SEMIMETRIC SPACE における不動点定理

九州工業大学　　鈴木 智成　(Tomonari SUZUKI)

1. 序

本稿の議論の基礎となる semimetric space の定義をまず述べる.

定義 1. Let X be a nonempty set and let d be a function from $X \times X$ into $[0, \infty)$. Then (X, d) is said to be a *semimetric space* if the following hold:

(D1) $d(x, x) = 0$.
(D2) If $d(x, y) = 0$, then $x = y$.
(D3) $d(x, y) = d(y, x)$.

条件 (D1)〜(D3) に, 次の条件 (D4) を加えたものが距離空間の定義である. 条件 (D4) は三角不等式と呼ばれる. したがって, semimetric space は距離空間の公理から三角不等式を取り除いた枠組みに相当する.

(D4) $d(x, z) \le d(x, y) + d(y, z)$.

次の不動点定理は「Banach の縮小原理 (the Banach contraction principle)」としてよく知られている. この定理は, 距離空間論における最も基本的かつ有用な定理の一つである. 後に紹介する補助定理 9 および定理 10 の証明と対比するため, ここでその証明も与えておく.

定理 2 ([1, 2])**.** Let (X, d) be a complete metric space and let T be a *contraction* on X, that is, there exists $r \in [0, 1)$ satisfying

$$d(Tx, Ty) \le r\, d(x, y) \tag{1}$$

for all $x, y \in X$. Then T has a unique fixed point z. Moreover, the sequence $\{T^n x\}$ converges to z for all $x \in X$.

MSC (2020). 54H25.
キーワード. The Banach contraction principle, semimetric space.
住所. 〒 804–8550 北九州市戸畑区 九州工業大学工学研究院.
電子メール. `suzuki-t@mns.kyutech.ac.jp`.

証明. Fix $u \in X$. Then we have

$$\begin{aligned} d(T^n u, T^{n+1}u) &\le r\, d(T^{n-1}u, T^n u) \le r^2\, d(T^{n-2}u, T^{n-1}u) \\ &\le \cdots \le r^n\, d(u, Tu) \end{aligned}$$

for any $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned} d(T^n u, T^m u) &\le d(T^n u, T^{n+1}u) + d(T^{n+1}u, T^m u) \\ &\le d(T^n u, T^{n+1}u) + d(T^{n+1}u, T^{n+2}u) + d(T^{n+2}u, T^m u) \\ &\le \cdots \le \sum_{k=n}^{m-1} d(T^k u, T^{k+1}u) \le \sum_{k=n}^{m-1} r^k\, d(u, Tu) \\ &\le \sum_{k=n}^{\infty} r^k\, d(u, Tu) = \frac{r^n}{1-r}\, d(u, Tu), \end{aligned}$$

which implies that $\{T^n u\}$ is Cauchy. Since X is complete, $\{T^n u\}$ converges to some z. Since

$$\begin{aligned} d(z, Tz) &\le \limsup_{n\to\infty} \bigl(d(z, T^n u) + d(T^n u, Tz)\bigr) \\ &\le \lim_{n\to\infty} \bigl(d(z, T^n u) + r\, d(T^{n-1}u, z)\bigr) = 0 \end{aligned}$$

holds, z is a fixed point of T. In order to show the uniqueness of a fixed point, we let w be a fixed point of T. Then we have

$$d(z, w) = d(Tz, Tw) \le r\, d(z, w).$$

Since $r < 1$ holds, we obtain $z = w$. Hence z is a unique fixed point of T. This completes the proof. $\square$

三角不等式 (D4) が有効に用いられており, 簡潔な証明になっていることが分かる.

Banach の縮小原理（定理 2）には多くの拡張が知られている. その中には, 距離空間を semimetric space にまで一般化したものもある. 以下に示す定理は, そのような拡張定理の代表例の一つである.

定理 3 (Jachymski, Matkowski and Świątkowski [3]). Let (X, d) be a Hausdorff complete semimetric space. Assume the following:

(D5) There exist $\varepsilon > 0$ and $\delta > 0$ such that $d(x, y) < \delta$ and $d(y, z) < \delta$ imply $d(x, z) < \varepsilon$.

Let T be a *contraction* on X. Then T has a unique fixed point z. Moreover, the sequence $\{T^n x\}$ converges to z for all $x \in X$.

semimetric space では三角不等式 (D4) を仮定していないため, 単に semimetric space が完備であるだけでは, 不動点の存在を保証できない. 実際, それを示す反例（例 4）がある. したがって, (D5) のような付加的条件を課す必要がある.

例 4 ([3])**.** Put $X = \mathbb{N}$ and let $r \in (0,1)$. Define a function d from $X \times X$ into $[0,\infty)$ by

$$d(x,y) = r^{\min\{x,y\}}\,|x-y|.$$

Define a mapping T on X by

$$Tx = x+1.$$

Then the following hold:

(i) (X,d) is a complete semimetric space.

(ii) T is a contraction, however, it does not have a fixed point.

semimetric space では三角不等式を仮定しないため, 開球全体が位相の基底をなすとは限らない. したがって, 通常の意味での距離位相を導入することは一般にはできない. 定理 3 では Hausdorff 性が仮定され, 結論部分でも収束について述べられているが, そこで用いられている Hausdorff 性および収束の概念は, semimetric space に適合するように定義されたものである. 以下にこれらの定義を示す.

定義 5. Let (X,d) be a semimetric space. Let $\{x_n\}$ be a sequence in X and let $x, y \in X$.

- $\{x_n\}$ is said to *converge* to x if $\lim_n d(x_n, x) = 0$.
- $\{x_n\}$ is said to be *Cauchy* if $\lim_n \sup\{d(x_n, x_m) : m > n\} = 0$.
- X is said to be *complete* if every Cauchy sequence converges in X.
- X is said to be *Hausdorff* if $\lim_n d(x_n, x) = 0$ and $\lim_n d(x_n, y) = 0$ imply $x = y$.

2. 主結果

条件 (D5) は三角不等式 (D4) に比べて, 非常に弱い条件である. そのため, 筆者は定理 3 を初めて目にしたとき, 大変驚いたことを今でも覚えている. 一見しただけでは, 定理 3 が正しいとはとても思えなかったからである.

一つ気になったのは completeness である. というのも completeness という条件の背後に, 三角不等式の影が見えたからである. そして, この点に着目することにより, 新たな結果を得ることができた.

定義 6 ([7])**.** Let (X, d) be a semimetric space. Then X is said to be *semicomplete* if every Cauchy sequence has a convergent subsequence.

以下は, semicomplete であって complete でない semimetric space の例である.

例 7 ([7])**.** Put $X = [-1, 1] \cup \{\alpha\}$. Define a function d from $X \times X$ into $[0, \infty)$ by

$$\begin{aligned}
&d(x, y) = |x - y| && \text{if } x, y \in X \setminus \{0, \alpha\}, x < y \\
&d(0, y) = y && \text{if } y > 0 \\
&d(0, y) = 1 && \text{if } y \in X \setminus [0, 1] \\
&d(\alpha, y) = |y| && \text{if } y < 0 \\
&d(\alpha, y) = 1 && \text{if } y > 0 \\
&d(x, y) = d(y, x) && \text{if } d(y, x) \text{ is defined above} \\
&d(x, x) = 0.
\end{aligned}$$

Then the following hold:

(i) (X, d) is a Hausdorff semimetric space.
(ii) X is semicomplete but not complete.

次の補助定理はほぼ自明であるが, 念のため証明を与える.

補助定理 8 ([7])**.** Let X be a nonempty set and let d be a function from $X \times X$ into $[0, \infty)$. Let T be a mapping on X and define a mapping S on X by $S = T^\kappa$, where κ is a positive integer. Then the following hold:

(i) If S has a unique fixed point z, then z is also the unique fixed point of T.
(ii) If $\lim_n d(S^n x, z) = 0$ holds for any $x \in X$, then $\lim_n d(T^n x, z) = 0$ holds for any $x \in X$.

証明. We first show (i). Since $S \circ Tz = T \circ Sz = Tz$ holds, Tz is a fixed point of S. Since the fixed point z of S is unique, we have $Tz = z$. Let $w \in X$ be a fixed point of T. Then we have $Sw = T^{\kappa-1}w = \cdots = Tw = w$. Since the fixed point z of S is unique, we have $z = w$.

We next show (ii). For any $j \in \{0, \cdots, \kappa - 1\}$, we have

$$\lim_{n\to\infty} d(T^{\kappa n+j}x, z) = \lim_{n\to\infty} d(S^n \circ T^j x, z) = 0.$$

So we obtain the desired result. □

以下の補助定理は, 主結果を導く上で本質的な役割を果たす.

補助定理 9 ([7])**.** Let (X,d) be a Hausdorff, semicomplete semimetric space. Let T be a contraction on X. Assume that there exist $M>0$, $\kappa\in\mathbb{N}$ and $u\in X$ satisfying the following:

(i) $d(u,T^n u)<M$.

(ii) If a subsequence of $\{T^n u\}$ converges to some $x\in X$, then $d(x,T^\kappa x)<M$ holds.

Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x\in X$.

証明. Since

$$d(T^m u,T^{m+n}u)\le r^m\,d(u,T^n u)\le r^m\,M$$

holds for any $m,n\in\mathbb{N}$, $\{T^n u\}$ is Cauchy. Since X is semicomplete, there exist $z_0\in X$ and a subsequence $\{f(n)\}$ of the sequence $\{n\}$ in $\mathbb{N}$ such that $\lim_n d(T^{f(n)}u,z_0)=0$ holds. Since $\{T^{f(n)-1}u\}$ is also Cauchy, there exist $z_1\in X$ and a subsequence $\{g_1(n)\}$ of $\{n\}$ in $\mathbb{N}$ such that $\lim_n d(T^{f\circ g_1(n)-1}u,z_1)=0$ holds. Then we have

$$\lim_{n\to\infty} d(T^{f\circ g_1(n)}u,Tz_1)\le \lim_{n\to\infty} r\,d(T^{f\circ g_1(n)-1}u,z_1)=0.$$

Note that $\{T^{f\circ g_1(n)}u\}$ is a subsequence of $\{T^{f(n)}u\}$. Since X is Hausdorff, we have $Tz_1=z_0$. Since $\{T^{f\circ g_1(n)-2}u\}$ is also Cauchy, there exist $z_2\in X$ and a subsequence $\{g_2(n)\}$ of $\{n\}$ in $\mathbb{N}$ such that $\lim_n d(T^{f\circ g_1\circ g_2(n)-2}u,z_2)=0$ holds. As above, we can prove $Tz_2=z_1$. Continuing this argument, we can define a sequence $\{z_m\}_{m=0}^\infty$ in X satisfying the following:

- $Tz_m=z_{m-1}$ holds for $m\in\mathbb{N}$.
- For any $m\in\mathbb{N}\cup\{0\}$, there exists a subsequence of $\{T^n u\}$ converging to z_m.

From the assumption, we have $d(z_m,T^\kappa z_m)<M$ for $m\in\mathbb{N}$. Hence, putting $z=z_0$, we have

$$d(z,T^\kappa z)=d(T^m z_m,T^{m+\kappa}z_m)\le r^m\,d(z_m,T^\kappa z_m)\le r^m\,M$$

for any $m\in\mathbb{N}$. Since $m\in\mathbb{N}$ is arbitrary, we obtain $d(z,T^\kappa z)=0$. Thus, z is a fixed point of T^κ. We let $w\in X$ be a fixed point of T^κ. Then we have

$$d(z,w)=d(T^\kappa z,T^\kappa w)\le r^\kappa\,d(z,w).$$

Since $r^\kappa < 1$ holds, we obtain $d(z,w) = 0$, thus, $z = w$ holds. So we have shown that z is a unique fixed point of T^κ. By Lemma 8 (i), z is also a unique fixed point of T. Let $x \in X$ be arbitrary. Then we have

$$\lim_{n\to\infty} d(T^n x, z) = \lim_{n\to\infty} d(T^n x, T^n z) \le \lim_{n\to\infty} r^n\, d(x,z) = 0.$$

Thus, $\{T^n x\}$ converges to z. □

次の定理が本稿の主結果である.

定理 10 ([7])**.** Let (X,d) be a Hausdorff, semicomplete semimetric space satisfying (D5). Let T be a contraction on X. Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.

証明. Let ε and δ be as in (D5). By replacing δ with $\min\{\delta,\varepsilon\}$ if necessary, we may assume $\delta \le \varepsilon$. Let $r \in [0,1)$ satisfy (1) for all $x, y \in X$. Let $\kappa \in \mathbb{N}$ satisfy $\varepsilon\, r^\kappa < \delta$. Define a function f from X into $[0,\infty)$ by

$$f(x) = \max\left\{d(T^i x, T^j x) : i, j \in \{0,\cdots,\kappa\}\right\},$$

where T^0 is the identity mapping on X. It is obvious that $f(Tx) \le r\, f(x)$ holds. Hence $\lim_n f(T^n x) = 0$ holds.

Fix $u \in X$ and choose $\mu \in \mathbb{N}$ satisfying $f(T^\mu u) < \delta$. We will show

$$d(T^\mu u, T^{\mu+n} u) < \varepsilon \tag{2}$$

for any $n \in \mathbb{N}$. It is obvious that (2) holds in the case where $n \in \{1,\cdots,\kappa\}$. We fix $n \in \mathbb{N}$ with $n > \kappa$ and assume that (2) holds for positive integers less than n. Then we have

$$d(T^{\mu+\kappa} u, T^{\mu+n} u) \le r^\kappa\, d(T^\mu u, T^{\mu+n-\kappa} u) \le r^\kappa\, \varepsilon < \delta.$$

We also have

$$d(T^\mu u, T^{\mu+\kappa} u) \le f(T^\mu u) < \delta.$$

By (D5), we obtain (2). So (i) of Lemma 9 holds with $u := T^\mu u$ and $M := \varepsilon$. For $i, j \in \mathbb{N}$ with $\mu + \kappa \le i \le j$, we have by (2)

$$d(T^i u, T^j u) \le r^{i-\mu}\, d(T^\mu u, T^{\mu+j-i} u) \le r^\kappa\, \varepsilon < \delta. \tag{3}$$

In order to prove (ii) of Lemma 9, we assume that $\lim_n d(T^{g(n)} u, z) = 0$ holds for some $z \in X$ and some subsequence $\{g(n)\}$ of $\{n\}$ in $\mathbb{N}$. Choose $\lambda \in \mathbb{N}$ satisfying

$$d(T^{g(\lambda)} u, z) < \delta \quad \text{and} \quad g(\lambda) \ge \mu + 2\,\kappa. \tag{4}$$

By (D5), (3) and (4), we have $d(T^j u, z) < \varepsilon$ for $j \in \mathbb{N}$ with $\mu + \kappa \le j$. Hence

$$d(T^j u, T^\kappa z) \le r^\kappa\, d(T^{j-\kappa} u, z) \le r^\kappa\, \varepsilon < \delta \tag{5}$$

holds for $j \in \mathbb{N}$ with $\mu + 2\,\kappa \le j$. By (D5), (4) and (5), we have $d(z, T^\kappa z) < \varepsilon$. Therefore we have shown (ii) of Lemma 9. □

参考文献

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., 3 (1922), 133–181.

[2] R. Caccioppoli, *Un teorema generale sull'esistenza di elementi uniti in una transformazione funzionale*, Rend. Accad. Naz. Lincei, 11 (1930), 794–799.

[3] J. Jachymski, J. Matkowski and T. Świątkowski, *Nonlinear contractions on semimetric spaces*, J. Appl. Anal., 1 (1995), 125–134. MR1395268

[4] T. Suzuki, *Caristi's fixed point theorem in semimetric spaces*, J. Fixed Point Theory Appl., 20, Art 30, 15 pp. MR3761386

[5] ______, *Characterization of $\sum$-semicompleteness via Caristi's fixed point theorem in semimetric spaces*, J. Funct. Spaces, 2018, Art. ID 9435470, 7 pp. MR3773458

[6] ______, *Fixed point theorem for a kind of Ćirić type contractions in complete metric spaces*, Adv. Theory Nonlinear Anal., 2 (2018), 33–41.

[7] ______, *Fixed point theorems for contractions in semicomplete semimetric spaces*, Carpathian J. Math., 34 (2018), 269–275. MR3880370

[8] ______, *Edelstein's fixed point theorem in semimetric spaces*, J. Nonlinear Var. Anal., 2 (2018), 165–175.

[9] ______, *Generalizations of Edelstein's fixed point theorem in compact metric spaces*, Fixed Point Theory, 20 (2019), 703–714.

[10] ______, *Jachymski-Matkowski-Świątkowski's fixed point theorem*, Appl. Anal. Discrete Math., 13 (2019), 632–642. MR4033791