On weak*-weak points of continuity in the dual unit ball of Banach spaces

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1 Introduction

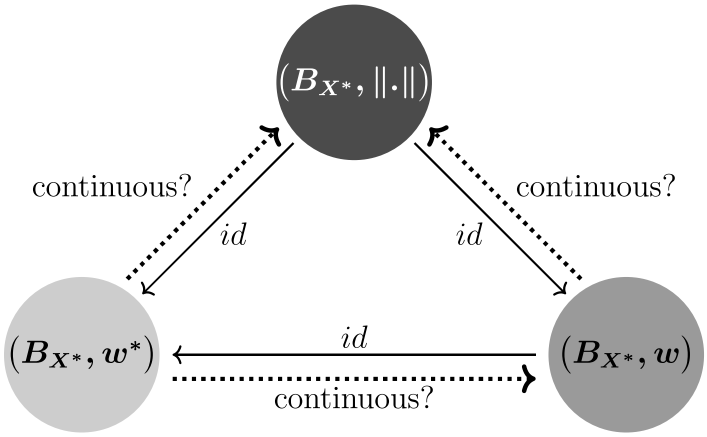
This survey article is devoted to the study of the geometric aspects of the topological notion of weak*-weak points of continuity in the dual unit ball of Banach spaces, following [4]. We develop the theory in detail, emphasizing its geometric interpretation, structural consequences, and key implications.

Let X be a Banach space. We denote by $S_X = \{x \in X : \|x\| = 1\}$ the unit sphere of X and by $B_X = \{x \in X : \|x\| \leq 1\}$ the closed unit ball. We write X^* for the dual space of X. Let $J_X : X \longrightarrow X^{**}$ be the canonical embedding defined by $J_X(x)(x^*) = x^*(x)$ for all $x^* \in X$. Recall that a Banach space X is replexive if $J_X(X) = X^{**}$. Every finite dimensioanal space is a reflexive space where c_0 space of all null sequence with supremum norm is not an replexive space.

In this study, we mainly use three canonical topologies on X^*, namely: the *norm topology* (in short, the $\|\cdot\|$-topology) – the topology induced by the dual norm; the *weak topology* (in short, the w-topology) – the weakest topology on X^* such that all $x^{**} \in X^{**}$ are continuous; and the *weak* topology* (in short, the w^*-topology) – the weakest topology on X^* such that all evaluation maps $J_X(x) \in X^{**}$, for $x \in X$, are continuous.

Definition 1.1. Let τ and σ denote any two of the topologies $\|\cdot\|$, w, and w^*. A point $x^* \in B_{X^*}$ is called a point of τ-σ continuity (in short, a τ-σ-pc) in B_{X^*} if the identity map $id : (B_{X^*}, \tau) \longrightarrow (B_{X^*}, \sigma)$ is continuous at x^*.

It is clear that if τ is a finer topology than σ, then the identity map $id : (B_{X^*}, \tau) \longrightarrow (B_{X^*}, \sigma)$ is always continuous. Consequently, we have the following situations, where solid arrows indicate continuity of the identity map, while dotted arrows indicate cases where continuity does not hold in general. In the diagram below, the depth of the black color of the unit ball represents the relative strength of the topology.



The continuity of

$$id : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, \|\cdot\|) \quad \text{and} \quad id : (B_{X^*}, w) \longrightarrow (B_{X^*}, \|\cdot\|)$$

has been extensively studied in the geometry of Banach spaces. These mappings are closely related to the well-known geometric properties known as the *Mazur Intersection Property (MIP)* and the *weak* Mazur Intersection Property (w*-MIP)*[11, 6]. A Banach space X is said to have the Mazur (respectively, weak* Mazur) Intersection Property if every closed (respectively, weak*-closed) bounded convex subset of X can be represented as the intersection of closed balls containing it. A point $x^* \in B_{X^*}$ is called an extreme point if it cannot be expressed as the midpoint of two distinct points of B_{X^*}.

It is known that a Banach space X has the Mazur (respectively, weak* Mazur) Intersection Property if and only if the set of points of continuity of

$$id : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, \|\cdot\|) \quad (\text{respectively } id : (B_{X^*}, w) \longrightarrow (B_{X^*}, \|\cdot\|))$$

which are also extreme points of B_{X^*}, is norm-dense in S_{X^*}. This highlights the importance of studying the continuity properties of these identity mappings. We refer [6, 10, 9] for detailed discussion.

It therefore remains natural to investigate the continuity of

$$id : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w),$$

and to ask whether it reflects any geometric property of the Banach space X. In order to study this, we formally state the following definition.

Definition 1.2 (weak*-weak point of continuity)**.** Let $x^* \in B_{X^*}$ is said to be a weak*-weak point of continuity (w^*-w-pc in short) in B_{X^*} if $id : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w)$ is continuous at x^*.

We denote by w^*-w-pc(B_{X^*}) the set of all weak*-weak points of continuity of B_{X^*}; that is, the set of all points of B_{X^*} at which the identity mapping $id : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w)$ is continuous.

This survey article is organized as follows. Section 2 describes the basic properties on weak*-weak point of continuity. We discuss that this study is link with study of preserved extreme points. In Section 3, we discuss that notions of unique Hahn-Banach extension property and Chebyshev subspace are well connected with weak*-weak point of continuity. In Section 4, we consider this study in higher dual spaces under certain condition on the original space.

2 A few basic facts on weak*-weak point of continuity

In this section, we examine some basic properties of the weak*-to-weak points of continuity in the dual unit ball of a Banach space. First, we observe that w^*-w-pc(B_{X^*}) is a convex set; in fact, it is a face of B_{X^*}. Recall that a set D is called convex if for any $x, y \in D$ and any $\lambda \in [0,1]$, the point $\lambda x + (1-\lambda)y$ also belongs to D. Moreover, if D is a convex set and $C \subseteq D$ is convex, then C is called a face of D if the following holds: whenever $a \in C$ can be written as a convex combination $a = \lambda b + (1-\lambda)c$ for some $b, c \in D$ and $\lambda \in (0,1)$, it follows that $b, c \in C$.

To show that w^*-w-pc(B_{X^*}) is a convex set, let $x^*, y^* \in w^*$-w-pc(B_{X^*}) and $\lambda \in (0,1)$. Suppose $(z_\alpha)_{\alpha\in\Lambda}$ is a net such that $z_\alpha \longrightarrow \lambda x^* + (1-\lambda)y^*$ in the w^*-topology. Then $\frac{z_\alpha-(1-\lambda)y^*}{\lambda} \longrightarrow x^*$ in the w^*-topology. Since x^* is a w^*-w point of continuity in B_{X^*}, it follows that $\frac{z_\alpha-(1-\lambda)y^*}{\lambda} \longrightarrow x^*$ in the weak topology. Multiplying back by λ and adding $(1-\lambda)y^*$, we obtain $z_\alpha \longrightarrow \lambda x^* + (1-\lambda)y^*$ in the weak topology. Hence, $\lambda x^* + (1-\lambda)y^*$ is a w^*-w point of continuity in B_{X^*}. The cases $\lambda = 0$ or $\lambda = 1$ hold trivially. Next, we check that w^*-w-pc(B_{X^*}) is a face of B_{X^*}.

Lemma 2.1 ([4])**.** *Let $x^* \in S_{X^*}$ be a point of continuity of $id : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w)$, and $x^* = \lambda y^* + (1-\lambda)z^*$ for some $y^*, z^* \in B_{X^*}$, for $\lambda \in (0,1)$, then both y^* and z^* are points of continuity of $id : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w)$.*

In particular, w^*-w-pc(B_{X^*}) is a face of B_{X^*}.

As an application of Lemma 2.1, together with the fact that the weak and weak* topologies differ on B_{X^*} for a non-reflexive space, we have the following.

Lemma 2.2. *Let X be a nonreflexive space and let $x^* \in B_{X^*}$ be a point of continuity of $id : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w)$, then $x^* \in S(X^*)$. In particular, w^*-w-pc$(B_{X^*}) \subset S_{X^*}$.*

Hence, every weak*-to-weak point of continuity lies on the unit sphere of the dual space.

By the weak*–lower semicontinuity of the norm, it is enough to consider the identity map restricted to the unit sphere.

Lemma 2.3. *Let $x^* \in S_{X^*}$. Then, x^* is a point of continuity of $id : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w)$ if and only if x^* is a point of continuity of $id : (S_{X^*}, w^*) \longrightarrow (S_{X^*}, w)$.*

By Goldstine's theorem, which asserts that S_X is weak*-dense in $S_{X^{**}}$, we conclude that weak*–weak points of continuity in the bidual must belong to the original space.

Lemma 2.4. *Let $x^{**} \in B_{X^{**}}$ be a point of continuity of $id : (B_{X^{**}}, w^*) \longrightarrow (B_{X^{**}}, w)$, then $x^{**} \in S(X)$.*

We now turn to the connection between weak*–weak points of continuity and preserved extreme points. Recall that extreme points of B_X which remain extreme in $B_{X^{**}}$ are called *preserved extreme points*. Recall that a point x in B_X of a Banach space X is called an *extreme point* if it cannot be written as a nontrivial convex combination of two other points in B_X. We denote the set of all extreme points of B_X by $\mathrm{Ext}(B_X) = \{\, x \in B_X : x \text{ is extreme in } B_X \,\}$. It is clear that any extreme point of $B_{X^{**}}$ that belongs to X is also an extreme point of B_X. In general, however, the converse fails. This line of investigation was initiated by James, who proved that $Ext(B_{X^{**}}) = Ext(B_X)$ if and only if X is reflexive. We refer to [7] for further discussion on this topic. In what follows, we show that the converse does hold for weak*–weak points of continuity in $B_{X^{**}}$.

Lemma 2.5. *Let $x^{**} \in B_{X^{**}}$ be a point of continuity of $id : (B_{X^{**}}, w^*) \longrightarrow (B_{X^{**}}, w)$. Then $x^{**} \in Ext(B_{X^{**}})$ if and only if $x^{**} \in Ext(B_X)$.*

3 Connection with unique Hahn–Banach extensions

In this section, we first discuss the unique Hahn–Banach extension property, which can be characterized as a geometric property of Banach spaces. Then we discuss the connection between the w^*–w point of continuity property and the unique Hahn–Banach extension property.

3.1 Unique Hahn–Banach extensions property

Let Y be a subspace of a Banach space X. The Hahn-Banach theorem ensures that for every bounded linear functional $y^* \in Y^*$, there exists an extension $x^* \in X^*$ such that

$$x^*(y) = y^*(y) \quad \forall y \in Y, \quad \text{and} \quad \|x^*\| = \|y^*\|.$$

Such an extension x^* is called a Hahn-Banach extension or norm-preserving extension. While existence is guaranteed, uniqueness is not automatic. Early work by Taylor (1939) and Foguel (1958) linked the uniqueness of Hahn-Banach extensions to the strict convexity of the dual space X^* [14, 5]. However, this is a global condition and does not provide information about uniqueness restricted to a particular subspace.

To address this, Phelps (1960) introduced property-(U), a notion of uniqueness on a subspace. The idea is that a subspace Y may allow each of its functionals to have a *unique* norm-preserving extension to the ambient space X, even when X^* is not strictly convex.

Definition 3.1 (Property-(U))**.** [12] Let Y be a subspace of a Banach space X. We say that Y has *property-(U)* in X if for every $y^* \in Y^*$ there exists a *unique* $x^* \in X^*$ such that

$$x^*(y) = y^*(y) \quad \forall y \in Y, \quad \text{and} \quad \|x^*\| = \|y^*\|.$$

Equivalently, property-(U) means that each bounded linear functional on Y has a unique norm-preserving extension to X.

The geometric interpretation of property-(U) arises naturally through the concept of Chebyshev subspaces. Recall that a subspace Y of a Banach space X is Chebyshev if every point in X has a unique best approximation in Y; that is, the set

$$P_Y(x) := \{y \in Y : \|x - y\| = d(x, Y)\}$$

is a singleton for all $x \in X$. For a subspace Y of X, we denote $Y^\perp = \{x^* \in X^* : x^*(y) = 0 \, \forall \, y \in Y\}$, the annihilator of the subspace Y [13]. The geometric characterization of property-(U) is as follows.

Theorem 3.2 ([12])**.** *Let Y be a subspace of X. Then the following are equivalance:*

1. *Y has property-(U) in X.*

2. *$Y^{\perp}$ is a Chebyshev subspace of X^*.*

Thus, property-(U) ensures that each functional on Y has a unique norm-preserving extension to X, which is equivalent to the uniqueness of best approximations in the dual space. Recent work on this topic can be found in [1, 2, 3]. We provide several examples illustrating the behavior of subspaces with respect to property-(U) in finite-dimensional spaces, considering both strictly convex and non-strictly convex settings.

We first consider the classical Euclidean space, where the dual space is strictly convex.

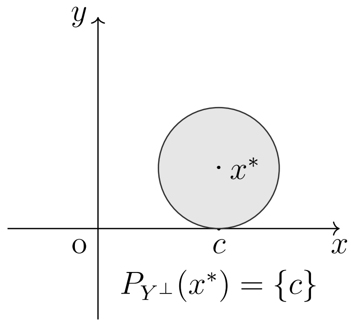
Example 3.3. Let the space and subspace be:

$$X = (\mathbb{R}^2, \|\cdot\|_2), \quad Y = \{(0, y) : y \in \mathbb{R}\}.$$

The dual space and the annihilator of Y are:

$$X^* = (\mathbb{R}^2, \|\cdot\|_2), \quad Y^{\perp} = \{(x, 0) : x \in \mathbb{R}\}.$$

For $x^* \in X^*$, one can check that $P_{Y^{\perp}}(x^*) = Y^{\perp} \bigcap B[x^*, d(x^*, Y^{\perp})]$.



It is evident that the annihilator $Y^{\perp}$ is a Chebyshev subspace in X^*. Therefore Y has property-(U) in X.

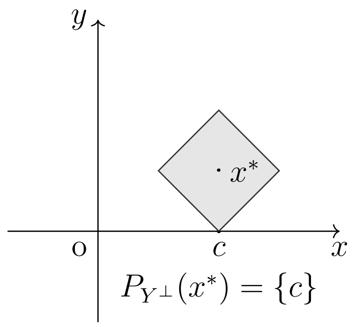
We now consider a space whose dual is not strictly convex but still exhibits property-(U).

Example 3.4. Let

$$X = (\mathbb{R}^2, \|\cdot\|_\infty), \quad Y = \{(0, y) : y \in \mathbb{R}\}.$$

Then

$$X^* = (\mathbb{R}^2, \|\cdot\|_1), \quad Y^{\perp} = \{(x, 0) : x \in \mathbb{R}\}.$$



Even though X^* is not strictly convex, $Y^{\perp}$ is still a Chebyshev subspace. Hence Y has property-(U) in X.

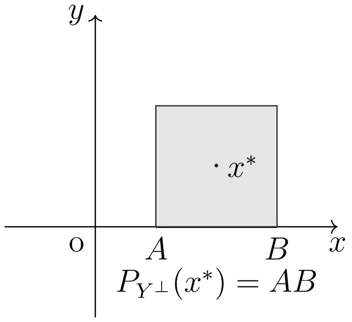
Finally, we examine a case where the dual space is not strictly convex, and property-(U) fails.

Example 3.5. Let

$$X = (\mathbb{R}^2, \|\cdot\|_1), \quad Y = \{(0, y) : y \in \mathbb{R}\}.$$

Then

$$X^* = (\mathbb{R}^2, \|\cdot\|_\infty), \quad Y^\perp = \{(x, 0) : x \in \mathbb{R}\}.$$



Here, $Y^\perp$ is not a Chebyshev subspace in X^*. Thus Y does not have property-(U) in X.

We now consider this notion in the special case where a Banach space X is regarded as a subspace of its bidual X^{**} via the canonical embedding $J_X : X \longrightarrow X^{**}$.

Let X be an infinite-dimensional Banach space, let X^{**} denote its bidual, and let $J_{X^*} : X^* \longrightarrow X^{***}$ be the canonical embedding.

Definition 3.6 (Hahn–Banach smooth)**.** An non-reflexive Banach space X is said to be *Hahn–Banach smooth* if $J_X(X)$ has property-(U) in X^{**}; that is, for every $x^* \in X^*$, $J_{X^*}(x^*) \in X^{***}$ is the only unique norm-preserving extension of x^* to X^{**}.

For instance, the Banach space c_0 of sequences converging to zero, equipped with the supremum norm, is Hahn–Banach smooth. Similarly, if H is an infinite-dimensional Hilbert space, the space $\mathcal{K}(H)$ of compact operators on $\mathcal{H}$, endowed with the operator norm, is also Hahn–Banach smooth. As a counterexample, the Banach space ℓ^1 of absolutely summable sequences with equipped with the standard ℓ^1-norm is not Hahn–Banach smooth.

3.2 How weak*-weak point of continuity connected with unique Hahn–Banach extensions

We observe that the notion of weak*-weak point of continuity admits a geometric interpretation. As an immediate consequence of Goldstine's theorem together with the weak*-compactness of the unit ball of the triple dual, we obtain the following result.

Lemma 3.7 ([8])**.** *Let X be a Banach space. Then $x^* \in S_{X^*}$ is a point of continuity of*

$$\mathrm{id} : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w)$$

if and only if x^* admits a unique norm-preserving extension to X^{**}.

This local phenomenon extends to a global characterization.

Theorem 3.8. *For a Banach space X, the following assertions are equivalent:*

1. *The identity map*
$$\mathrm{id} : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w)$$
is continuous at every $x^* \in S_{X^*}$.

2. *X is Hahn–Banach smooth in X^{**}.*

3. *$X^\perp$ is a Chebyshev subspace of X^{***}.*

This equivalence shows that the global continuity of the identity map is precisely equivalent to the uniqueness of norm-preserving extensions to the bidual, which in turn is equivalent to the Chebyshev property of the annihilator in the triple dual.

In particular, what initially appears to be a purely topological condition on the dual ball turns out to encode a strong geometric requirement on the canonical embedding $X \hookrightarrow X^{**}$.

4 Weak*–Weak Points of Continuity in higher duals of an M-Embedded Space

In this section, we focus on the weak*–weak points of continuity in the unit ball of the first, second, and third duals of M-embedded spaces. We provide an exact count of all weak*–weak points of continuity in the unit ball for these duals. Throughout this section, let X be a non-reflexive Banach space.
A closed subspace $Y \subset X$ is called an M-ideal in X if there exists a bounded linear projection
$$P : X^* \longrightarrow X^*$$
such that $\ker P = Y^\perp$ and
$$\|x^*\| = \|Px^*\| + \|(I - P)x^*\| \quad \text{for all } x^* \in X^*.$$
It is known that every M-ideal in X has property (U) in X. Indeed, for each $x^* \in X^*$, the element $(I - P)x^*$ is the unique nearest point to x^* in the annihilator $Y^\perp$ (see [8, Proposition 1.12]). There are several classical examples of M-ideals. In Banach space theory, the space c_0 is an M-ideal in ℓ_∞, the space of all bounded sequences equipped with the supremum norm. In operator algebra, the algebra of compact operators $\mathcal{K}(H)$ is an M-ideal in the algebra of bounded linear operators $\mathcal{B}(H)$ on a Hilbert space H. In fact, every two-sided ideal of a C^*-algebra is an M-ideal (see [8, Example I.1.4(d)]). For additional standard examples in classical Banach spaces, see [8, Example I.1.4(a)], [8, Example I.1.4(b)], and [8, Example I.1.4(c)].

It is know that if Y is a subspace of X, in general, we do not know that Y^* is a subspace of X^*, but if it is an M-ideal. Then it happens that Y^* can be viewed as a subspace of X^*. Under this situation, we now observe that weak*-weak point of continuity lifted from the dual of a subspace to dual of the entire space.

Proposition 4.1 ([4])**.** *Let Y be an M-ideal in X and let $y^* \in S_{Y^*}$. If $id : (B_{Y^*}, w^*) \longrightarrow (B_{Y^*}, w)$ is continuous at y^*, then $id : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w)$ is continuous at y^*.*

We now recall the notion of an M-embedded space from [8].

Definition 4.2 (M-embedded space)**.** A Banach space X is said to be an M-embedded space if $J_X(X)$ is an M-ideal in X^{**}.

It is well known that $c_0^{**} = \ell_\infty$ and $\mathcal{K}(H)^{**} = \mathcal{B}(H)$. Therefore, from the discussion above concerning examples of M-ideals, it follows that both c_0 and $\mathcal{K}(H)$ are M-embedded spaces. For further examples in classical Banach spaces, we refer to [8, Example III.1.4]. In fact, Chapter III of [8] is devoted entirely to this topic. We have the following as an immediate consequence of the discussion preceding Definition 4.2.

Lemma 4.3 ([8])**.** *Let X be an M-embedded space. Then X is a Hahn–Banach smooth space.*

Hence the following result follows from Lemma 4.3 and Theorem 3.8. Recall the notation w^*-w-pc$(B_{X^{**}}) = \{x^{**} \in B_{X^{**}} : id : (B_{X^{**}}, w^*) \longrightarrow (B_{X^{**}}, w)$ is continuous at $x^{**}\}$.

Theorem 4.4 ([4])**.** *Let X be an M-embedded space. Then $id : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w)$ is continuous at each $x^* \in S_{X^*}$.*

In particular, $S_{X^*} \subseteq w^*$-w-pc(B_{X^*}).

Thus the following follows from Lemma 2.2.

Corollary 4.5. *Let X be an M-embedded space and let $x^* \in X^*$. Then $id : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w)$ is continuous at x^* if and only if $x^* \in S_{X^*}$. Therefore, w^*-w-pc$(B_{X^*}) = S_{X^*}$.*

We now consider the discussion in triple dual of Banach spaces. We apply Proposition 4.1 for the situation when a Banach space embedded in its bidual as an M-ideal.

Theorem 4.6 ([4])**.** *Let X be an M-embedded space. Then $id : (B_{X^{***}}, w^*) \longrightarrow (B_{X^{***}}, w)$ is continuous at each $x^* \in S_{X^*}$.*

In particular, $S_{X^*} \subseteq w^*$-w-pc$(B_{X^{***}})$.

Applying Lemma 2.4 and Theorem 4.6, we have the following. We denote w^*-w-pc$(B_{X^{***}}) := \{x^{***} \in B_{X^{***}} : id : (B_{X^{***}}, w^*) \longrightarrow (B_{X^{***}}, w)$ is continuous at $x^{***}\}$.

Corollary 4.7. *Let X be an M-embedded space and let $x^* \in X^*$. Then $id : (B_{X^{***}}, w^*) \longrightarrow (B_{X^{***}}, w)$ is continuous at x^* if and only if $x^* \in S_{X^*}$. Therefore, w^*-w-pc$(B_{X^{***}}) = S_{X^*}$.*

We now consider the double dual case. From Lemma 2.4, it follows that w^*-w-pc$(B_{X^{**}}) \subseteq S_X$. It remains to investigate whether the any point in S_X considering element in X^{**} is a point of continuity for $id : (B_{X^{**}}, w^*) \longrightarrow (B_{X^{**}}, w)$. In this direction, the following example says that this situation is difference from the cases of first and third dual cases. We denote c by the space of all convergent sequence with supremum norm.

Example 4.8 ([4])**.** Let $x \in c \setminus c_0$ with $\|x\| = 1$. Then $id : (B_{c^{**}}, w^*) \longrightarrow (B_{c^{**}}, w)$ is not continuous at x.

Note that c^{**} is isometrically isomorphic to ℓ_∞ which is bidual of c_0. We also note that c_0 is an M-ideal in its bidual ℓ_∞ and ℓ_1 is the unique predual of ℓ_∞ as ℓ_∞ is a von Neumann algebra. It is also known that $\mathcal{K}(H)$ is an M-ideal in its bidual $\mathcal{B}(H)$ and $\mathcal{L}(H)$, space of trace class operators with trace norm, is the unique predual of $B(H)$ as $B(H)$ again a von Neumann algebra. In order to investigate Example 4.8, in non commutative setup, we first show existence of a subspace $Y \subset \mathcal{B}(H)$ that contains at least one non-compact operator and Y^{**} is isometrically isomorphic to $\mathcal{B}(H)$. Fortunately, we have have this type of subspace, in fact, we have something more in Banach space theory set up.

Proposition 4.9 ([4]). *Let X be an M-embedded space. Then, there exists a subspace $Y \subset X^{**}$ such that Y^{**} is isometrically isomorphic to X^{**} and $Y \setminus X \neq \{\phi\}$.*

It is worth to mention that in the proof of Proposition 4.9, we used the fact that if X is an M-embedded space then X^* is strongly predual of X^{**} and X is not strongly predual of X^*, where strongly predual is a stronger than the notion of predual, see [8, Proposition III.2.10] and the discussion before it.

As an application of Lemma 2.4, we have the following.

Proposition 4.10 ([4]). *Let X be a Banach space such that X^* be the unique predual X^{**} and $Y \subset X^{**}$ be a subspace such that Y^{**} is isometrically isomorphic to X^{**}. If $id : (B_{X^{**}}, w^*) \longrightarrow (B_{X^{**}}, w)$ is continuous at x^{**}, then $x^{**} \in Y$.*

Now for every $x \in X$, we can construct a subspace $Y \subset X^{**}$ such that $x \notin Y$ and Y^{**} is isometrically isomorphic to X^{**}; see the last part of the proof of [4, Theorem 6.7] for detailed construction. Hence we have the following as an application of Proposition 4.10.

Theorem 4.11 ([4]). *Let X be a M-embedded space. Then, no $x \in S_X$ is a point of continuity of $id : (B_{X^{**}}, w^*) \longrightarrow (B_{X^{**}}, w)$.*

Hence we have the following. We denote w^*-w-pc$(B_{X^{**}}) = \{x^{**} \in B_{X^{**}} : id : (B_{X^{**}}, w^*) \longrightarrow (B_{X^{**}}, w)$ is continuous at $x^{**}\}$.

Corollary 4.12. *Let X be an M-embedded space and let $x^* \in X^*$. Then w^*-w-pc$(B_{X^{**}}) = \phi$.*

We conclude our discussion with the following summary of this section.

SUMMARY: Let X be an M-embedded space. Then we have the following;

1. w^*-w-pc$(B_{X^*}) = S_{X^*}$.
2. w^*-w-pc$(B_{X^{**}}) = \phi$.
3. w^*-w-pc$(B_{X^{***}}) = S_{X^*}$.

Acknowledgements

This survey article is based on [4] and on the talk entitled "*On weak*-weak points of continuity in the dual unit ball of Banach spaces*", presented at the conference "`Theory of function spaces and related topics`," held on 15–17 December 2025 at the Research Institute for Mathematical Sciences (RIMS), Kyoto University, Japan.

The author would like to thank Tokyo University of Science for supporting this talk. The author also sincerely thanks Prof. Ryotaro Tanaka for his valuable support and suggestions.

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