Log-majorization on spectral geometric mean

Masatoshi Fujii
Osaka Kyoiku University

1 Introduction.

First of all, this note is based on a joint work [4] with R. Nakamoto. Throughout this note, we denote by $A \geq 0$ if a square matrix A is positive semidefinite, and $A > 0$ if A is positive definite. The matrix geometric mean is defined by

$$A \# B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}} \quad \text{for } A, B > 0.$$

For a fixed $\alpha \in [0,1]$, α-spectral geometric mean is defined by

$$A \ \mathrm{sp}_\alpha \ B = (A^{-1} \# B)^\alpha A (A^{-1} \# B)^\alpha$$

for $A, B > 0$.

For two positive semidefinite matrices X and Y, the log–majorization $X \prec_{(\log)} Y$ means that

$$\prod_{j=1}^{k} \lambda_j(X) \leq \prod_{j=1}^{k} \lambda_j(Y), \ (k = 1, \cdots, n-1)$$

and $\det X = \det Y$, where $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$ are the eigenvalues of X in decreasing order. Recently Furuichi-Seo proposed the following conjecture on spectral geometric mean in [6].

Conjecture FS. If $-1 \leq \alpha \leq -\dfrac{1}{2}$, then

$$A \ \mathrm{sp}_\alpha B \prec_{(\log)} A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}}$$

holds for matrices $A, B > 0$.

The purpose of this note, we prove it affirmatively and discuss some applications to relative entropies.

2 Proof of the conjecture.

The main tool of the proof is a reverse inequality (RBLP) of BLP-inequality cited in [9] and [10].

(RBLP) If $A, B > 0$, then

$$\|A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}}\| \geq \|A^{\frac{1}{2}}(A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{1}{2}}\|$$

holds for $t \geq s \geq 1$.

In our discussion below, we use it for the simplest case $s = 1$. That is,

(RBLP-1) If $A, B > 0$, then

$$\|A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}}\| \geq \|A^{\frac{1}{2}}(A^{\frac{1}{2}} B A^{\frac{1}{2}})^t A^{\frac{1}{2}}\|$$

holds for $t \geq 1$.

Theorem 2.1. *If $-1 \leq \alpha \leq -\frac{1}{2}$, then*

$$A \ \mathrm{sp}_\alpha B \prec_{(\log)} A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}}$$

holds for $A, B > 0$.

Proof. It suffices to prove that

$$A^{1-\alpha} \leq B^{-\alpha} \Longrightarrow A^{\frac{1}{2}}(A \ \# \ B^{-1})^{-2\alpha} A^{\frac{1}{2}} \leq 1,$$

or equivalently, putting $\beta = -\alpha$,

$$A^{1+\beta} \leq B^\beta \Longrightarrow A^{\frac{1}{2}}(A \ \# \ B^{-1})^{2\beta} A^{\frac{1}{2}} \leq 1 \quad \text{for } \beta \in [\frac{1}{2}, 1].$$

Now, since $2\beta \geq 1$, it follows from (RBLP-1) that

$$\begin{aligned}
&\|A^{\frac{1}{2}}(A \sharp B^{-1})^{2\beta} A^{\frac{1}{2}}\| \\
&= \|A^{\frac{1}{2}}\{A^{\frac{1}{2}}(A^{-\frac{1}{2}} B^{-1} A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}\}^{2\beta} A^{\frac{1}{2}}\| \\
&= \|A^{\frac{1}{2}}\{A^{\frac{1}{2}} C A^{\frac{1}{2}}\}^{2\beta} A^{\frac{1}{2}}\| \quad (C = (A^{-\frac{1}{2}} B^{-1} A^{-\frac{1}{2}})^{\frac{1}{2}}) \\
&\leq \|A^{\frac{1}{2}+\beta} C^{2\beta} A^{\frac{1}{2}+\beta}\| \\
&= \|A^{\frac{1}{2}+\beta}(A^{-\frac{1}{2}} B^{-1} A^{-\frac{1}{2}})^\beta A^{\frac{1}{2}+\beta}\| \\
&= \|A^\beta (A \ \#_\beta \ B^{-1}) A^\beta\|.
\end{aligned}$$

We here suppose that $A^{1+\beta} \leq B^\beta$. Then the Löwner-Heinz inequality ensures that $A \leq B^{\frac{\beta}{1+\beta}}$ and $A^{2\beta} \leq B^{\frac{2\beta^2}{1+\beta}}$ by $0 < \frac{1}{1+\beta}, \frac{2\beta}{1+\beta} \leq 1$, respectively. Hence we have

$$A^\beta (A \ \#_\beta \ B^{-1}) A^\beta \leq A^\beta (B^{\frac{\beta}{1+\beta}} \ \#_\beta \ B^{-1}) A^\beta = A^\beta B^{\frac{-2\beta^2}{1+\beta}} A^\beta \leq I.$$

Combining with the norm inequality obtained in above, we have the conclusion. $\square$

3 The case $-\frac{1}{2} \le \alpha \le 0$.

The following result has been proved by Furuichi-Seo. We here give a simple proof in the frame of operator inequalities. As a matter of fact, we only use the Löwner-Heinz inequality, i.e.,

$$A \ge B \ge 0 \quad \Rightarrow \quad A^t \ge B^t \quad \text{for} \quad t \in [0,1].$$

Theorem 3.1. *If* $-\dfrac{1}{2} \le \alpha \le 0$, *then*

$$A \ \mathrm{sp}_\alpha B \prec_{(\log)} A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}}$$

holds for $A, B > 0$.

Proof. We show that if $A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}} \le I$, then $A^{\frac{1}{2}}(A \,\#\, B^{-1})^{-2\alpha} A^{\frac{1}{2}} \le I$. Putting $\beta = -\alpha$, we have $0 \le 2\beta \le 1$. Since $A^{1+\beta} \le B^\beta$ by the assumption, it follows from Löwner-Heinz inequality that $A \le B^{\frac{\beta}{1+\beta}}$ and so

$$(A \,\#\, B^{-1})^{-2\alpha} = (A \,\#\, B^{-1})^{2\beta} \le (B^{\frac{\beta}{1+\beta}} \,\#\, B^{-1})^{2\beta} = B^{-\frac{\beta}{1+\beta}} \le A^{-1}$$

by the monotonicity of # and Löwner-Heinz inequality again. Hence we have

$$A^{\frac{1}{2}}(A \,\#\, B^{-1})^{-2\alpha} A^{\frac{1}{2}} \le A^{\frac{1}{2}} A^{-1} A^{\frac{1}{2}} = I,$$

as desired. □

Remark 3.2. We note that Bhatia, Lim and Yamazaki showed in [2, Theorem 2] that

$$A^{\frac{1}{2}}(A \,\#\, B)A^{\frac{1}{2}} \ \prec_{(\log)} \ A^{\frac{3}{4}} B^{\frac{1}{2}} A^{\frac{3}{4}}$$

for positive definite matrices A, B by the use of the Furuta inequality.
We here remark that it is just the case $\alpha = -\frac{1}{2}$ in both Theorem 2.1 and Theorem 3.1 by replacing B^{-1} by B.

Combining with Theorems 2.1 and 3.1, we obtain that *if* $-1 \le \alpha \le 0$, *then*

$$A \ \mathrm{sp}_\alpha B \prec_{(\log)} A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}}$$

holds for $A, B > 0$.

4 An application to relative entropies.

For $0 \neq \alpha \in \mathbb{R}$, α-Tsallis relative entropy and α-divergence are defined as follows:

$$NT_\alpha(A||B) = -\mathrm{Tr}\ [T_\alpha(A|B)] = -\mathrm{Tr}\ \left[\frac{A \natural_\alpha B - A}{\alpha}\right], \text{ and}$$
$$D_\alpha(A||B) = -\mathrm{Tr}\ \left[\frac{A^{1-\alpha}B^\alpha - A}{\alpha}\right]$$

for $A, B > 0$, respectively. If A and B commute, then $A \natural_\alpha B = A^{1-\alpha}B^\alpha$ and hence

$$NT_\alpha(A||B) = D_\alpha(A||B)$$

We here propose a generalized relative entropy of Tsallis type which includes both $NT_\alpha(A||B)$ and $D_\alpha(A||B)$. A family of binary operations G_α among positive operators is called a generalized geometric mean if $G_\alpha(A, B) = A^{1-\alpha}B^\alpha$ for all commuting pair $\{A, B\}$. (If necessary, we add suitable conditions.) For a family of generalized geometric mean G_α, a relative entropy TG_α is defined by

$$TG_\alpha(A||B) = -\mathrm{Tr}\ \left[\frac{G_\alpha(A, B) - A}{\alpha}\right].$$

For convenience, we denote the transpose of the spectral geometric mean by sg_α, that is,

$$A \mathrm{\ sg}_\alpha\ B = A^{\frac{1}{2}}(A^{-1}\#B)^{2\alpha}A^{\frac{1}{2}}$$

for $A, B > 0$. Then the generalized relative entropy for it is given by

$$TSG_\alpha(A||B) = -\mathrm{Tr}\ \left[\frac{A \mathrm{\ sg}_\alpha\ B - A}{\alpha}\right]$$

for $A, B > 0$. Thus we have the following inequality as an application of Theorems 2.1 and 3.1:

Theorem 4.1. *For $-1 \leq \alpha < 0$,*

$$TSG_\alpha(A||B) \leq D_\alpha(A||B).$$

holds for $A, B > 0$.

Now we recall an important fact that Tsallis relative operator entropy $T_\alpha(A|B)$ converges to the relative operator entropy

$$T_\alpha(A||B) = \frac{A \#_\alpha B - A}{\alpha} \longrightarrow S(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$$

as $\alpha \to 0$ strongly. Its spectral geometric mean version

$$TS_\alpha(A||B) = \frac{A \text{ sg}_\alpha B - A}{\alpha}$$

is called the α-spectral Tsallis relative operator entropy. A result corresponding to it is as follows:

Proposition 4.2. *A family of α-spectral Tsallis relative operator entropies converges to $2S(A|A\hat{\#}B)$ as $\alpha \to 0$, where $A\hat{\#}B = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ is the operator fidelity of A and B.*

Proof. It follows that

$$\begin{aligned} TS_\alpha(A||B) &= \frac{A \text{ sg}_\alpha B - A}{\alpha} \\ &= \frac{1}{\alpha}(A^{\frac{1}{2}}((A^{-1} \# B)^{2\alpha} - I)A^{\frac{1}{2}}) \\ &= 2A^{\frac{1}{2}} \cdot \frac{(A^{-1} \# B)^{2\alpha} - I}{2\alpha} \cdot A^{\frac{1}{2}} \\ &\longrightarrow 2A^{\frac{1}{2}} \log(A^{-1} \# B)A^{\frac{1}{2}}. \end{aligned}$$

Noting that $A^{-1} \# B = A^{-\frac{1}{2}}(A\hat{\#}B)A^{-\frac{1}{2}}$, we have

$$2A^{\frac{1}{2}} \log(A^{-1} \# B)A^{\frac{1}{2}} = 2A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}(A\hat{\#}B)A^{-\frac{1}{2}})A^{\frac{1}{2}} = 2S(A|A\hat{\#}B),$$

as required. □

Concluding this section, we remark log-majorization between the geometric operator mean and the operator fidelity.

Proposition 4.3. *For $A, B > 0$, $A \# B \prec_{(\log)} A \hat{\#} B$.*

Proof. If $A \hat{\#} B \le I$, i.e., $(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}} \le I$, then we have $A^{\frac{1}{2}}BA^{\frac{1}{2}} \le I$ and so $B \le A^{-1}$. It implies that

$$A \# B \le A \# A^{-1} = I,$$

by which we have the conclusion. □

5 Appendix - Proof of (RBLP).

For reader's convenience, we mention a proof of (RBLP) in this section. For this, we start to cite a complementary inequality of (FI) due to Kamei [8]:

Theorem K. *Suppose that $A \geq B > 0$ and $0 \leq t < p \leq 1$.*

(1) *If $\frac{1}{2} \leq p \leq 1$, then $A^t \natural_{\frac{1-t}{p-t}} B^p \leq B$.*

(2) *If $0 < p \leq \frac{1}{2}$, then $A^t \natural_{\frac{2p-t}{p-t}} B^p \leq B^{2p}$.*

To prove (RBLP), we need Theorem K (1). We note that it is proved by the use of the following tools:

Lemma 5.1. *For $X, Y, Z > 0$,*

(i) $X \geq Y > 0 \ \Rightarrow \ X \natural_\alpha Z \leq Y \natural_\alpha Z$ *holds for* $1 \leq \alpha \leq 2$.

(ii) $X \natural_\alpha Y = YX^{-1}(X \natural_{\alpha-2} Y)X^{-1}Y$.

We prepare another elementary lemma to prove Theorem K.

Lemma 5.2. *Suppose that $\frac{1}{2} \leq p \leq 1$ and $0 \leq t < p$. If $2 \leq k \leq \frac{1-t}{p-t}$, then the followings hold:*

(i) $0 \leq t \leq 1 - k(p-t) \leq 1$,

(ii) $-1 \leq -t \leq 1 - k(p-t) - 2t \leq 0$.

Proof. (i) is easily checked. (ii) is shown by

$$\begin{aligned} -t = 1 - 2t - (1-t) &\leq 1 - 2t - k(p-k) \\ &\leq 2p - 2t - k(p-t) \\ &= (2-k)(p-t) \leq 0, \end{aligned}$$

as desired. □

Proof of Theorem K (1). First of all, we suppose that $1 < \frac{1-t}{p-t} \leq 2$. Then it follows from (i) in Lemma 6.1 that

$$A^t \natural_{\frac{1-t}{p-t}} B^p \leq B^t \natural_{\frac{1-t}{p-t}} B^p = B.$$

Next, if $2n - 1 < \frac{1-t}{p-t} \leq 2n$, then it follows from the preceding lemmas that

$$\begin{aligned} &A^t \natural_{\frac{1-t}{p-t}} B^p \\ &= (B^pA^{-t})^{n-1}(A^t \natural_{\frac{1-t}{p-t}-2(n-1)} B^p)(A^{-t}B^p)^{n-1} \\ &\leq (B^pA^{-t})^{n-1}(B^t \natural_{\frac{1-t}{p-t}-2(n-1)} B^p)(A^{-t}B^p)^{n-1} \\ &= (B^pA^{-t})^{n-1}B^{1-t-2(n-1)(p-t)+t}(A^{-t}B^p)^{n-1} \\ &\leq (B^pA^{-t})^{n-1}A^{1-2(n-1)(p-t)}(A^{-t}B^p)^{n-1} \end{aligned}$$

$$
\begin{aligned}
&= (B^pA^{-t})^{n-2}B^pA^{1-2(n-1)(p-t)-2t}B^p(A^{-t}B^p)^{n-2}\\
&\le (B^pA^{-t})^{n-2}B^pB^{1-2(n-1)(p-t)-2t}B^p(A^{-t}B^p)^{n-2}\\
&= (B^pA^{-t})^{n-2}B^{1-2(n-2)(p-t)}(A^{-t}B^p)^{n-2}\\
&\vdots\\
&= (B^pA^{-t})B^{1-2(p-t)}(A^{-t}B^p)\\
&\le (B^pA^{-t})A^{1-2(p-t)}(A^{-t}B^p)\\
&= B^pA^{1-2p}B^p\\
&\le B^pB^{1-2p}B^p = B.
\end{aligned}
$$

On the other hand, if $2n < \frac{1-t}{p-t} \le 2n+1$, then it follows from the preceding lemmas that

$$
\begin{aligned}
&A^t \natural_{\frac{1-t}{p-t}} B^p\\
&= (B^pA^{-t})^n(A^t \natural_{\frac{1-t}{p-t}-2n} B^p)(A^{-t}B^p)^n\\
&\le (B^pA^{-t})^n(A^t \natural_{\frac{1-t}{p-t}-2n} A^p)(A^{-t}B^p)^n\\
&= (B^pA^{-t})^nA^{1-t-2n(p-t)+t}(A^{-t}B^p)^n\\
&= (B^pA^{-t})^nA^{1-2n(p-t)}(A^{-t}B^p)^n\\
&= (B^pA^{-t})^{n-1}B^pA^{1-2n(p-t)-2t}B^p(A^{-t}B^p)^{n-1}\\
&\le (B^pA^{-t})^{n-1}B^pB^{1-2n(p-t)-2t}B^p(A^{-t}B^p)^{n-1}\\
&= (B^pA^{-t})^{n-1}B^{1-2(n-1)(p-t)}(A^{-t}B^p)^{n-1}\\
&\vdots\\
&= (B^pA^{-t})B^{1-2(p-t)}(A^{-t}B^p)\\
&\le (B^pA^{-t})A^{1-2(p-t)}(A^{-t}B^p)\\
&= B^pA^{1-2p}B^p\\
&\le B^pB^{1-2p}B^p = B,
\end{aligned}
$$

which completes the proof.

Finally we show a road to (RBLP) from Theorem K (1).

Proof of (RBLP). Fix $t \ge s \ge 1$. It suffices to show that

$$A^{-(1+t)} \ge B^t \;\Rightarrow\; A^{-s} \natural_{\frac{t}{s}} B^s \le A^{-(1+s)}.$$

The assumption $A^{-(1+t)} \ge B^t$ is weakened to $A_1 = A^{-(1+s)} \ge B^{\frac{t(1+s)}{1+t}} = B_1$ by Löwner-Heinz inequality. By the use of Theorem K (1) for $A_1 \ge B_1$, it is ensured that

$$A_1^{t_1} \natural_{\frac{1-t_1}{p_1-t_1}} B_1^{p_1} \le A_1 = A^{-(1+s)}$$

if both (i) $\frac{1}{2} \le p_1 \le 1$ and (ii) $0 < t_1 < p_1$ are satisfied. Now we take

$$t_1 = \frac{s}{1+s} \quad \text{and} \quad p_1 = \frac{s(1+t)}{t(1+s)}.$$

Then (i) and (ii) are assured by $\frac{t}{t+2} \le 1 \le s \le t$ and $p_1 - t_1 = \frac{s}{t(1+s)}$, respectively. As a consequence, it follows that

$$A^{-s} \natural_{\frac{t}{s}} B^s \le A^{-(1+s)} \quad \text{by} \quad \frac{1-t_1}{p_1-t_1} = \frac{t}{s},$$

which is the desired conclusion. $\square$

In addition, we cite a proof of Theorem K (2). It is similar to that of Theorem K (1), but.

Proof of Theorem K (2). Since $\dfrac{2p-t}{p-t} = 2 + \dfrac{t}{p-t} \ge 2$, it is divided into the two cases; (i) $2n < \dfrac{2p-t}{p-t} \le 2n+1$ for $n = 1, 2, \cdots$, and (ii) $2n-1 < \dfrac{2p-t}{p-t} \le 2n$ for $n = 2, 3, \cdots$.

(i) We first note that under the case (i) for some n,

$$0 \le 2k(p-t) \le 1, \text{ and } 0 \le 2kt - 2(k-1)p \le 1$$

hold for $k = 1, 2, \cdots, n$. As a matter of fact, the former is ensured by

$$2k(p-t) \le 2n(p-t) \le 2p - t \le 2p \le 1,$$

and the latter is done as follows: Since $2k(p-t) \le 2p-t$ as in above, we have $2(k-1)p < (2k-1)t < 2kt$, so that $2kt - 2(k-1)p \ge 0$, and

$$2kt - 2(k-1)p \le 2kp - 2(k-1)p = 2p \le 1.$$

Now it follows that

$$
\begin{aligned}
& A^t \natural_{\frac{2p-t}{p-t}} B^p \\
&= (B^p A^{-t})^n (A^t \#_{\frac{2p-t}{p-t}-2n} B^p)(A^{-t}B^p)^n \\
&\le (B^p A^{-t})^n (A^t \#_{\frac{2p-t}{p-t}-2n} A^p)(A^{-t}B^p)^n \\
&= (B^p A^{-t})^n A^{2p-t-2n(p-t)+t}(A^{-t}B^p)^n \\
&= (B^p A^{-t})^n A^{2p-2n(p-t)}(A^{-t}B^p)^n \\
&= (B^p A^{-t})^{n-1} B^p A^{2p-2n(p-t)-2t} B^p (A^{-t}B^p)^{n-1} \\
&\le (B^p A^{-t})^{n-1} B^p B^{2p-2n(p-t)-2t} B^p (A^{-t}B^p)^{n-1} \\
&= (B^p A^{-t})^{n-1} B^{2p-2(n-1)(p-t)}(A^{-t}B^p)^{n-1} \\
&\vdots \\
&= (B^p A^{-t}) B^{2p-2(p-t)}(A^{-t}B^p) \\
&\le (B^p A^{-t}) A^{2t}(A^{-t}B^p) \\
&= B^{2p}.
\end{aligned}
$$

(ii) Next, for the case (ii) $2n-1 < \dfrac{2p-t}{p-t} \le 2n$ for some $n \ge 2$, we note that

$0 \le (2n-1)(p-t) \le 2p-t \le 2p \le 1$, and

$0 \le 2kt - 2(k-1)p \le 2kp - 2(k-1)p = 2p \le 1$.

Since $1 < \frac{2p-t}{p-t} - 2(n-1) \le 2$, it follows from Lemma 5.1 (1) that

$$
\begin{aligned}
& A^t \natural_{\frac{2p-t}{p-t}} B^p \\
&= (B^p A^{-t})^{n-1} (A^t \natural_{\frac{2p-t}{p-t}-2(n-1)} B^p)(A^{-t}B^p)^{n-1} \\
&\le (B^p A^{-t})^{n-1} (B^t \natural_{\frac{2p-t}{p-t}-2(n-1)} B^p)(A^{-t}B^p)^{n-1} \\
&= (B^p A^{-t})^{n-1} B^{2p-t-2(n-1)(p-t)+t}(A^{-t}B^p)^{n-1} \\
&\le (B^p A^{-t})^{n-1} A^{2p-2(n-1)(p-t)}(A^{-t}B^p)^{n-1} \\
&= (B^p A^{-t})^{n-2} B^p A^{2p-2(n-1)(p-t)-2t} B^p (A^{-t}B^p)^{n-2} \\
&\le (B^p A^{-t})^{n-2} B^p B^{2p-2(n-1)(p-t)-2t} B^p (A^{-t}B^p)^{n-2} \\
&= (B^p A^{-t})^{n-2} B^{2p-2(n-2)(p-t)}(A^{-t}B^p)^{n-2} \\
&\vdots \\
&= (B^p A^{-t}) B^{2p-2(p-t)}(A^{-t}B^p) \\
&\le (B^p A^{-t}) A^{2t}(A^{-t}B^p) \\
&= B^{2p}.
\end{aligned}
$$

Hence Theorem K (2) is proved. $\square$

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Masatoshi Fujii, Department of Mathematics, Osaka Kyoiku University, Asahi-gaoka, Kashiwara, Osaka 582-8582, Japan
mfujii@cc.osaka-kyoikuAc.jp