Fixed point results for some generalized nonexpansive mappings

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1 Introduction

Let E be a real Banach space, let C be a nonempty subset of E. For a mapping $T : C \to E$, we denote by $F(T)$ the set of *fixed points* of T, i.e.,

$$F(T) = \{z \in C : Tz = z\}.$$

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. The fixed point theory for such mappings is rich and varied. It finds many applications in nonlinear functional analysis. The existence of fixed points for nonexpansive mappings in Banach and metric spaces has been investigated since the early 1960s (For example, see [7, 8, 9, 11, 15]).

In recent years, a new direction has been very active essentially after the publication of Ran and Reurings results [20]. They proved an analogue of the classical Banach contraction principle [6] in metric spaces endowed with a partial order. In particular, they show how this extension is useful when dealing with some special matrix equations (see also [14, 18, 28, 29]).

Mann [17] introduced an iteration process for approximation of fixed points of a mapping T in a Hilbert space as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n$$

for all $n \geq 1$, where $\{\alpha_n\}$ is sequences in $[0, 1]$. Later, Reich [16] discussed this iteration process in a uniformly convex Banach space whose norm is Frechet differentiable. Bin Dehaish and Khamsi [13] proved a weak convergence theorem of Mann's type [17] iteration for monotone nonexpansive mappings in Banach spaces endowed with a partial order (see also [1, 23]). Takahashi and Tamura [27]

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proved some weak convergence theorems for a pair of nonexpansive mappings in a Banach space by using the iteration process considered by Das and Debata [12]. Takahashi and Shimoji [26] introduced an iteration process, given by finite nonexpansive mappings, which generalizes Das and Debata's iteration, and then proved weak convergence theorems for finite nonexpansive mappings in a Banach space (see also [5].

The concept of mean nonexpansive mapping was introduced by Zhang [31] in 1975, as a generalization of a nonexpansive mapping in Banach space. He [31] provided the existence of fixed points for mean nonexpansive mappings in Banach space setting under the normal structure assumption. Wu and Zhang [30] investigated some properties of these mappings. Zuo [32] investigated some properties of these mappings and he [32] proved fixed point results for the mappings. So many fixed point results for these mappings are established by many researchers.

In this paper, we provide weak convergence theorems for finite monotone nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order. Further, we provide weak and strong convergence theorems for mean nonexpansive mappings.

2 Preliminaries and notations

We denote by E^* the topological dual space of E. We denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the set of all positive integers and the set of all nonnegative integers, respectively. We also denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and the set of all nonnegative real numbers, respectively. We write $x_n \to x$ (or $\lim\limits_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges weakly to x. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset A of E, co$A$ and $\overline{\text{co}}A$ mean the convex hull of A and the closure of convex hull of A, respectively.

A Banach space E is said to be strictly convex if

$$\frac{\|x+y\|}{2} < 1$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\|\frac{x+y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex.

3 Convergence theorems for finite monotone nonexpansive mappings

In this section, we give convergence theorems for a family of monotone nonexpansive mapping in ordered Banach spaces. We assume that E is a real Banach space with norm $\|\cdot\|$ and endowed with a *partial order* $\preceq$ compatible with the linear structure of E, that is,

$$x \preceq y \text{ implies } x+z \preceq y+z,$$

$$x \preceq y \text{ implies } \lambda x \preceq \lambda y$$

for every $x,y,z \in E$ and $\lambda \geq 0$. We will say that this Banach space $(E,\|\cdot\|,\preceq)$ is an *ordered Banach space*. As usual we adopt the convention $x \succeq y$ if and only if $y \preceq x$. It follows that all *order intervals* $[x,\rightarrow) = \{z \in E : x \preceq z\}$ and $(\leftarrow,y] = \{z \in E : z \in E : z \preceq y\}$ are convex. Moreover, we assume that each order intervals $[x,\rightarrow)$ and $(\leftarrow,y]$ are closed. Recall that an order interval is any of the subsets $[a,\rightarrow) = \{x \in X; a \preceq x\}$ or $(\leftarrow,a] = \{x \in X; x \preceq a\}$. for any $a \in E$. As a direct consequence of this, the subset

$$[a,b] = \{x \in X; a \preceq x \preceq b\} = [a,\rightarrow) \cap (\leftarrow,b]$$

is also closed and convex for each $a,b \in E$.

Let E be a real Banach space with norm $\|\cdot\|$ and endowed with a partial order $\preceq$ compatible with the linear structure of E. We will say that this Banach space $(E,\|\cdot\|,\preceq)$ is an *ordered Banach space*. Let C be a nonempty subset of E. A mapping $T : C \to C$ is called *monotone* if

$$Tx \preceq Ty$$

for each $x,y \in C$ such that $x \preceq y$. For a mapping $T : C \to C$, we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$.

Let C be a nonempty convex subset of a Banach space E. Let $T_1,T_2,\ldots,T_r$ be finite mappings of C into itself and let $\alpha_1,\alpha_2,\ldots,\alpha_r$ be real numbers such that $0 \leq \alpha_i \leq 1$ for every $i = 1,2,\ldots,r$. Then, we define a mapping W of C into itself as follows (see [24, 26]):

$$\begin{aligned} U_1 &= \alpha_1 T_1 + (1-\alpha_1)I, \\ U_2 &= \alpha_2 T_2 U_1 + (1-\alpha_2)I, \\ &\vdots \\ U_{r-1} &= \alpha_{r-1} T_{r-1} U_{r-2} + (1-\alpha_{r-1})I, \\ W = U_r &= \alpha_r T_r U_{r-1} + (1-\alpha_r)I. \end{aligned} \tag{3.1}$$

Such a mapping W is called the W-mapping generated by $T_1,T_2,\ldots,T_r$ and α_1, $\alpha_2,\ldots,\alpha_r$ (see also [5]). Let $\alpha_{n1},\alpha_{n2},\ldots,\alpha_{nr}(n = 1,2,\ldots)$ be real numbers such that $0 \leq \alpha_{ni} \leq 1$ for every $i = 1,2,\ldots,r$. Let $W_n(n = 1,2,\ldots)$ be the W-mappings generated by $T_1,T_2,\ldots,T_r$ and $\alpha_{n1},\alpha_{n2},\ldots,\alpha_{nr}$. Now consider the following iteration process:

$$x_1 \in C, \quad x_{n+1} = W_n x_n \quad \text{for every} \quad n = 1,2,\ldots.$$

The following lemma was proved in [5].

Lemma 3.1 ([5]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots, T_r$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i) \neq \emptyset$ and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i = 1, 2, \ldots, r-1$ and $0 < \alpha_r \leq 1$. Let W be the W-mapping of C into itself generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$. Then, $F(W) = \bigcap_{i=1}^r F(T_i)$.*

We study approximate fixed point sequences and monotone sequences. Let C be a nonempty subset of an ordered Banach space E and let T be a mapping of C into itself. A sequence $\{x_n\}$ in C is said to be an *approximate fixed point sequence* of a mapping T of C into itself if

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

(see also [16, 25]). Let $T_1, T_2, \ldots, T_r$ be mappings of C into itself. A sequence $\{x_n\}$ in C is said to be an *approximate common fixed point sequence* of mappings $T_1, T_2, \ldots, T_r$ of C if for every $k = 1, 2, \ldots, r$,

$$\lim_{n\to\infty} \|x_n - T_k x_n\| = 0.$$

A sequence $\{x_n\}$ in E is said to be *monotone* if

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots$$

(see also [13]).

Lemma 3.2 ([2]). *Let E be an ordered uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T_1, T_2, \ldots, T_r$ be finite monotone nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i)$ is nonempty. Assume that $T_k x \succeq x (k = 1, 2, \ldots, r)$ for every $x \in C$. Let $\alpha_{n,1}, \ldots, \alpha_{n,r}$ $(n = 1, 2, \ldots)$ be real numbers such that $0 < \alpha_{n,i} \leq 1$ $(n = 1, 2, \ldots)$ for every $i = 1, 2, ..., r$. Let $W_n (n = 1, 2, \ldots)$ be W-mappings generated by $T_1, T_2, \ldots, T_r$ and $\alpha_{n,1}, \ldots, \alpha_{n,r}$. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = W_n x_n$$

for every $n = 1, 2, \ldots$. Then, the sequence $\{x_n\}$ is monotone.

Lemma 3.3 ([2]). *Let E be an ordered uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T_1, T_2, \ldots, T_r$ be finite monotone nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i)$ is nonempty. Fix $u \in C$ such that $T_k u \succeq u (k = 1, 2, \ldots, r)$. Let $\alpha_1, \ldots, \alpha_r$ be real numbers such that $0 < \alpha_i \leq 1$ for every $i = 1, 2, ..., r$. Let W be a W-mapping generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \ldots, \alpha_r$. Suppose $x_1 = u \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = W^n x_1$$

for every $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ is monotone.

Now, we give weak convergence theorems for finite noncommutative monotone nonexpansive mappings.

Theorem 3.4 ([2]). *Let E be an ordered uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T_1, T_2, \ldots, T_r$ be finite monotone nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i)$ is nonempty. Assume that $T_k x \succeq x$ $(k = 1, 2, \ldots, r)$ for every $x \in C$. Let a, b be real numbers with $0 < a \leq b < 1$. Let $\alpha_{n,1}, \ldots, \alpha_{n,r} (n = 1, 2, \ldots)$ be real numbers such that $a \leq \alpha_{n,i} \leq b$ for every $i = 1, 2, ..., r$. Let $W_n (n = 1, 2, \ldots)$ be W-mappings generated by $T_1, T_2, \ldots, T_r$ and $\alpha_{n,1}, \ldots, \alpha_{n,r}$. Then, the sequence $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, \ldots, T_r$.*

By Theorem 3.4, we get the following theorem.

Theorem 3.5 ([2]). *Let E be an ordered uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T_1, T_2, \ldots, T_r$ be finite monotone nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i)$ is nonempty. Fix $x_1 = u \in C$ such that $T_k x_1 \succeq x_1$ $(k = 1, 2, \ldots, r)$. Let a, b be real numbers with $0 < a \leq b < 1$. Let $\alpha_{n,1}, \ldots, \alpha_{n,r} (n = 1, 2, \ldots)$ be real numbers such that $a \leq \alpha_{n,i} \leq b$ for every $i = 1, 2, ..., r$. Let W be a W-mapping generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \ldots, \alpha_r$. Suppose $x_1 = u \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = W^n x_1$$

for every $n = 1, 2, \ldots$. Then, the sequence $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, \ldots, T_r$.

By Theorem 3.4, we get some convergence theorems (see also [23]).

Theorem 3.6 ([2]). *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let T and S be monotone nonexpansive mappings of C into itself such that $F(T) \cap F(S) \neq \emptyset$. Assume that $x \preceq Tx$ and $x \preceq Sx$ for each $x \in C$. Let a, b be real numbers with $0 < a \leq b < 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be a sequence of real numbers such that $a < \alpha_n < b$ and $a < \beta_n < b$ for each $n = 1, 2, \ldots$, respectively. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n S[\beta_n T x_n + (1 - \beta_n) x_n] + (1 - \alpha_n) x_n$$

for every $n = 1, 2, \ldots$. Then, $\{x_n\}$ converges weakly to a common fixed point of T and S.

Theorem 3.7 ([2]). *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let T be a monotone nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Assume that $x \preceq Tx$. Let a, b be real numbers with $0 < a \leq b < 1$. Let $\{\alpha_n\}$ be the sequence of real numbers such that $a < \alpha_n < b$ for each $n \in \mathbb{N}$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n T x_n + (1 - \alpha_n) x_n$$

for every $n = 1, 2, \ldots$. Then, $\{x_n\}$ converges weakly to a fixed point of T.

Now, we study the more general class of operators, namely monotone Reich–Suzuki-type mappings ([19]). Let C be a nonempty subset of a Banach space E. A mapping $T : C \to C$ is called Reich–Suzuki-type nonexpansive if there exists $k \in [0, 1)$ such that,

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\|$$

implies

$$\|Tx - Ty\| \leq (1 - 2k)\|x - y\| + k\|x - Tx\| + k\|y - Ty\|$$

for each $x, y \in C$ (see [19]).

Theorem 3.8 ([3]). *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E and let T and S be monotone Reich–Suzuki-type nonexpansive mappings of C into itself such that $ST = TS$ and $F(T) \cap F(S) \neq \emptyset$. Assume that $x \preceq Tx$ and $x \preceq Sx$ for $x \in C$. Then, the sequence $\frac{1}{(n+1)^2}\sum_{i=0}^{n}\sum_{j=0}^{n} T^i S^j x$ converges weakly to a point of $F(T) \cap F(S)$.*

4 Convergence theorems for mean nonexpansive mappings

In this section, we give weak and strong convergence theorems for mean nonexpansive mappings.

Let C be a nonempty subset of a Banach space E. A mapping T of C into itselft is called mean nonexpansive if for $x, y \in C$ there are non-negative real numbers a, b such that $a + b \leq 1$, we have $\|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\|$.

Theorem 4.1 ([4]). *Let E be a uniformly convex Banach space which satisdies Opial's condition and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_r$ be finite mean nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i)$ is nonempty. Let c, d be real numbers with $0 < c \leq d < 1$. Let $\alpha_1, \ldots, \alpha_r$ be real numbers such that $c \leq \alpha_i \leq d$ for every $i = 1, 2, ..., r$. Let W be a W-mappings generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \ldots, \alpha_r$. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = W^n x_n$$

for every $n = 1, 2, \ldots$. Then, x_n converges weakly to an element of $\bigcap_{i=1}^{r} F(T_i)$

Theorem 4.2 ([4]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T_1, T_2, \ldots, T_r$ be finite mean nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i)$ is nonempty. Let $\alpha_1, \ldots, \alpha_r$ be real numbers such that $0 \leq \alpha_i \leq 1$ for every $i = 1, 2, ..., r$. Let W be a W-mappings generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \ldots, \alpha_r$. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = W^n x_n$$

for every $n = 1, 2, \ldots$. Then, x_n converges strongly to an element of $\bigcap_{i=1}^{r} F(T_i)$ if and only if

$$\lim_{n\to\infty} d(x_n, \bigcap_{i=1}^{r} F(T_i)) = 0,$$

where

Theorem 4.3 ([4]). *Let C be a nonempty compact convex subset of a uniformly convex Banach space E. Let $T_1, T_2, \ldots, T_r$ be finite mean nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i)$ is nonempty. Let c, d be real numbers with $0 < c \leq d < 1$. Let $\alpha_1, \ldots, \alpha_r$ be real numbers such that $c \leq \alpha_i \leq d$ for every $i = 1, 2, ..., r$. Let W be a W-mappings generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \ldots, \alpha_r$. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = W^n x_n$$

for every $n = 1, 2, \ldots$. Then, x_n converges strongly to an element of $\bigcap_{i=1}^{r} F(T_i)$.

By Theorem 4.3, we get the followsing theorems.

Theorem 4.4 ([4]). *Let C be a nonempty compact convex subset of a uniformly convex Banach space E. Let T be a mean nonexpansive mapping of C into itself. Let c,d be real numbers with $0 < c \le d < 1$. Let α and β be real numbers such that $c < \alpha < d$ and $c < \beta < d$. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha T[\beta T x_n + (1-\beta)x_n] + (1-\alpha)x_n$$

for every $n = 1,2,\ldots$. Then, $\{x_n\}$ converges strong to a fixed point of T.

Theorem 4.5 ([4]). *Let C be a nonempty compact convex subset of a uniformly convex Banach space E. Let T be a mean nonexpansive mapping of C into itself. Let c,d be real numbers with $0 < c \le d < 1$. Let α be a real number $c < \alpha < d$. . Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha T x_n + (1-\alpha)x_n$$

for every $n = 1,2,\ldots$. Then, $\{x_n\}$ converges strong to a fixed point of T.

A mapping T of C into itself is said to be **compcat** if T is continuous and maps bounded sets into relatively compact sets. A mapping T of C into itself is said to be **demicompcat at** z if for any bounded sequence $\{y_n\}$ in C such that $y_n - Ty_n \to z$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $y \in C$ such that $y_{n_k} \to z$ and $y - Ty = z$. A mapping T of C into itself T is said to be **demicompcat on** C if T is demicompcat for each $y \in C$. If T is compcat on C, then T is demicompcat on C. We obtain a strong convergence theorem.

Theorem 4.6 ([4]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T_1, T_2, \ldots, T_r$ be finite mean nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i)$ is nonempty. Let c,d be real numbers with $0 < c \le d < 1$. Let $\alpha_1, \ldots, \alpha_r$ be real numbers such that $0 < c \le \alpha_i \le d < 1$ for every $i = 1,2,\ldots,r$. Let W be a W-mappings generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \ldots, \alpha_r$. Assume that W is demicompact at 0. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = W^n x_n$$

for every $n = 1,2,\ldots$. Then, x_n converges strongly to an element of $\bigcap_{i=1}^r F(T_i)$

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