}B^{-p}A^{\frac{p}{2}})] \tag{1.2}$$

for all $p>0$. Moreover, the LHS and RHS of (1.2) converge to the Umegaki relative entropy $S_U(A|B)$ as $p\to 0$. We call (1.2) the Hiai-Petz inequality. In terms of the relative operator entropy, (1.2) can be rephrased as

$$\frac{1}{p}\mathrm{Tr}[B^{-\frac{p}{2}}AB^{-\frac{p}{2}}S(B^p|A^p)] \le S_U(A|B) \le -\frac{1}{p}\mathrm{Tr}[A^{1-p}S(A^p|B^p)] \tag{1.3}$$

for all $p>0$. In particular, if we put $p=1$ in (1.3), then we have

$$S_U(A|B) \le S_{FK}(A|B). \tag{1.4}$$

Moreover, we have the following 1-parameter extension of (1.1): For each $\alpha\in[-1,1]$,

$$D_\alpha(A|B) \le -\frac{1}{p}\mathrm{Tr}[A^{1-p}T_{\frac{\alpha}{p}}(A^p|B^p)] \qquad \text{for all } p\ge|\alpha|>0, \tag{1.5}$$

see also [8, 9, 7, 17]. If we put $\alpha\to\pm 0$ in (1.5), then we have the second inequality of the Hiai-Petz inequality (1.2). Refer to Figure 1 below:

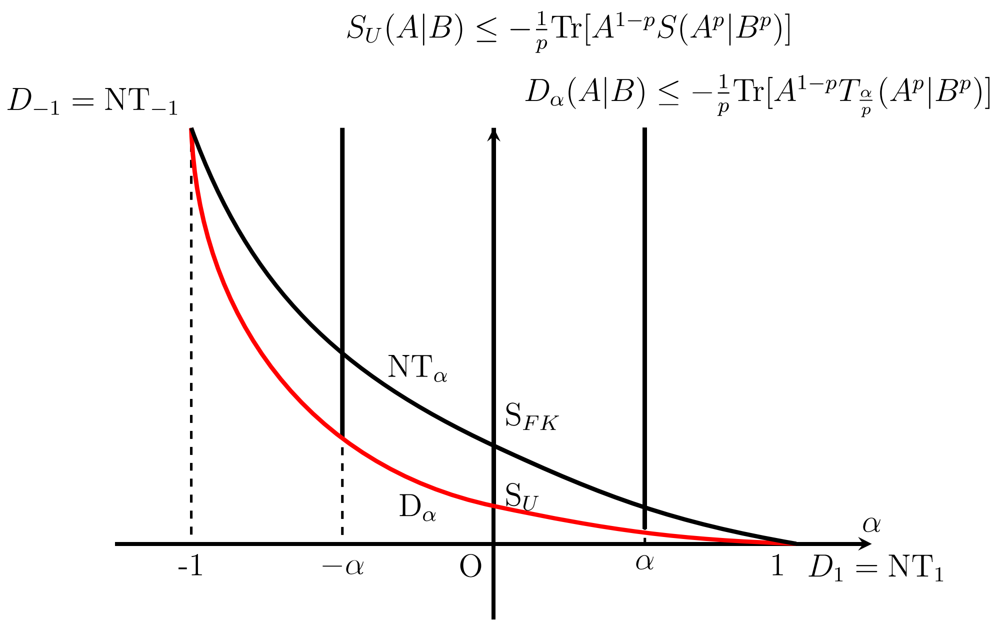

FIGURE 1. The relation between D_α and NT_α

Thus, based on the study in the second inequality of Hiai-Petz one (1.2), we observe the order relation among the Tsallis type trace inequalities and the Umegaki relative entropy.

In this paper, we concentrate on the first inequality of the Hiai-Petz one (1.2) and show the lower bound of the quantum Tsallis relative entropy $D_\alpha(A|B)$ as in (1.5).

2. Results

We firstly rephrase the first inequality of (1.2) in terms of the relative operator entropy:

$$\frac{1}{p}\mathrm{Tr}[A(\log B^{-\frac{p}{2}}A^pB^{-\frac{p}{2}})] = \frac{1}{p}\mathrm{Tr}[B^{-\frac{p}{2}}AB^{-\frac{p}{2}}S(B^p|A^p)] \le S_U(A|B)$$

for all $p > 0$. In particular, if $p = 1$, then

$$\begin{aligned}\mathrm{Tr}[A(\log B^{-\frac{1}{2}}AB^{-\frac{1}{2}})] &= \mathrm{Tr}[B^{-\frac{1}{2}}AB^{-\frac{1}{2}}S(B|A)]\\ &= -\mathrm{Tr}[B^{-\frac{1}{2}}AB^{-\frac{1}{2}}A^{-1}S(A|B)] \le S_U(A|B) \le -\mathrm{Tr}[S(A|B)].\end{aligned}$$

For convenience, we define quantum relative entropies by

$$S_{CFK}(A|B) = \mathrm{Tr}[B^{-\frac{1}{2}}AB^{-\frac{1}{2}}S(B|A)]$$

and

$$NL_\alpha(A|B) = \mathrm{Tr}[B^{-\frac{1}{2}}AB^{-\frac{1}{2}}T_{-\alpha}(B|A)] \quad \text{for each order } \alpha \in [-1,1]\setminus\{0\}.$$

Indeed, $\lim_{\alpha\to 0} NL_\alpha(A|B) = S_{CFK}(A|B)$. If A and B commute, then we have $S_{CFK}(A|B) = S_U(A|B) = S_{FK}(A|B)$ and $NL_\alpha(A|B) = D_\alpha(A|B) = NT_\alpha(A|B)$ for any order $\alpha \in [-1,1]\setminus\{0\}$.

We show the following properties of $NL_\alpha(A|B)$ of order $\alpha \in [-1,1]\setminus\{0\}$:

Proposition 2.1. *Let A and B be positive definite matrices in $\mathbb{P}_n$ and $\alpha \in [-1,1]\setminus\{0\}$. Then the following properties of $NL_\alpha(A|B)$ hold:*

(i) (Positivity) *If $A \ge B$, then $NL_\alpha(A|B) \ge 0$ for all $\alpha \in [-1,1]\setminus\{0\}$. Under the condition $A \ge B$, the equality holds if and only if $A = B$.*

(ii) (Unitary invariance) *$NL_\alpha(U^*AU|U^*BU) = NL_\alpha(A|B)$ for any unitary U and $\alpha \in [-1,1]\setminus\{0\}$.*

(iii) (Pseudoadditivity) *For all $\alpha \in [-1,1]\setminus\{0\}$*

$$\begin{aligned}NL_\alpha(A_1\otimes A_2|B_1\otimes B_2) &= NL_\alpha(A_1|B_1)\mathrm{Tr}[A_2] + NL_\alpha(A_2|B_2)\mathrm{Tr}[A_1]\\ &\quad - \alpha NL_\alpha(A_1|B_1)NL_\alpha(A_2|B_2).\end{aligned}$$

(iv) (Monotonicity) *$NL_\alpha(A|B)$ is non-increasing for $\alpha \in [-1,1]\setminus\{0\}$.*

Then we show the order relation between $D_\alpha(A|B)$ and $NL_\alpha(A|B)$ of real order:

Theorem 2.2. *Let A and B be positive definite matrices in $\mathbb{P}_n$. Then*

$$NL_\alpha(A|B) \le D_\alpha(A|B) \qquad \text{for all } \alpha \in [-1,\tfrac{1}{2}]\setminus\{0\}\cup\{1\}.$$

Remark 2.3. *Theorem 2.2 implies*

$$\frac{1}{\alpha}\mathrm{Tr}[A - A(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^\alpha] \le \frac{1}{\alpha}\mathrm{Tr}[A - A^{1-\alpha}B^\alpha]$$

or equivalently if $0 < \alpha \le \frac{1}{2}$, then

$$\mathrm{Tr}[A^{1-\alpha}B^\alpha] \le \mathrm{Tr}[A(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^\alpha]$$

and if $-1 \le \alpha < 0$, then

$$\mathrm{Tr}[A(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^\alpha] \le \mathrm{Tr}[A^{1-\alpha}B^\alpha].$$

To prove Theorem 2.2, we need the following log-majorization, which is a variant of the BLP inequality [3]:

Lemma 2.4. *Let A and B be two positive definite matrices in $\mathbb{P}_n$. Then*

$$A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \prec_{(\log)} A^{\frac{1}{2}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} \qquad \text{for all } 0 < \alpha \leq \tfrac{1}{2}$$

and

$$A^{\frac{1}{2}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} \prec_{(\log)} A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \qquad \text{for all } -1 \leq \alpha < 0.$$

Remark 2.5. *If $\alpha = 1$, then*

$$B \prec_{(\log)} A^{\frac{1}{2}} B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}}.$$

Proof of Lemma 2.4. Suppose that $-1 \leq \alpha < 0$. It suffices to prove the following implication:

$$A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \leq I \implies A^{\frac{1}{2}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} \leq I.$$

Since $0 < \frac{1}{1-\alpha} < 1$, we have

$$B^{\frac{\alpha}{1-\alpha}} \leq A^{-1} \qquad \text{or} \qquad A \leq B^{\frac{\alpha}{\alpha-1}}.$$

Hence, it follows form $0 < -\alpha < 1$ that

$$\begin{aligned} & A^{\frac{1}{2}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} = A^{\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{-\alpha} A^{\frac{1}{2}} \\ & \leq A^{\frac{1}{2}} (B^{\frac{1}{2}} B^{\frac{\alpha}{\alpha-1}} B^{\frac{1}{2}})^{-\alpha} A^{\frac{1}{2}} = A^{\frac{1}{2}} B^{\frac{-\alpha}{\alpha-1}} A^{\frac{1}{2}} \\ & \leq A^{\frac{1}{2}} A^{-1} A^{\frac{1}{2}} = I. \end{aligned}$$

For each $\alpha \in (0, \frac{1}{2}]$

$$A^{\frac{1}{2}} (B^{\frac{p}{2}} A^{-p} B^{\frac{p}{2}})^{\frac{\alpha}{p}} A^{\frac{1}{2}} \leq I \quad \text{implies} \quad A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \leq I \quad \text{for all } 1 \geq p \geq 2\alpha.$$

Putting $A_1 = A^{-1}$ and $B_1 = (B^{\frac{p}{2}} A^{-p} B^{\frac{p}{2}})^{\frac{\alpha}{p}}$, we have $B_1 \leq A_1$. Since $B_1^{\frac{p}{\alpha}} = B^{\frac{p}{2}} A^{-p} B^{\frac{p}{2}}$, it follows from the Riccati equation that $B^{\frac{p}{2}} = A^p \sharp B_1^{\frac{p}{\alpha}} = A_1^{-p} \sharp B_1^{\frac{p}{\alpha}}$. Since $A_1^{-p} \leq B_1^{-p}$ by $-1 \leq -p < 0$, it follows that

$$\begin{aligned} B^{\alpha} = B^{\frac{p}{2}\frac{2\alpha}{p}} &= (A_1^{-p} \sharp B_1^{\frac{p}{\alpha}})^{\frac{2\alpha}{p}} \leq (B_1^{-p} \sharp B_1^{\frac{p}{\alpha}})^{\frac{2\alpha}{p}} \quad \text{by } 0 < \tfrac{2\alpha}{p} < 1 \\ &= B_1^{1-\alpha} \leq A_1^{1-\alpha} = A^{\alpha-1} \quad \text{by } 0 < 1-\alpha < 1. \end{aligned}$$

□

By Lemma 2.4, we have the following norm inequalities:

Lemma 2.6. *Let A and B be two positive definite matrices in $\mathbb{P}_n$.*

$$\left|\left|\left| A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \right|\right|\right| \leq \left|\left|\left| A^{\frac{1}{2}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} \right|\right|\right| \qquad \text{for all } 0 < \alpha \leq \tfrac{1}{2}$$

and

$$\left|\left|\left| A^{\frac{1}{2}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} \right|\right|\right| \leq \left|\left|\left| A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \right|\right|\right| \qquad \text{for all } -1 \leq \alpha < 0,$$

for every unitarily invariant norm $|||\cdot|||$.

Remark 2.7. *If $\alpha = 1$, then*

$$|||B||| \leq \left|\left|\left| A^{\frac{1}{2}} B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}} \right|\right|\right|.$$

Proof of Theorem 2.2. If $0 < \alpha \le \frac{1}{2}$, then it follows from Lemma 2.6 that

$$\begin{aligned} NL_\alpha(A|B) &= \frac{1}{\alpha}\mathrm{Tr}[A - A^{\frac{1}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^\alpha A^{\frac{1}{2}}] \\ &\le \frac{1}{\alpha}\mathrm{Tr}[A - A^{\frac{1-\alpha}{2}}B^\alpha A^{\frac{1-\alpha}{2}}] \\ &= D_\alpha(A|B). \end{aligned}$$

Similarly, we have the desired result in the case of $-1 \le \alpha < 0$. □

Unfortunately, in the case of $\frac{1}{2} \le \alpha < 1$, it is unclear whether the inequality in Theorem 2.2 holds or not.

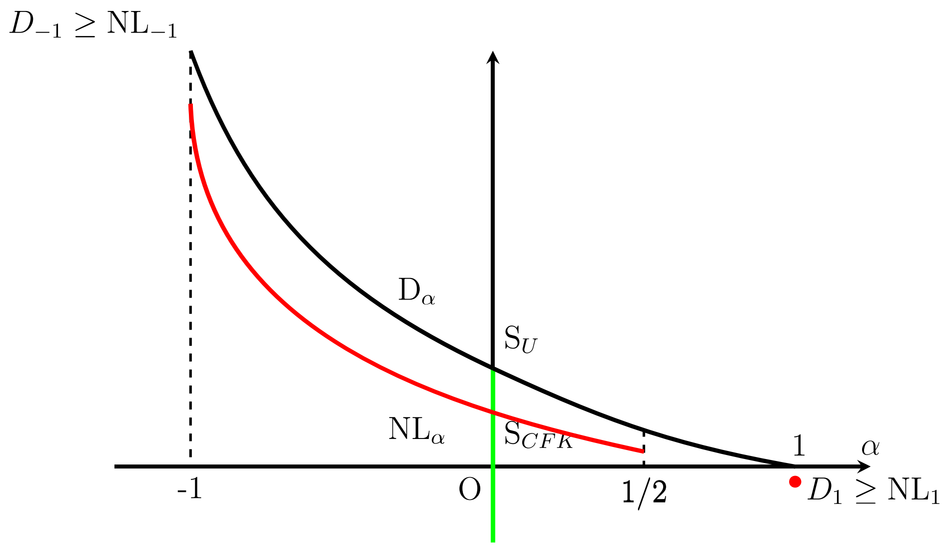

FIGURE 2. The relation between D_α and NL_α

3. Quantum relative entropy

We have the following lower bound of the quantum Tsallis relative entropy $D_\alpha(A|B)$ as a complementary part of (1.5):

Theorem 3.1. *Let A and B be positive definite matrices in $\mathbb{P}_n$. Then*

$$\frac{1}{p}\mathrm{Tr}[B^{-\frac{p}{2}}AB^{-\frac{p}{2}}T_{-\frac{\alpha}{p}}(B^p|A^p)] \le D_\alpha(A|B) \tag{3.1}$$

for all $\alpha \in [-1, 0)$ and $1 - \alpha \ge p \ge -\alpha > 0$, or $\alpha \in (0, \frac{1}{2}]$ and $1 \ge p \ge 2\alpha > 0$.

Remark 3.2. *If we put $\alpha \to \pm 0$ in* (3.1) *of Theorem 3.1, then $T_{-\frac{\alpha}{p}}(B^p|A^p)] \to S(B^p|A^p)$ implies*

$$\frac{1}{p}\mathrm{Tr}[A(\log B^{-\frac{p}{2}}A^pB^{-\frac{p}{2}})] = \frac{1}{p}\mathrm{Tr}[B^{-\frac{p}{2}}AB^{-\frac{p}{2}}S(B^p|A^p)] \le S_U(A|B)$$

for all $0 < p \le 1$, and so we get the first inequality of the Hiai-Petz inequality (1.2) *for all $0 < p \le 1$.*

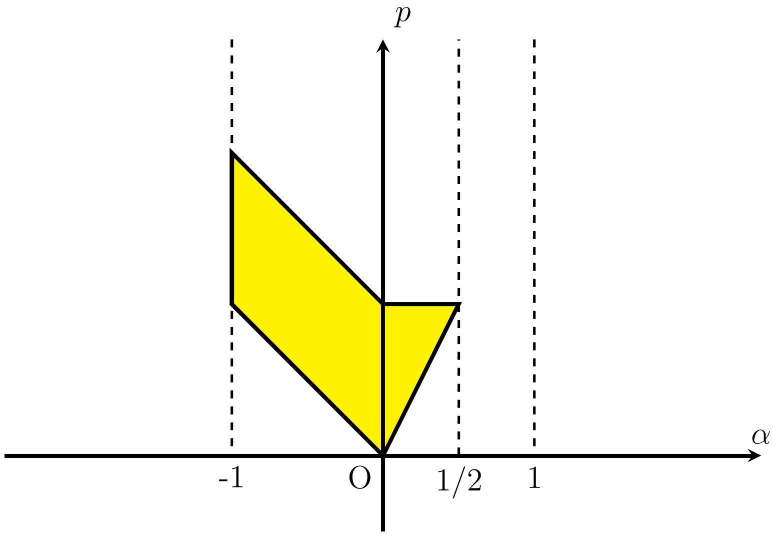

FIGURE 3. The range in which Theorem 3.1 holds

As shown in Figure 3, the range (α, p) where Theorem 3.1 holds is not symmetric. Further discussion is required.

Finally, we propose another version of 1-parameter extension of Umegaki relative entropy $S_U(A|B)$: Put

$$L_\alpha(A|B) = \frac{1}{\alpha}\mathrm{Tr}[A - AB^{\frac{\alpha}{2}}A^{-\alpha}B^{\frac{\alpha}{2}}]$$

and

$$NL_\alpha(A|B) = \frac{1}{\alpha}\mathrm{Tr}[A - A(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^\alpha] = \mathrm{Tr}[B^{-\frac{1}{2}}AB^{-\frac{1}{2}}T_{-\alpha}(B|A)]$$

for $\alpha \in \mathbb{R}$. If A and B commute, then $D_\alpha(A|B) = L_\alpha(A|B) = NLT_\alpha(A|B)$. Moreover, it follows that

$$\lim_{\alpha\to 0} L_\alpha(A|B) = S_U(A|B) = \mathrm{Tr}[A(\log A - \log B)]$$

and

$$\lim_{\alpha\to 0} NL_\alpha(A|B) = \mathrm{Tr}[B^{-\frac{1}{2}}AB^{-\frac{1}{2}}S(B|A)].$$

Since $D_\alpha(A|B) \geq L_\alpha(A|B)$ for all $\alpha \in \mathbb{R}$, it follows that

$$S_U(A|B) \geq L_\alpha(A|B) \qquad \text{for all } \alpha > 0.$$

Regarding the first inequality of the HP inequality (1.2), further research is expected.

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