Refined Non-Hermite Robertson-Schrödinger Uncertainty Relation

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Abstract

We show that non-hermite Robertson-Schrödinger uncertainty relation can be refined by using the eigenvalues of ρ.

1 Introduction

In quantum mechanical system, the expectation value of an observable A in a quantum state ρ is expressed by $E_\rho(A) = Tr[\rho A]$. It is natural that the variance for a quantum state ρ and an observable A is defined by $V_\rho(A) = Tr[\rho A^2] - (Tr[\rho A])^2 = Tr[\rho A_0^2]$, where $A_0 = A - (Tr[\rho A])I$. It is well known that Robertson uncertainty relation is represented by the commutator $[A, B] = AB - BA$ in the following inequality.

$$\frac{1}{4}|Tr[\rho[A,B]]|^2 \leq V_\rho(A)V_\rho(B).$$

It is well known that Schrödinger uncertainty relation is stronger than Robertson uncertainty relation in the following.

$$\frac{1}{4}|Tr[\rho[A,B]]|^2 + (Re\{Tr[\rho A_0 B_0]\})^2 \leq V_\rho(A)V_\rho(B).$$

In this paper we obtain some kinds of non-hermite extensions of Robertson-Schrödinger uncertainty relation.

2 Non-Hermite Robertson-Schrödinger Uncertainty Relation

Definition 2.1 *Let $M_n(\mathbb{C}), M_{n,1}(\mathbb{C})$ and $M_{n,sa}(\mathbb{C})$ be the set of all n dimensioanl complex matrices, the set of all n dimensional density matrices an the set of all* *n dimensional hermite matrices. For $A, B \in M_n(\mathbb{C})$, we define commutator by $[A, B] = AB - BA$. Then we define the following expectation $E_\rho(A)$ and variance $Var_\rho(A)$.*

$$E_\rho(A) = Tr[\rho A],$$
$$Var_\rho(A) = \frac{1}{2}Tr[\rho(A_0A_0^* + A_0^*A_0)],$$

Let $\rho = \sum_{i=1}^{n} \lambda_i|\phi_i\rangle\langle\phi_i|$ be the spectral decomposition. Then it is easy to show

$$Var_\rho(A) = \frac{1}{2}\sum_{i,j}(\lambda_i + \lambda_j)|a_{ij}|^2,$$

where $a_{ij} = \langle\phi_i|A_0|\phi_i\rangle$. The non-hermite Robertson Uncertainty relation is given as follows:

Theorem 2.1 ([1],[2]) *Let $\rho \in M_{n,1}(\mathbb{C})$ and $A, B \in M_n(\mathbb{C})$. Then the following uncertainty relation holds.*

$$\frac{1}{4}|Tr[\rho[A, B]]|^2 \leq Var_\rho(A)Var_\rho(B),$$

where $[A, B] = AB - BA$ is commutator.

Proof of Theorem 2.1 Since $Tr[\rho[A, B]] = \sum_{i,j}(\lambda_i - \lambda_j)a_{ij}b_{ji}$, we have

$$\begin{aligned}
|Tr[\rho[A, B]]|^2 &\leq (\sum_{i,j}|\lambda_i - \lambda_j||a_{ij}||b_{ji}|)^2 \\
&\leq (\sum_{i,j}(\lambda_i + \lambda_j)|a_{ij}||b_{ji}|)^2 \\
&\leq \sum_{i,j}(\lambda_i + \lambda_j)|a_{ij}|^2\sum_{i,j}(\lambda_i + \lambda_j)|b_{ji}|^2 \\
&= 4Var_\rho(A)Var_\rho(B).
\end{aligned}$$

□

The stronger non-hermite Schrödinger uncertainty relation is given as follows.

Theorem 2.2 *Let $\rho \in M_{n,1}(\mathbb{C})$ be a density matrix and $A, B \in M_n(\mathbb{C})$. Then the following uncertainty relation holds.*

$$\frac{1}{4}|Tr[\rho[A, B]]|^2 + \frac{1}{4}|Tr[\rho(A_0^*B_0 + B_0A_0^*)]|^2 \leq Var_\rho(A)Var_\rho(B).$$

The result is given by $c = 1$ in Lemma 3.1 stated in next section.

3 Refined Non-Hermite Robertson-Schrödinger Uncertainty Relation I

The refined non-hermite Robertson uncertainty relation is given as follows:

Theorem 3.1 *Let $\rho = \sum_{i=1}^{n} \lambda_i |\phi_i\rangle\langle\phi_i|$ be the spectral decomposition of $\rho \in M_{n,1}(\mathbb{C})$ with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ satisfying $\lambda_1 \neq \lambda_n$ and $A, B \in M_n(\mathbb{C})$. Then the following refined non-hermite uncertainty relation holds.*

$$\frac{(\lambda_n + \lambda_1)^2}{4(\lambda_n - \lambda_1)^2} |Tr[\rho[A,B]]|^2 \leq Var_\rho(A) Var_\rho(B). \tag{1}$$

Proof of Theorem 3.1 Let $a_{ij} = \langle\phi_i|A_0|\phi_j\rangle$ and $b_{ji} = \langle|\phi_j|B_0|\phi_i\rangle$.

$$\begin{aligned}
\frac{1}{(\lambda_n - \lambda_1)^2} |Tr[\rho[A,B]]|^2 &\leq \left(\sum_{i,j} \left| \frac{\lambda_i - \lambda_j}{\lambda_n - \lambda_1} \right| |a_{ij}||b_{ji}| \right)^2 \\
\leq & \left(\sum_{i,j} \left| \frac{\lambda_i - \lambda_j}{\lambda_n - \lambda_1} \right| |a_{ij}|^2 \right) \left(\sum_{i,j} \left| \frac{\lambda_i - \lambda_j}{\lambda_n - \lambda_1} \right| |b_{ji}|^2 \right) \\
\leq & \left(\sum_{i,j} \left| \frac{\lambda_i + \lambda_j}{\lambda_n + \lambda_1} \right| |a_{ij}|^2 \right) \left(\sum_{i,j} \left| \frac{\lambda_i + \lambda_j}{\lambda_n + \lambda_1} \right| |b_{ji}|^2 \right).
\end{aligned} \tag{2}$$

(2) is obtained by the reason why $\left(\frac{t-1}{t+1}\right)^2$ is monotone increasing for $t \geq 1$ and monotone decreasing for $0 < t < 1$. Then

$$\begin{aligned}
& \frac{(\lambda_n + \lambda_1)^2}{(\lambda_n - \lambda_1)^2} |Tr[\rho[A,B]]|^2 \\
\leq \ & \left(\sum_{i,j} (\lambda_i + \lambda_j)|a_{ij}|^2\right)\left(\sum_{i,j} (\lambda_i + \lambda_j)|b_{ji}|^2\right) = 4Var_\rho(A)Var_\rho(B).
\end{aligned}$$

$\square$

Remark 3.1 *Let $\rho = \frac{1}{n} I$ be the maximally mixed state. Then by the same method of [12]*

$$|Tr[A_0 B_0]|^2 \leq Tr[A_0^* A_0] Tr[B_0^* B_0].$$

Let ρ be non-faithful state. That is $\lambda_1 = 0$. Then by the same method of [12]

$$\frac{1}{4} |Tr[\rho[A,B]]|^2 \leq Var_\rho(A) Var_\rho(B).$$

By putting $A, B \in M_{n,sa}(\mathbb{C})$ in (1), we have refined Robertson uncertainty relation.

Corollary 3.1

$$\frac{(\lambda_n - \lambda_1)^2}{4(\lambda_n - \lambda_1)^2} |Tr[A, B]]|^2 \leq V_\rho(A) V_\rho(B). \tag{3}$$

The following refined nonhermite Schrödinger uncertainty relation is obtained.

Theorem 3.2 *Let $\rho = \sum_{i=1}^{n} \lambda_i |\phi_i\rangle\langle\phi_i|$ be the spectral decomposition of $\rho \in M_{n,1}(\mathbb{C})$ with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ satisfying $\lambda_1 \neq \lambda_n$ and $A, B \in M_n(\mathbb{C})$. Then the following refined uncertainty relation holds.*

$$\frac{(\lambda_n + \lambda_1)^2}{4(\lambda_n - \lambda_1)^2} |Tr[\rho[A, B]]|^2 + \frac{1}{4} |Tr[\rho(A_0^* B_0 + B_0 A_0^*)]|^2 \leq Var_\rho(A) Var_\rho(B). \tag{4}$$

The proof of the uncertainty relation (4) is base on the following lemma.

Lemma 3.1 *Let $\|X\|_\rho^2 = Tr[\rho X^* X]$ for any $X \in M_n(\mathbb{C})$. For $c > 0$, if the relation*

$$(\|X\|_\rho^2 + \|X^*\|_\rho^2)(\|Y\|_\rho^2 + \|Y^*\|_\rho^2) \geq c Tr[\rho[X, Y]]|^2 \tag{5}$$

holds for any $X, Y \in M_n(\mathbb{C})$, then the stronger relation

$$(\|X\|_\rho^2 + \|X^*\|_\rho^2)(\|Y\|_\rho^2 + \|Y^*\|_\rho^2) - |Tr[\rho(X^* Y + Y X^*)]|^2 \geq c |Tr[\rho[X, Y]])^2 \tag{6}$$

also holds.

Proof of Lemma 3.1 The schwarz inequality for the inner prouct $\langle X|Y\rangle_\rho = Tr[\rho X^* Y]$ implies that if $\|Y\|_\rho = \|Y^*\|_\rho = 0$, then $Tr[\rho(X^* Y + Y X^*)] = 0$. Therefore, in the case of $\|Y\|_\rho = \|Y^*\|_\rho = 0$, the statement holds trivially. We henceforth assume $Tr[\rho(Y^* Y + Y Y^*)] \neq 0$. In view of the identity $[X + tY, Y] = [X, Y]$ for any $t \in \mathbb{C}$, the assumption (5) implies

$$(\|X + tY\|_\rho^2 + \|X^* + \bar{t} Y^*\|_\rho^2)(\|Y\|_\rho^2 + \|Y^*\|_\rho^2) \geq c |Tr[\rho[X, Y]]|^2. \tag{7}$$

Letting $t = -\frac{\overline{Tr[\rho(X^* Y + Y X^*)]}}{\|Y\|_\rho^2 + \|Y^*\|_\rho^2}$, we obtain

$$\begin{aligned}
& \|X + tY\|_\rho^2 + \|X^* + \bar{t} Y^*\|_\rho^2 \\
= \; & \|X^*\|_\rho^2 + \|X\|_\rho^2 + |t|^2 (\|Y^*\|_\rho^2 + \|Y\|_\rho^2) + 2Re\{t Tr[\rho(X^* Y + Y X^*)]\} \\
= \; & \|X^*\|_\rho^2 + \|X\|_\rho^2 - \frac{|Tr[\rho(X^* Y + Y X^*)]|^2}{\|Y\|_\rho^2 + \|Y^*\|_\rho^2}
\end{aligned}$$

Substituting this into (7) yields (6). □

Proof of Theorem 3.2 We put $X = A, Y = B$ in (6) for $A, B \in M_n(\mathbb{C})$. Since $Var_\rho(X) = \frac{1}{2}Tr[\rho(X_0^*X_0 + X_0X_0^*)] = \frac{1}{2}(\|X_0\|_\rho^2 + \|X_0^*\|_\rho^2)$ and $c = \frac{(\lambda_n+\lambda_1)^2}{(\lambda_n-\lambda_1)^2}$, we obtain

$$Var_\rho(A)Var_\rho(B) \geq \frac{1}{4}|Tr[\rho(A_0^*B_0 + B_0A_0^*)]|^2 + \frac{(\lambda_n+\lambda_1)^2}{4(\lambda_n-\lambda_1)^2}|Tr[\rho[A,B]]|^2.$$

□

By putting $A, B \in M_{n,sa}(\mathbb{C})$ in (4), we have refined Schrödinger uncertainty relation.

Corollary 3.2

$$\frac{(\lambda_n+\lambda_1)^2}{4(\lambda_n-\lambda_1)^2}|Tr[\rho[A,B]]|^2 + (Re\{Tr[\rho A_0B_0]\})^2 \leq V_\rho(A)V_\rho(B).$$

Since $\frac{(\lambda_n+\lambda_1)^2}{4(\lambda_n-\lambda_1)^2} = \frac{\lambda_1\lambda_n}{(\lambda_n-\lambda_1)^2} + \frac{1}{4}$, *we can rewrite as follows:*

$$\begin{aligned} & \frac{\lambda_1\lambda_n}{(\lambda_n-\lambda_1)^2}|Tr[\rho[A,B]]|^2 + \frac{1}{4}|Tr[\rho[A,B]]|^2 + (Re\{Tr[\rho A_0B_0]\})^2 \\ = \; & \frac{\lambda_1\lambda_n}{(\lambda_n-\lambda_1)^2}|Tr[\rho[A,B]]|^2 + |Tr[\rho A_0B_0]|^2 \leq V_\rho(A)V_\rho(B). \end{aligned} \tag{8}$$

On the other hand, the another refined non-hermite Schrödinger uncertainty relation is also given as follows:

Theorem 3.3 ([5]) *Let $\rho \in M_{n,1}(\mathbb{C})$ with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ satisfying $\lambda_1 \neq \lambda_n$ and $A, B \in M_n(\mathbb{C})$. Then the following refined uncertainty relation holds.*

$$\frac{\lambda_1\lambda_2^*}{\lambda_1+\lambda_2^*}Tr[\rho[A,B]^*[A,B]] + |Tr[\rho A_0^*B_0]|^2 \leq Tr[\rho A_0^*A_0]Tr[\rho B_0^*B_0], \tag{9}$$

where λ_2^* is the second smallest eigenvalue of ρ.

Remark 3.2 *In the case of $A, B \in M_{n,sa}(\mathbb{C})$, (4) and (9) are represented as follows.*

$$\frac{(\lambda_n+\lambda_1)^2}{4(\lambda_n-\lambda_1)^2}|Tr[\rho[A,B]]|^2 + (Re\{Tr[\rho A_0B_0]\})^2 \leq V_\rho(A)V_\rho(B).$$

That is

$$\frac{\lambda_n\lambda_1}{(\lambda_n-\lambda_1)^2}|Tr[\rho[A,B]]|^2 + \frac{1}{4}|Tr[\rho[A,B]]|^2 + (Re\{Tr[\rho A_0B_0]\})^2 \leq V_\rho(A)V_\rho(B).$$

And also

$$\frac{\lambda_1\lambda_2^*}{\lambda_1+\lambda_2^*}Tr[\rho[A,B]^*[A,B]]+\frac{1}{4}|Tr[\rho[A,B]]|^2+(Re\{Tr[\rho A_0B_0]\})^2\leq V_\rho(A)V_\rho(B).$$

We compare between

$$[1]=\frac{\lambda_1\lambda_n}{(\lambda_n-\lambda_1)^2}|Tr[\rho[A,B]]|^2 \quad \text{and} \quad [2]=\frac{\lambda_1\lambda_2^*}{\lambda_1+\lambda_2^*}Tr[\rho[A,B]^*[A,B]].$$

Let

$$\rho=\begin{pmatrix}1/4 & 0 & 0\\ 0 & 1/4 & 0\\ 0 & 0 & 1/2\end{pmatrix},\quad A=\begin{pmatrix}1 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 1\end{pmatrix},\quad B=\begin{pmatrix}2 & -i & 0\\ i & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Then $[1]=0<[2]=1/2$.

Let

$$\rho=\begin{pmatrix}1/6 & 0 & 0\\ 0 & 1/3 & 0\\ 0 & 0 & 1/2\end{pmatrix},\quad A=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix},\quad B=\begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix}.$$

Then $[1]=9/27>[2]=8/27$.

4 Refined Non-Hermite Robertson-Schrödinger Uncertainty Relation II

The another refined non-hermite Robertson uncertainty relation is given as follows:

Theorem 4.1 *Let* $\rho\in M_{n,1}(\mathbb{C})$ *with eigenvalues* $0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n$ *satisfying* $\lambda_1\neq\lambda_n$ *and* $A,B\in M_n(\mathbb{C})$. *Then the following refined non-hermite uncertainty relation holds.*

$$\frac{\lambda_1\lambda_n}{(\lambda_n-\lambda_1)^2}|Tr[\rho[A,B]]|^2\leq Tr[\rho A_0^*A_0]Tr[\rho B_0^*B_0]. \tag{10}$$

Proof of Theorem 4.1 Let $\rho = \sum_{i=1}^{n} \lambda_i |\phi\rangle\langle\phi_i|$ be the spectral decomposition. Then

$$Tr[\rho[A, B]] = \sum_{i,j} (\lambda_i - \lambda_j) a_{ij} b_{ji},$$

where $a_{ij} = \langle\phi_i|A_0|\phi_j\rangle$ and $b_{ji} = \langle\phi_j|B_0|\phi_i\rangle$. Then we have

$$\begin{aligned}
|Tr[\rho[A, B]]|^2 &\leq \left(\sum_{i,j} |\lambda_i - \lambda_j||a_{ij}||b_{ji}|\right)^2 \\
&= \left(\sum_{i,j} \left|\frac{\lambda_i - \lambda_j}{\lambda_i^{1/2}\lambda_j^{1/2}}\right| \lambda_i^{1/2}\lambda_j^{1/2}|a_{ij}||b_{ji}|\right)^2 \\
&\leq \max_{i,j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i\lambda_j} (\sum_{i,j} \lambda_i^{1/2}\lambda_j^{1/2}|a_{ij}||b_{ji}|)^2 \\
&\leq \max_{i,j} \left(\frac{\lambda_i^{1/2}}{\lambda_j^{1/2}} - \frac{\lambda_j^{1/2}}{\lambda_i^{1/2}}\right)^2 \sum_{i,j} \lambda_j |a_{ij}|^2 \sum_{i,j} \lambda_i |b_{ji}|^2 \\
&= \left(\frac{\lambda_n^{1/2}}{\lambda_1^{1/2}} - \frac{\lambda_1^{1/2}}{\lambda_n^{1/2}}\right)^2 Tr[\rho A_0^* A_0] Tr[\rho B_0^* B_0] \\
&= \frac{(\lambda_n - \lambda_1)^2}{\lambda_1\lambda_n} Tr[\rho A_0^* A_0] Tr[\rho B_0^* B_0].
\end{aligned}$$

□

Remark 4.1 *Let $\rho = \frac{1}{n} I$ be the maximally mixed state. Since $|Tr[\rho[A, B]]|^2 = 0$ and $\frac{(\lambda_n - \lambda_1)^2}{\lambda_1\lambda_n} Tr[\rho A_0^* A_0] Tr[\rho B_0^* B_0] = 0$ in the proof of Theorem 4.1, we can state that the result holds. And let ρ be non-faithfull state. Since $\lambda_1 = 0$, Theorem 4.1 holds.*

By putting $A, B \in M_{n,sa}(\mathbb{C})$ in (10), we have refined Robertson uncertainty relation.

Corollary 4.1

$$\frac{\lambda_1\lambda_n}{(\lambda_n - \lambda_1)^2} |Tr[\rho[A, B]]|^2 \leq V_\rho(A) V_\rho(B). \tag{11}$$

The following another refined non-hermite Schrödinger uncertainty relation is obtained.

Theorem 4.2 *Let $\rho \in M_{n,1}(\mathbb{C})$ with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ satisfying $\lambda_1 \neq \lambda_n$ and $A, B \in M_n(\mathbb{C})$. Then the following refined non-hermite uncertainty relation holds.*

$$\frac{\lambda_1\lambda_n}{(\lambda_n-\lambda_1)^2}|Tr[\rho[A,B]]|^2 + |Tr[\rho A_0^* B_0]|^2 \leq Tr[\rho A_0^* A_0]Tr[\rho B_0^* B_0]. \qquad (12)$$

The proof of non-hermite uncertainty relation (12) is based on the following lemma.

Lemma 4.1 *Let $\|X\|_\rho^2 = Tr[\rho X^* X]$ for any $X \in M_n(\mathbb{C})$. For $c > 0$, if the relation*

$$\|X\|_\rho^2 \|Y\|_\rho^2 \geq c|Tr[\rho[X,Y]]|^2 \qquad (13)$$

holds for any $X, Y \in M_n(\mathbb{C})$, then the stronger relation

$$\|X\|_\rho^2 \|Y\|_\rho^2 - |Tr[\rho X^* Y]|^2 \geq c|Tr[\rho[X,Y]])^2 \qquad (14)$$

also holds.

Proof of Lemma 4.1 The schwarz inequality for the inner prouct $\langle X|Y\rangle_\rho = Tr[\rho X^* Y]$ implies that if $\|Y\|_\rho = 0$, then $Tr[\rho X^* Y] = 0$. Therefore, in the case of $\|Y\|_\rho = 0$, the statement holds trivially. We henceforth assume $Tr[\rho Y^* Y] \neq 0$. In view of the identity $[X + tY, Y] = [X, Y]$ for any $t \in \mathbb{C}$, the assumption (13) implies

$$\|X + tY\|_\rho^2 \|Y\|_\rho^2 \geq c|Tr[\rho[X,Y]]|^2. \qquad (15)$$

Letting $t = -\frac{\overline{Tr[\rho X^* Y]}}{\|Y\|_\rho^2}$, we obtain

$$\begin{aligned} & \|X + tY\|_\rho^2 \\ = \;& \|X\|_\rho^2 + 2Re\{tTr[\rho X^* Y]\} + Tr[|t|^2 \rho Y^* Y] \\ = \;& \|X\|_\rho^2 - \frac{|Tr[\rho X^* Y]|^2}{\|Y\|_\rho^2}. \end{aligned}$$

Substituting this into (15) yields (14). □

By putting $A, B \in M_{n,sa}(\mathbb{C})$ in (12), we have refined Schrödinger uncertainty relation.

Corollary 4.2

$$\frac{\lambda_1\lambda_n}{(\lambda_n-\lambda_1)^2}|Tr[\rho[A,B]]|^2 + |Tr[\rho A_0 B_0]|^2 \leq V_\rho(A)V_\rho(B). \qquad (16)$$

Remark 4.2 *Though the LHS of (3) is larger than the LHS of (11), the LHS of (8) is equal to the LHS of (16).*

Remark 4.3 *It is easy to show that the LHS of (1) is larger than the LHS of (10). But when $A \in M_{n,sa}(\mathbb{C})$, $B \in M_n(\mathbb{C})$ and $\|B_0^*\|_\rho^2 \leq \|B_0\|_\rho^2$, the RHS of (1) is less than the RHS of (10) and when $A \in M_{n,sa}(\mathbb{C})$, $B \in M_n(\mathbb{C})$ and $\|B_0^*\|_\rho^2 \geq \|B_0\|_\rho^2$, the RHS of (1) is larger than the RHS of (10). Then we can't compare two inequalities.*

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