臨界ソボレフ熱流の集中現象と速い拡散型非線形放物型方程式に対する有限時間消滅

Masashi Misawa (三沢 正史) †, Kumamoto University (熊本大学)

In this paper we shall report the finite-time extinction behavior of solutions to the Cauchy-Dirichlet problem for the doubly nonlinear fast diffusive parabolic type equation

$$\begin{cases} \partial_t v^q - \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) = 0 & \text{in } \Omega_\infty := \Omega \times (0,\infty) \\ v = v_0 & \text{on } \Omega \times \{0\} \\ v = 0 & \text{on } \partial\Omega \times (0,\infty) \end{cases} \tag{1}$$

Here Ω is an open bounded domain in $\mathbb{R}^n$, $n \geqq 2$, with a smooth boundary $\partial\Omega$. The exponents p, q are chosen as $1 < p < n$, $q+1 = p^*$ with $p^* := \frac{np}{n-p}$. Moreover, $u : \Omega_\infty \to [0,\infty)$ is a nonnegative function and, as usual, $\nabla v = (\partial_{x_\alpha} v)_{1 \leqslant \alpha \leqslant n}$ denotes the spatial gradient of a function v and $\partial_t v^q$ means the derivative in time of v^q. The initial data $v_0 = v_0(x)$ belongs to the Sobolev space $W_0^{1,p}(\Omega)$ and is bounded, nonnegative and not identically zero.

This paper is based on a collaborative research with Tuomo Kuusi, Professor in University of Helsinki, Finland, and Kenta Nakamura, Academic Researcher in Nihon University.

The following theorem states on the finite-time extinction of solutions to (1) [*M. Misawa, K. Nakamura: J. Geom. Anal. 33: 33 (2023); M. Misawa, K. Nakamura, Md Abu Hanif Sarkar: Nonlinear Differ. Eqn. Appl. 30 ; 43 (2023)*].

Theorem 1 (Finite-time extinction)**.** *Let $p > 1$, $q > 0$ satsify $p < q+1 \leqq p^*$. Let $v \in L^\infty(0,\infty; W_0^{1,p}(\Omega)) \cap C([0,\infty); L^{q+1}(\Omega))$ be a nonnegative weak solution of (1). Then, there exists a positive number t^* such that $v > 0$ in $\Omega \times (0, t^*)$ and $v = 0$ in $\Omega \times [t^*, \infty)$. Moreover, we have positive constants $C_i = C_i(p,q)$, $i = 1, 2$, such that*

$$C_1(t^* - t)^{1/(q+1-p)} \leqq \|v(t)\|_{q+1}, \ \|\nabla v(t)\|_p \leqq C_2(t^* - t)^{1/(q+1-p)}, \tag{2}$$

where $\|\cdot\|_r$ denotes the $L^r(\Omega)$-norm.

We now shall exhibit the outline of proof.

Proof of Theorem 1. Multiplying the equation $(1)_1$ by the solution u, by integration-by-parts we have

$$\frac{d}{dt}\|u(t)\|_{q+1}^{q+1} + \frac{q+1}{q}\|\nabla u(t)\|_p = 0. \tag{3}$$

For derivation of the upper-estimate of the volume we need a boundedness of the Rayleigh quotient $R(w) := \|\nabla w\|_p / \|w\|_{q+1}$ for $w \in W_0^{1,p}(\Omega) \setminus \{0\}$. In fact, it holds true that $R(v(t) \leqq$

† Department of Mathematics, Faculty of Advanced Science and Technology, Kumamoto University (熊本大学大学院先端科学研究部（理学系）数学コース), 2-39-1 Kurokami, Chuo-ku, Kumamoto-shi, Kumamoto 860-8555, Japan; *E-mail address*: mmisawa@kumamoto-u.ac.jp; M.M was partially supported by the Grant-in-Aid for Scientific Research (C) No.21K03330, (C) No.24K06798 at Japan Society for the Promotion of Science (JSPS for short).

$R(v_0)$ for the weak solution v of (1) and $t \in [0, t^*)$. This estimation is obtained from the volume equality (3) and the energy inequality

$$\|\partial_t v^{(q+1)/2}\|_2^2 + \frac{d}{dt}\|\nabla v(t)\|_p^p \leqq 0. \tag{4}$$

The energy inequality (4) follows from testing $(1)_1$ by time-derivative $\partial_t v$ through some appropriate approximation (see [Appendix D, Page 39-43, *Nonlinear Differ. Eqn. Appl. 30 ; 43 (2023)*] and [*Adv. Calc. Var. 16, no.2 (2023)*]).
For brevity we put the notation as $\Psi(t) = \|u(t)\|_{q+1}^{q+1}$ and $\prime = d/dt$. Substituting the boundedness of Rayleigh quotient above into (3) we have

$$\frac{q}{q+1}\Psi'(t) + R(v_0)^p \Psi^{p/(q+1)}(t) \geqq 0,$$

which is solved as

$$\Psi(t)^{(q+1-p)/(q+1)} \leqq \frac{q+1-p}{q} R(v_0)^p (t^* - t)^{1/(q+1-p)} \tag{5}$$

and thus, we have the upper-boundedness of volume in (2).
Now, we use the Sobolev-Poincaré inequality $\|w\|_{q+1} \leqslant C_{p,q}\|\nabla w\|_p$ being true for a positive constant $C_{p,q}$ and any $w \in W_0^{1,p}(\Omega)$. From the equality (3) we have

$$\frac{q}{q+1}\Psi'(t) + C_{p,q}^{-p}\Psi^{p/(q+1)}(t) \leqq 0,$$

which is integrated over $(0, t^*)$ and yields

$$\|u(t)\|_{q+1} = \Psi^{1/(q+1)}(t) \geqq \big(\frac{q+1-p}{qC_{p,q}^p}\big)^{1/(q+1-p)} (t^* - t)^{1/(q+1-p)}.$$

This is the lower-estimate of the volume of (2).
The estimations of energy in (2) easily follow from applying the definition of Rayleigh quotient and the Sobolev-Poincaré inequality for the inequalities of volume above.
The interior positivity of the weak solution v before the extinction-time t^* is obtained from that of the transformed solution u below, which is the solution of the $p-$Sobolev flow, satisfying the *volume* conservation. □

Our main theorem, Theorem 2, exhibits the so-called volume and energy concentration at finite-time extinction of positive solutions to the evolution equation (1). This phenomenon of concentration seems being peculiar to the Sobolev critical case that $q + 1 = p^*$. To state the result, let us define the parametrized family of functions $U[x_0, \mu]$ with a point $x_0 \in \mathbb{R}^n$ and $\mu \in (0, \infty)$, by

$$U[x_0,\, \mu](x) := \left[\frac{\mu^{1/(p-1)} n^{1/p} \left(\frac{n-p}{p-1}\right)^{(p-1)/p}}{\mu^{p/(p-1)} + |x - x_0|^{p/(p-1)}}\right]^{(n-p)/p}, \quad x \in \mathbb{R}^n.$$

Such functions as above are called as *Talenti bubbles* and the extremal functions attaining the best constant S of Sobolev's inequality on the whole space $\mathbb{R}^n$

$$S = \inf\left\{\frac{\|\nabla\phi\|_p}{\|\phi\|_{q+1}} : \phi \in L^{p^*}(\mathbb{R}^n),\, \|\nabla\phi\|_p < \infty\right\}.$$

Theorem 2 (Concentration extinction profile). *Let $\frac{2n}{n+2} \leqq p < n$, $q+1 = p^*$ and $r_0 = 1 \wedge \lambda_0^{-1/p}$ with $\lambda_0 = \|\nabla v_0\|_p^p / \|v_0\|_{q+1}^p$. For a positive number $r \leqq r_0$, define $\{\psi_{x_i}\}_{1 \leqq i \leqq N}$ as Lipschitz bump functions such that*

$$\begin{cases} 0 \leqq \psi_{x_i} \leqq 1, \quad \mathrm{supp}(\psi_{x_i}) \subseteq B(x_i, r) =: B_i \\ \|\nabla \psi_{x_i}\|_n^n \leqq C(n,p) \end{cases}$$

Suppose that v is a solution to (1) as in Theorem 1. Let $\{t_j\}_{j \in \mathbb{N}}$ be a sequence of times so that $t_j \nearrow \infty$ as $j \to \infty$. Then, there exist a sequence $\{\tau_j\}_{j \in \mathbb{N}}$ depending on $\{t_j\}_{j \in \mathbb{N}}$ and satisfying $\tau_j \nearrow \infty$ as $j \to \infty$, a positive number λ_∞ and a set $\{x_i\}_{i=1}^N$ of at most finitely many distinct points, where $N := \lceil \varepsilon_0^{-1} \rceil$ (the maximal value not exceeding ε_0^{-1}) and $\varepsilon_0 \leqq \left(\frac{S}{\lambda_\infty}\right)^{\frac{n}{p}}$ with S being the best constant of Sobolev's inequality on $\mathbb{R}^n$, a sequence $\{\mu_i\}_{i=1}^N$ of positive numbers such that we have the convergence as $r \searrow 0$ and $j \to \infty$

$$\frac{v(\cdot, h(\tau_j))}{(t^* - h(\tau_j))^{1/(q+1-p)}} \longrightarrow \sum_{i=1}^N \psi_{x_i}(\cdot) \lambda_\infty^{-(n-p)/p^2} k_{x_i,j} U[0, \lambda_\infty^{-1/p} \mu_i] \left(k_{x_i,j}^{p/(n-p)} (\cdot - x_i) \right) + u_\infty(\cdot)$$

in $W_0^{1,p}(\Omega)$, where t^* is the extinction-time as in Theorem 1, the function $h(\tau)$, $\tau \geqq 0$, is defined below, $1 \leqq k_{x_i,j} \nearrow \infty$ as $j \to \infty$ and u_∞ is either a trivial function $u_\infty \equiv 0$ or a positive weak solution in $W_0^{1,p}(\Omega)$ of

$$-\mathrm{div}(|\nabla u|^{p-2} \nabla u) = \lambda_\infty u^q \quad \text{in } \Omega$$

with a positive constant λ_∞ so that both u_∞ and its spatial gradient ∇u_∞ are uniformly Hölder continuous in Ω.

Theorem 2 is established by a scaling transformation intrinsic to (1) (which was kindly advised by Professor J. Vázquez in 2013) as follows: First, let $\Lambda(\theta) \in C^1([0,\infty))$ and $g(\tau) \in C^1([0,\infty))$ be the solutions of the ordinary differential equations, respectively,

$$\begin{cases} \Lambda'(\theta) = (t^*)^{-1} \|v(t^*(1 - \exp(-\Lambda(\theta))\|_{q+1}^{q+1-p}, \ \theta \geqq 0 \\ \Lambda(0) = 0 \end{cases} \qquad \begin{cases} g'(\tau) = \exp(\Lambda(g(\tau))), \ \tau \geqq 0 \\ g(0) = 0 \end{cases}$$

where we note by the volume identity (3) that the function $\|v(s)\|_{q+1}^{q+1-p}$ is continuous on $s \geqq 0$. Using the functions above we define the transformation of the solution of (1) by

$$h(t) = t^* \{1 - \exp(-\Lambda(g(t)))\}, \quad u(x,t) = \frac{v(x, h(t))}{\|v(h(t))\|_{q+1}}, \quad t \geqq 0. \tag{6}$$

The function $h(t)$ above is increasing and extends the finite-interval $[0, t^*)$ from 0 up to the extinction-time t^* into the infinite-interval $[0, \infty)$.

Theorem 3. *The function u defined by (6) belongs to $L^\infty(0,\infty; W_0^{1,p}(\Omega)) \cap C([0,\infty); L^{q+1}(\Omega))$ and a weak solution of the $p-$Sobolev flow*

$$\begin{cases} \partial_t u^q - \mathrm{div}\left(|\nabla u|^{p-2} \nabla u\right) = \|\nabla u(t)\|_p^p u^q & \text{in } \Omega_\infty := \Omega \times (0, \infty) \\ \|u(t)\|_{q+1} = 1 & t \geqq 0 \end{cases} \tag{7}$$

with the initial and boundary conditons: $u = u_0 := v_0 / \|v_0\|_{q+1}$ on $\Omega \times \{0\}$, $u = 0$ on $\partial\Omega \times (0, \infty)$.

The solution u satisfies the energy inequality

$$\int_{\Omega_\infty} \left|\partial_t u^{(q+1)/2}\right|^2 \, dxdt + \sup_{0<t<\infty} \frac{1}{p}\|\nabla u(t)\|_p^p \leqq \frac{1}{p}\|\nabla u_0\|_p^p. \tag{8}$$

Furthermore, the solution u is positive and bounded, $0 < u \leqq \exp(\|u_0\|_p^p T/q)\|u_0\|_\infty$ almost everywhere in $\Omega_T := \Omega \times (0,T)$ for any positive $T < \infty$, and u and its spatial gradient ∇u are locally in space-time Hölder continuous in Ω_T for any positive $T < \infty$.

Proof of Theorem 3. Now we confirm the first statement of Theorem 3 by direct computation. In the following verification, for brevity we write $\|\cdot\| = \|\cdot\|_{q+1}$ and $' = d/dt$.
It follows from that of v assumed in Theorem 1 that the transformed function u belongs to the indicated function-space.
Since $0 \leqq h(t) < t^*$ for all $t \in [0,\infty)$, $\|v(h(t))\|_{q+1} > 0$ for all $t \in [0,\infty)$. The volume identity (3) measns that the function $\|v(s)\|^{q+1}$ is Lipschitz continuous on $s \geqq 0$ and thus, so is $\|v(h(t))\|$. Now we calculate as

$$\begin{aligned} &(\Lambda(g(t)))' = \exp(g(t))(t^*)^{-1}\|v(h(t)\|^{q+1-p}, \\ &(h(t))' = t^* \exp(-\Lambda(g(t)))(\Lambda(g(t)))' = \|v(h(t)\|^{q+1-p}. \end{aligned} \tag{9}$$

and then, we have

$$\partial_t u^q = \partial_s v^q \|v(h(t))\|^{1-p} - q\,\|v(h(t))\|^{-1}(\|v(h(t))\|)' u^q. \tag{10}$$

Here, by the volume identity (3) and (9) for $(h(t))'$ the right-side of (10) is computed as follows:

$$\begin{aligned} \partial_s v^q \|v(h(t))\|^{1-p} &= \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right)\|v(h(t))\|^{1-p}, \\ &= \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \\ (\|v(h(t))\|)' &= (q+1)^{-1}\|v(h(t))\|^{-q/(q+1)} \left.\frac{d}{dh}\|v(h)\|^{q+1}\right|_{h=h(t)} (h(t))' \\ &= -q^{-1}\|v(h(t))\|\|\nabla u(t)\|_p^p. \end{aligned} \tag{11}$$

Combining (10) and (11) we obtain the equation (7) for the transformed function u. The volume constraint for u follows from the definition (6).

We also see that the volume constraint $(7)_2$ is natural in the following sense. In fact, we multiply both sides of $(7)_1$ by the solution u itself and integrate by parts to find that $\|\nabla u(t)\|_p^p$ in the right-side of $(7)_1$ is the Lagrange multiplier stemming from the integral condition $(7)_2$.

The boundedness of the $p-$Sobolev flow naturally follows from the maximum principle for the doubly nonlinear parabolic type operator of (7), together with the bounedness of initial datum. In virtue of the *volume constaint* in $(7)_2$ we can show the positivity of the p-Sobolev flow (7), where the important point is that the so-called expansion of positivity holds true for the nonnegative weak solution of (7), by the volume constraint $(7)_2$ and the boundedness of solution. The positivity-expansion is a sort of the weak Harnack inequality and the proof is based on the local energy estimates for the truncated solution and De Giorgi's measure-theoretical argument, under the scale-invariance intrinsic to the doubly nonlinear parabolic operator of (7)

[*E. Dibenedetto, U. Gianazza, V. Vespri: Acta Math. 200 (2008);* Springer Monographs in Math. 2012]. The boundedness and positivity further yields the gradient regularity of u [*T. Kuusi, M. Misawa, N. Nakamura: J. Geom. Anal. 30 no.2 (2020); J. Differ. Eq. 279 (2021); M. Misawa: Calc. Var. 62 (2023), no. 9, No. 265*]. $\square$

The concentration extinction profile for (1) as stated in Theorem 2 follows from the volume and energy concentration of (7) at infinite time. The equation in (7) describes the gradient flow associated with the constrained minimization problem determining the best constant of the usual Sobolev inequality for $W_0^{1,p}(\Omega)$. The case $q+1=p^*$ is called the *Sobolev critical* one, in the sense that any bounded sequence in $W_0^{1,p}(\Omega)$ is not always compact, and the so-called volume and energy concentration of (7) may appear at infinite time. The proof is the blowing-up argument based on a local boundedness with a local volume [*T. Kuusi, M. Misawa, K. Nakamura: submitted (2025)*].

From now on we shall show the outline of the proof of Theorem 2.

First we present the local boundedness with a local volume, which is a new ingredient obtained for the doubly nonlinear parabolic equation and is of its own interest. It holds true for interior and boundary local regions. The key of proof is the so-called "reverse Hölder type estimate", derived from De Giorgi's iteration method based on the local energy inequality for the truncated solution and the Sobolev-Poincaré inequality, under the scale-invarinace intrinsic to $(7)_1$. We omit the detail of the proof (Refer to [*T. Kuusi, G. Mingione, J. Eur. Math. Soc. 16 (2014)*]).

Theorem 4 (Local boundedness). *Let u be a nonnegative weak solution to* (7). *Let $r_0 \in (0,1]$ satisfy $r_0^p\lambda_0 \leqq 1$ with $\lambda_0 := \|\nabla u_0\|_p^p$ and $k_0 \geqq 1$, and set $\gamma := p(1+2/n)$. Take a point $x_0 \in \overline{\Omega}$ and a time t_0 so that $(t_0 - r_0^p, t_0) \subseteq (0,\infty)$, and define the space-time region $Q^{k_0}(x_0,t_0,r_0) = B(x_0, k_0^{(p-(q+1))/p} r_0) \times (t_0 - r_0^p, t_0)$ There exists a constant $\delta_0 \equiv \delta_0(n,p) \in (0,1)$ such that if*

$$\frac{1}{|Q(x_0,t_0,r_0)|}\int_{Q^{k_0}(x_0,t_0,r_0)\cap\Omega_\infty} u^{q+1}\,dxdt \leqq \delta_0^\gamma\,, \tag{12}$$

then it holds true that

$$\|u\|_{L^\infty(Q^{4k_0}(x_0,t_0,r_0/2)\cap\Omega_\infty)} \leqq 4k_0\,. \tag{13}$$

We now study the asymptotic behavior of the global-in-time solutions to (7), obtained in Theorem 3. The first one characterizes concentration at infinite time and derives the condition of the strong convergence in energy at infinite time.

Theorem 5 (ε_0-strong compactness). *Set $q+1=p^*$ for $p \in [\frac{2n}{n+2}, n)$. Let u be a nonnegative weak solution of* (7). *Fix $r_0 := 1 \wedge \lambda_0^{-1/p}$ with $\lambda_0 := \|\nabla u_0\|_p^p$. Let $\{t_j\}_{j\in\mathbb{N}}$ be a sequence of times such that $t_j \nearrow \infty$ as $j\to\infty$ and $\varepsilon_0 > 0$. Then, there exist a sequence $\{\tau_j\}_{j\in\mathbb{N}}$ depending on $\{t_j\}_{j\in\mathbb{N}}$ and satisfying $\tau_j \nearrow \infty$ as $j\to\infty$, and a set $E_{\rm int} \subseteq \Omega$ of distinct points of singularities, such that the following statements hold: there exist a positive number λ_∞ and a function $u_\infty \in W_0^{1,p}(\Omega)$ such that, as $j\to\infty$, $\lambda(\tau_j) \to \lambda_\infty$ and*

$$u(\tau_j) \to u_\infty \quad \begin{cases} \textit{weakly in} \quad W_0^{1,p}(\Omega) \\ \textit{strongly in} \quad L^\gamma(\Omega), \quad \forall \gamma \in [0,p^*) \\ \textit{almost everywhere in} \quad \Omega \end{cases}$$

where u_∞ is a nonnegative weak solution to $-\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda_\infty u^q$ in Ω. Moreover, there are only finitely many distinct points of E_{int} such that

$$\liminf_{j\to\infty} \int_{Q^k(x_0,\tau_j,r_0)\cap\Omega_\infty} u^{q+1}\,dxdt > \varepsilon_0$$

whenever space center $x_0 \in E_{\mathrm{int}}$ and $k \in [1,\infty)$, and the sequence $\{u(\tau_j)\}_{j\in\mathbb{N}}$ strongly converges to u_∞ in $W^{1,p}$ on any compact subset of $\Omega \setminus E_{\mathrm{int}}$.

We shall show the outline of proof.
Proof of Theorem 5. Let $\{t_j\}_{j\in\mathbb{N}}$ be an increasing sequence such that $t_j \nearrow \infty$ as $j \to \infty$. Set $I_j := (t_j, t_j+1)$ for $j \in \mathbb{N}$, which may be disjoint each other. Then, there exists a sequence $\{\tau_j\}_{j\in\mathbb{N}}$ (up to a zero Lebesgue measure set) such that $\tau_j \in I_j$ and

$$\lim_{j\to\infty} \int_\Omega |\partial_t u^q(\tau_j)|\,dx = 0, \tag{14}$$

Indeed, by the energy inequality (8) in Theorem 3 we have that

$$\sum_{j=1}^{\infty} \fint_{I_j}\int_\Omega \left|\partial_t u^{\frac{q+1}{2}}\right|^2 dxdt = \int_0^\infty \int_\Omega \left|\partial_t u^{\frac{q+1}{2}}\right|^2 dxdt < \infty,$$

and thus,

$$\lim_{j\to\infty} \fint_{I_j}\int_\Omega \left|\partial_t u^{\frac{q+1}{2}}\right|^2 dxdt = 0.$$

From the mean-value theorem, for each $j \in \mathbb{N}$, there exists $\tau_j \in I_j$ (up to a zero Lebesgue measure set) so that

$$\lim_{j\to\infty} \int_\Omega \left|\partial_t u^{\frac{q+1}{2}}(\tau_j)\right|^2 dx = 0. \tag{15}$$

By the chain rule of weak differential we have for $q \geqq 1$ that $\partial_t u^q = \frac{2q}{q+1} u^{\frac{q-1}{2}} \partial_t u^{\frac{q+1}{2}}$, since the function $[0,\infty) \ni z \mapsto z^{\frac{2q}{q+1}}$ is locally Lipschitz regular. From (15) and Hölder's inequality, it follows that

$$\begin{aligned}
\int_\Omega |\partial_t u^q(\tau_j)|\,dx &\leqq \frac{2q}{q+1}\|u^{\frac{q-1}{2}}(\tau_j)\|_{L^2(\Omega)} \left\|\partial_t u^{\frac{q+1}{2}}(\tau_j)\right\|_{L^2(\Omega)} \\
&\leqq \frac{2q}{q+1}|\Omega|^{\frac{1}{q+1}} \sup_{t\in(0,\infty)} \|u(t)\|_{L^{q+1}(\Omega)}^{\frac{q-1}{2}} \left\|\partial_t u^{\frac{q+1}{2}}(\tau_j)\right\|_{L^2(\Omega)} \\
&\leqq C(q,\Omega)\|u_0\|_{L^{q+1}(\Omega)}^{\frac{q-1}{2}} \left\|\partial_t u^{\frac{q+1}{2}}(\tau_j)\right\|_{L^2(\Omega)} \longrightarrow 0
\end{aligned}$$

as $j \to \infty$, which means (14).
The volume identity $(7)_2$ and the energy inequality (8) in Theorem 3 yield the weak convergence in some Lebesgue and Sobolev spaces of the sequences $u(\tau_j)$ and $\nabla u(\tau_j)$ along some subsequence $\tau_j \nearrow \infty$ as $j \to \infty$. We find by (14) that the limit function u_∞ of $\{u(\tau_j)\}$ belongs to the Sobolev space $W^{1,p}(\Omega)$ and is a nonnegative weak solution of $-\Delta_p u_\infty = \lambda_\infty u_\infty^q$ in Ω, which is exactly the stationary equation associated to (7).

Here we define the following dichotomy, which clarifies the phenomenon of bubbles.

Definition 6 (Ω_{reg} and $(\partial\Omega)_{\text{reg}}$)**.** Fix $r_0 \in (0,1]$ so that $\lambda_0 r_0^p \leqq 1$ with $\lambda_0 = \|\nabla u_0\|_p^p$. We may assume τ_j being so large that $(\tau_j - r_0^p, \tau_j) \subseteq (0,\infty)$.
Given a point $x_0 \in \Omega$, we say that $x_0 \in \Omega_{\text{reg}} \subseteq \Omega$ if there exists $k_{x_0} \in [1,\infty)$ such that

$$\liminf_{j\to\infty} \int_{Q^{k_{x_0}}(x_0,\tau_j,r_0)\cap\Omega_\infty} u^{q+1}\,dxdt \leqq \varepsilon_0\,. \tag{16}$$

Otherwise, we have

$$x_0 \in \Omega\setminus\Omega_{\text{reg}} \quad\Longleftrightarrow\quad \liminf_{j\to\infty} \int_{Q^{k}(x_0,\tau_j,r_0)\cap\Omega_\infty} u^{q+1}\,dxdt > \varepsilon_0, \quad \forall k \in [1,\infty)\,. \tag{17}$$

Similarly, given a boundary point $x_0 \in \partial\Omega$, x_0 belongs to $(\partial\Omega)_{\text{reg}} \subseteq \partial\Omega$ provided that, there exists $k_{x_0} \in [1,\infty)$ such that

$$\liminf_{j\to\infty} \int_{Q^{k_{x_0}}(x_0,\tau_j,r_0)\cap\Omega_\infty} u^{q+1}\,dxdt \leqq \varepsilon_0; \tag{18}$$

vice versa, $x_0 \in \partial\Omega\setminus(\partial\Omega)_{\text{reg}}$ means that

$$\liminf_{j\to\infty} \int_{Q^{k_{x_0}}(x_0,\tau_j,r_0)\cap\Omega_\infty} u^{q+1}\,dxdt > \varepsilon_0, \qquad \forall k \in [1,\infty). \tag{19}$$

Take δ_0^γ in Theorem 4 and write as $|Q(x_0,t_0,r_0)| = c(n)r_0^{n+p}$. We find that, if ε_0 is so small that $\varepsilon_0 \leqq \delta_0^\gamma c(n) r_0^{n+p}$, then Theorem 4 yields

$$\liminf_{j\to\infty} \|u\|_{L^\infty(Q^{4k_{x_0}}(x_0,\tau_j,r_0/2)\cap\Omega_\infty)} \leqq 4k_{x_0}\,, \qquad \forall x_0 \in \Omega_{\text{reg}}\,.$$

Similarly, when ε_0 is so small enough as above Theorem 4 gives that

$$\liminf_{j\to\infty} \|u\|_{L^\infty(Q^{4k_{x_0}}(x_0,\tau_j,r_0/2)\cap\Omega_\infty)} \leqq 4k_{x_0}\,, \qquad \forall x_0 \in (\partial\Omega)_{\text{reg}}\,.$$

Now we define as $E_{\text{int}} := \Omega\setminus\Omega_{\text{reg}}$ through Definition 6 and Theorem 4.
First, the volume constraint $(7)_2$ means that $\Omega\setminus\Omega_{\text{reg}}$ consists of at most $N := \lceil \varepsilon_0^{-1} \rceil$ many distinct points. Assume, on the contrary, that there are $N+1$ distinct points in $\Omega\setminus\Omega_{\text{reg}}$, say $\{x_i\}_{i=1}^{N+1}$. Then we obtain from (17) that there exists large enough k so that $\{Q^k(x_i,\tau_j,r_0)\}_{i=1}^{N+1}$ are disjoint and

$$\liminf_{j\to\infty} \int_{Q^{k}(x_i,\tau_j,r_0)\cap\Omega_\infty} u^{q+1}\,dxdt > \varepsilon_0\,.$$

Thus, by Fatou's lemma and the volume constraint $\|u(t)\|_{q+1} = 1$, we deduce that

$$(N+1)\varepsilon_0 < \sum_{i=1}^{N+1} \liminf_{j\to\infty} \int_{Q^{k}(x_i,\tau_j,r_0)\cap\Omega_\infty} u^{q+1}\,dxdt \leqq \liminf_{j\to\infty} \sum_{i=1}^{N+1} \int_{Q^{k}(x_i,\tau_j,r_0)\cap\Omega_\infty} u^{q+1}\,dxdt \leqq 1\,;$$

a contradiction. Therefore, there are at most N many distinct points in $\Omega\setminus\Omega_{\text{reg}}$.

Next, take any compact subset K of $\Omega \setminus E_{\text{int}} = \Omega_{\text{reg}}$. We may assume τ_j being so large that $(\tau_j - (r_0/2)^p, \tau_j) \subseteq (0, \infty)$. Since K is compact, we find sufficiently large $I_K \in \mathbb{N}$, finitely many points $\{y_i\}_{i=1}^{I_K} \subseteq K$ and numbers $k_{y_i} \geqq 1$, $i = 1, \ldots, I_K$, such that

$$K \times (\tau_j - (r_0/2)^p, \tau_j) \subseteq \bigcup_{i=1}^{I_K} Q^{4k_{y_i}}(y_i, \tau_j, r_0/2)(\subseteq \Omega_\infty) \tag{20}$$

and (16) with $x_0 := y_i$, $i = 1, \ldots I_K$, and thus, by Theorem 4,

$$\liminf_{j \to \infty} \|u\|_{L^\infty(Q^{4k_{y_i}}(y_i, \tau_j, r_0/2))} \leqq 4k_{y_i}\,. \tag{21}$$

Noting the inclusion (20) we obtain from (21) that

$$\liminf_{j \to \infty} \|u\|_{L^\infty(K \times (\tau_j - (r_0/2)^p, \tau_j))} \leqq 4 \max_{i \in I_K} k_{y_i} < \infty\,. \tag{22}$$

The estimation (22) provides the desired strong convergence on K and, since K can be taken arbitrarily, also strong convergence locally in $\Omega \setminus E_{\text{int}}$

$$\nabla u(\tau_j) \to \nabla u_\infty \qquad \text{strongly in } L^p_{\text{loc}}(\Omega \setminus E_{\text{int}}). \tag{23}$$

□

The next result renders the asymptotic profile at finite concentration points of the global solution of (7). As a consequence, the concentration of volume only appears at most finitely many points in the interior domain; in other words, the possibility of concentration points lying on the boundary is ruled out.

Theorem 7 (Volume concentration phenomenon)**.** *Let $\frac{2n}{n+2} \leqq p < n$, $q + 1 = p^*$. Let u be a nonnegative weak solution of (7). Fix $r_0 = 1 \wedge \lambda_0^{-1/p}$ with $\lambda_0 = \|\nabla u_0\|_p^p$. Let $\{t_j\}_{j \in \mathbb{N}}$ be a sequence of times so that $t_j \nearrow \infty$ as $j \to \infty$. Then, there exist a sequence $\{\tau_j\}_{j \in \mathbb{N}}$ depending on $\{t_j\}_{j \in \mathbb{N}}$ and satisfying $\tau_j \nearrow \infty$ as $j \to \infty$, a positive number λ_∞ and a set $E \subseteq \Omega$ of finitely many distinct points of singularities, say $\{x_i\}_{i=1}^N$, where $N := \lceil \varepsilon_0^{-1} \rceil$ and $\varepsilon_0 \leqq \left(\frac{S}{\lambda_\infty}\right)^{\frac{n}{p}}$ with S being the best constant of Sobolev's inequality on $\mathbb{R}^n$, and a sequence $\{\mu_i\}_{i=1}^N \subseteq \mathbb{R}_+$ such that we have, for any $i = 1, \ldots, N$, that*

$$\liminf_{j \to \infty} \int_{Q^{k_{x_i,j}}(x_i, \tau_j, r_0) \cap \Omega_\infty} u^{q+1}\, dxdt > \varepsilon_0 \tag{24}$$

whenever $k_{x_i,j} \in [1, \infty)$. Defining a sequence $\{k_{x_i,j}\}_{j \in \mathbb{N}}$ as $1 \leqq k_{x_i,j} \nearrow \infty$ as $j \to \infty$ and

$$v_{x_i,j}(x,t) := k_{x_i,j}^{-1} u\left(x_i + k_{x_i,j}^{(p-(q+1))/p} x,\ \tau_j + t\right)$$

we have

$$v_{x_i,j}(\cdot, 0) \quad \xrightarrow[j \to \infty]{} \quad \lambda_\infty^{-\frac{n-p}{p^2}} U\left[0, \lambda_\infty^{-1/p} \mu_i\right] \quad \text{in} \quad \mathcal{D}^{1,p}(\mathbb{R}^n), \tag{25}$$

where $U\left[0, \lambda_\infty^{-1/p} \mu_i\right]$ is the Talenti bubble having a peak at the origin $x = 0$.

We shall show the outline of proof.
Proof of Theorem 7. We shall treat the interior case.
Interior case. Let $E := \Omega \setminus \Omega_{\mathrm{reg}}$. By the proof of Theorem 5, E has only finitely many distinct points, that is, $E = \{x_i\}_{i=1}^N$ with $N := \lceil \varepsilon_0^{-1} \rceil$. Fix $i = 1, \dots N$ and $r_0 = 1 \wedge \lambda_0^{-1/p}$. Put $B^k(x_i, r_0) := B(x_i, r_0 k^{(p-(q+1)/p)})$. We may also assume τ_j so large that $(\tau_j - r_0^p, \tau_j) \subseteq (0, \infty)$. For $x_i \in E$, (17) implies that

$$\liminf_{j\to\infty} \int_{Q^k(x_i,\tau_j,r_0)\cap\Omega_\infty} u^{q+1}\,dxdt > \varepsilon_0\,, \qquad \forall k \in [1,\infty)\,. \tag{26}$$

By the mean-value theorem we can also take a number $\tau_j' \in (\tau_j - r_0^p, \tau_j)$ such that

$$\|u(\tau_j')\|^{q+1}_{L^{q+1}(B^k(x_i,r_0)\cap\Omega)} = \fint_{\tau_j - r_0^p}^{\tau_j} \|u(t)\|^{q+1}_{L^{q+1}(B^k(x_i,r_0)\cap\Omega)}\,dt$$

and thus, from (26) it follows that $\tau_j - r_0^p < \tau_j' < \tau_j$ and

$$\|u(\tau_j')\|^{q+1}_{L^{q+1}(B(x_i,r_0)\cap\Omega)} \geqq \|u(\tau_j')\|^{q+1}_{L^{q+1}(B^k(x_i,r_0)\cap\Omega)} > \varepsilon_0/r_0^p. \tag{27}$$

Subtracting a further subsequence, we may assume that $\tau_j' \nearrow \infty$ as $j \to \infty$.
Let y be any point in Ω and s be so large number that $(s - r_0^p, s) \subseteq (0, \infty)$. By the absolute continuity of integral, there holds for δ_0^γ in Theorem 4, and for sufficiently large $k(y,s) \in [1,\infty)$ that

$$\frac{1}{|Q(y,s,r_0)|} \int_{Q^{k(y,s)}(y,s,r_0)\cap\Omega_\infty} u^{q+1}\,dxdt \leqq \delta_0^\gamma$$

and then, letting $k_0 = k(y,s)$ and $t_0 = s$, Theorem 4 entails that

$$\sup_{Q^{4k(y,s)}(y,s,r_0/2)\cap\Omega_\infty} u \leqq 4k(y,s). \tag{28}$$

Since $\overline{Q(x_i,\tau_j',r_0)\cap\Omega_\infty}$ is compact, we have finitely many points $\{(y_l,s_l)\}_{l=1}^L \subseteq Q(x_i,\tau_j',r_0)\cap\Omega_\infty$ with a finite $L = L(x_i,\tau_j') \in \mathbb{N}$ such that $\overline{Q(x_i,\tau_j',r_0)\cap\Omega_\infty} \subseteq \bigcup_{l=1}^L Q^{4k(y_l,s_l)}(y_l,s_l,r_0/2)\cap\Omega_\infty$. Thus, by (28) we have the boundedness as

$$\|u\|_{L^\infty(Q(x_i,\tau_j',r_0)\cap\Omega_\infty)} \leqslant 4 \max_{l=1,\dots,L(x_i,\tau_j')} k(y_l,s_l) =: 4k_{x_i,j}. \tag{29}$$

Since $k(y_l,s_l)$ can be chosen as arbitrarily large. we may assume that $k_{x_i,j} \nearrow \infty$ as $j \to \infty$.
Letting the scaled functions

$$v_{x_i,j}(x,t) := k_{x_i,j}^{-1} u\big(x_i + k_{x_i,j}^{(p-(q+1))/p} x,\ \tau_j' + t\big), \qquad \nu_j(t) := \lambda(\tau_j' + t) = \int_\Omega \left|\nabla u(\tau_j' + t)\right|^p\,dx\,,$$

$v_{x_i,j}$ is a weak solution to

$$\partial_t v_{x_i,j}^q - \Delta_p v_{x_i,j} = \nu_j v_{x_i,j}^q \quad \text{in } \mathcal{Q}_{x_i,j} := \mathcal{B}_{x_i,j} \times (-r_0^p, 0)$$

with $\mathcal{B}_{x_i,j} := k_{x_i,j}^{(q+1-p)/p}((B(x_i, r_0) \cap \Omega) - x_i)$. By straightforward computations, it is clear that

$$\nu_j(t) = \int_\Omega |\nabla u(\tau'_j + t)|^p \, dx = \int_{\Omega_{x_i,j}} |\nabla v_{x_i,j}(t)|^p \, dx \tag{30}$$

with $\Omega_{x_i,j} := k_{x_i,j}^{(q+1-p)/p}(\Omega - x_i)$ and, (29) implies that

$$\|v_{x_i,j}\|_{L^\infty(\mathcal{Q}_{x_i,j})} \leqslant 4\,. \tag{31}$$

Moreover, one can check that,

$$\|v_{x_i,j}(t)\|_{L^{q+1}(\mathcal{B}_{x_i,j})} = \|u(\tau'_j + t)\|_{L^{q+1}(B(x_i,r_0)\cap\Omega)} \tag{32}$$

and

$$\left\|\partial_t v_{x_i,j}^{\frac{q+1}{2}}(t)\right\|_{L^2(\mathcal{B}_{x_i,j})} = \left\|\partial_t u^{\frac{q+1}{2}}(\tau'_j + t)\right\|_{L^2(B(x_i,r_0)\cap\Omega)}\,. \tag{33}$$

The family $\{\mathcal{B}_{x_i,j}\}_{j\in\mathbb{N}}$ of domains is enlarging, so that $\mathcal{B}_{x_i,j} \nearrow \mathbb{R}^n$ as $j \to \infty$. Similarly as the convergences and (23) in the proof of Theorem 5, by (30) there exist a limit function $w^{(i)} \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ and a limit number λ_∞ so that, we have the weak convergence in $W^{1,p}(\mathbb{R}^n)$ of $\{v_{x_i,j}(0)\}$ to $w^{(i)}$, the strong ones in some Lebesgue space on $\mathbb{R}^n$ of $\{v_{x_i,j}(0)\}$, $\{\nabla v_{x_i,j}(0)\}$ to $w^{(i)}$, $\nabla w^{(i)}$, respectively, and the convergence of $\{\nu_j(0)\}$ to λ_∞

By construction and the argument as in Theorem 5, the limit number λ_∞ is independent of $i \in \{1, \ldots, N\}$. We can use (33) and (15), and argue similarly as the proof of (15), we have for every $\varphi \in C_0^\infty(\mathcal{B}_{x_i,j})$ that $\left|\int_{\mathbb{R}^n} \partial_t v_{x_i,j}^q(0)\varphi\, dx\right| \to 0$ as $j \to \infty$.

Gathering the convergences above, we find that the limit function $w^{(i)}$ is a nonnegative weak solution of the stationary equation $-\mathrm{div}\left(|\nabla w^{(i)}|^{p-2}\nabla w^{(i)}\right) = \lambda_\infty (w^{(i)})^q$ in $\mathbb{R}^n$.

Similarly as in the proof of Theorem 5, the local boundedness (31) yields the strong convergences of $\{v_{x_i,j}(0)\}$ in $L^\gamma(\mathbb{R}^n)$ with $\forall\gamma \geqslant 1$, and $\{\nabla v_{x_i,j}(0)\}$ in $L^p(\mathbb{R}^n)$, respectively.

By the regularity results of the critical p-Laplace equation [Theorem 2.2, *P. Pucci and R. Servadei, Indiana Univ. Math. J. 57 (2008)*] and [Corollary, p. 830, *E. DiBenedetto, Nonlinear Anal. 7 (1983)*], $w^{(i)}$ is $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$-regular. In addition, by (27) with (32), and the strong convergences above, $w^{(i)}$ is not identically zero in $\mathbb{R}^n$. This, together with the strong maximum principle in [*J. L. Vázquez, Appl. Math. Optim. 12, (3) (1984)*] implies that $w^{(i)}$ is strictly positive in $\mathbb{R}^n$. By the fundamental result on the extremal function attaining the best constant of Sobolev inequality on $\mathbb{R}^n$ [*B. Sciunzi, Adv. Math. 291 (2016)*], $w^{(i)}$ has the form of $w^{(i)}(x) = \lambda_\infty^{(p-n)/p^2} U\left[0, \lambda_\infty^{-1/p}\mu_i\right]$ for some $\mu_i > 0$.

Boundary case. Finally, we consider the boundary singular point. Let $x_0 \in E' := \partial\Omega \setminus (\partial\Omega)_{\mathrm{reg}}$. A very similar argument leading to (29) in the interior case can be applied with the region $\mathbb{R}^n$ replaced by $\mathbb{R}^n_+$ to show that the limit function $w \in \mathcal{D}_0^{1,p}(\mathbb{R}^n_+)$ of the scaled-solutions $\{v_{x_0,j}(0)\}$ is a nonnegative weak solution of $-\mathrm{div}\left(|\nabla w|^{p-2}\nabla w\right) = \tilde{\lambda}_\infty w^q$ in $\mathbb{R}^n_+$ with $w = 0$ in $\mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$ for some positive number $\tilde{\lambda}_\infty$. Adapting (27) with (32) those on $B(x_0, r_0)\cap\Omega$, w is not identically zero in $\mathbb{R}^n_+$. From the Liouville type result in [Theorem 1.1, *C. Mercuri, M. Willem, Discrete contin. Dyn. Syst. A 28 (2010)*], w should be trivial, which contradicts to the above. Thus, $E' = \varnothing$. □

Acknowledgements. The auhor would like to express my sincere thanks to Professors Ohwada and Matuoka for giving me the nice chance to talk my recent result on the $p-$Sobolev flow.