FROM RESONANCE TO HYPOTHESIS S: A BRIEF SURVEY

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Abstract. This article is a brief survey, based on a talk given at the Research Institute for Mathematical Sciences (RIMS), Kyoto University, in January 2026, on several efforts toward Hypothesis S of Iwaniec, Luo, and Sarnak. The point of departure is Vinogradov's exponential sum over primes and its later refinements by zero-density methods. We then discuss Hypothesis S, its relation to low-lying zeros of automorphic L-functions, and the obstruction caused by resonance between automorphic Fourier coefficients and fractional exponential functions. The survey also recalls the role of Voronoi summation formulas and their asymptotic expansions, and ends with recent results in which the modular form is allowed to vary.

1. Introduction

A recurring theme in analytic number theory is to estimate exponential sums in which arithmetic coefficients are tested against oscillatory phases. One classical example is Vinogradov's exponential sum over primes,

$$\sum_{p\le X} e(\alpha\sqrt{p}),$$

where $e(x)=e^{2\pi ix}$, $\alpha\neq 0$ is fixed and X is large. Another is the automorphic analogue

$$\sum_{n} a_n e(\alpha n^{\beta})\phi\Big(\frac{n}{X}\Big), \tag{1.1}$$

where a_n is a sequence of Fourier coefficients, $0<\beta<1$, and $\phi\in C_c^\infty(1,2)$ is a smooth weight.

The motivation for this brief survey is Hypothesis S, introduced by Iwaniec, Luo, and Sarnak [7] in their study of low-lying zeros of families of automorphic L-functions. In a simplified form, Hypothesis S asks for cancellation beyond the exponent $3/4$ in the sum (1.1), for an arithmetic sequence a_n. Such bounds would enlarge the support in one-level density calculations and would have consequences for quasi-Riemann-hypothesis type results for holomorphic cusp form L-functions.

The purpose here is not to give a comprehensive history, but to explain the main line of thought from Vinogradov's estimate, through zero-density methods and Hypothesis S, to the resonance phenomena which impose natural barriers on some automorphic analogues. The article also reviews how Voronoi summation formulas and their asymptotic expansions provide the main technical mechanism behind resonance results, and how changing the modular form may offer a possible route around the resonance obstruction.

2. Vinogradov's exponential sum

In 1976 Vinogradov [19] proved the estimate

$$\sum_{p\le X} e(\alpha\sqrt{p}) \ll X^{7/8+\varepsilon} \tag{2.1}$$

for fixed $\alpha \neq 0$. In a dyadic form one may consider

$$S_X(\alpha, \beta) = \sum_{X<n\leq 2X} \Lambda(n) e(\alpha n^\beta),$$

where Λ is the von Mangoldt function. Vinogradov's estimate corresponds to $\beta = 1/2$.

Ren [13] improved Vinogradov's bound (2.1) in the square-root case to

$$S_X\Big(\alpha, \frac{1}{2}\Big) \ll X^{4/5} \log^{11} X,$$

and more generally proved, for $0 < \beta \leq 1/2$,

$$S_X(\alpha, \beta) \ll \Big(X^{3/4+\beta/10} + X^{1-\beta/2}\Big) \log^{11} X.$$

The proof uses analytic methods, including zero-density estimates for the Riemann zeta-function.

Let $N(\sigma, T)$ denote the number of zeros of $\zeta(s)$ in the region

$$\sigma < \operatorname{Re} s < 1, \qquad |\operatorname{Im} s| \leq T.$$

A zero-density estimate has the form

$$N(\sigma, T) \ll T^{B(\sigma)} \log^E T, \qquad \frac{1}{2} \leq \sigma < 1,$$

for a suitable function $B(\sigma)$ and constant $E > 0$. The zero-density hypothesis predicts the sharp exponent

$$B(\sigma) = 2(1 - \sigma).$$

Under this hypothesis, Iwaniec, Luo, and Sarnak [7] pointed out, and Ren [13] proved, that

$$S_X\Big(\alpha, \frac{1}{2}\Big) \ll X^{3/4} \log^E X.$$

Thus the exponent 3/4 naturally appears already in the prime exponential-sum problem.

As an application, put

$$A(\delta, X) = \#\{p \leq X : \{\alpha\sqrt{p}\} < \delta\}, \qquad 0 < \delta \leq 1.$$

Vinogradov's work [19] implies

$$A(\delta, X) = \delta\pi(X) + O(X^{9/10+\varepsilon}),$$

whereas Ren's estimates [13] give the improvement

$$A(\delta, X) = \delta\pi(X) + O(X^{5/6} \log^E X).$$

These estimates show how cancellation in fractional exponential sums over primes yields uniform distribution information for the fractional parts $\{\alpha\sqrt{p}\}$.

3. Hypothesis S

Iwaniec, Luo, and Sarnak [7, (1.62)] proposed the following hypothesis in their work on low-lying zeros of families of L-functions.

Conjecture 3.1 (Hypothesis S, prime version)**.** *Let $(a, c) = 1$. One expects*

$$\sum_{\substack{p\leq X \\ p\equiv a \,(\mathrm{mod}\, c)}} e\Big(\frac{2\sqrt{p}}{c}\Big) \ll_\varepsilon X^{\alpha+\varepsilon},$$

with $\alpha = 1/2$, or at least with some $\alpha < 3/4$.

They also formulated a more general version of Hypothesis S for automorphic coefficients ([7, Appendix C]). Let

$$S_q(X) = \sum_n a_n e(-2\sqrt{nq})\phi\Big(\frac{n}{X}\Big),$$

where $a_n \ll n^\varepsilon$ is an arithmetically defined sequence, $q \in \mathbb{Z}^+$, and $\phi \in C_c^\infty(1,2)$. A guiding question is whether one can obtain cancellation beyond the bound $O(X^{3/4})$. A fundamental example is

$$a_n = \lambda_f(n),$$

where $\lambda_f(n)$ denotes the normalized Fourier coefficient of a holomorphic cusp form f for $\mathrm{SL}_2(\mathbb{Z})$.

The relevance of Hypothesis S is that it would enlarge the crucial support in the study of the statistical distribution of low-lying zeros of a family of automorphic L-functions attached to holomorphic cusp forms. One consequence described by Iwaniec, Luo, and Sarnak is a quasi-Riemann-hypothesis statement: assuming Hypothesis S and assuming that the zeros of $L(s,f)$ for a holomorphic cusp form f for $\mathrm{SL}_2(\mathbb{Z})$ of weight k are either real or lie on the critical line $\operatorname{Re} s = 1/2$, then for k sufficiently large one obtains

$$L(s,f) \neq 0 \qquad \text{if } s > \frac{10}{11} + \varepsilon.$$

4. Resonance and the $X^{3/4}$ barrier

The automorphic version of Hypothesis S immediately encounters a phenomenon that is absent from a purely random model: resonance. Iwaniec, Luo, and Sarnak [7, Appendix C] discovered that for $a_n = \lambda_f(n)$ the sum $S_q(X)$ has a main term of size

$$q^{-1/4}\lambda_f(q)X^{3/4}.$$

This main term prevents one from proving, in general, a bound smaller than $O(X^{3/4})$ for fixed f.

Ren and Ye [14] studied this resonance phenomenon systematically. Let f be a fixed holomorphic cusp form of weight k for $\mathrm{SL}_2(\mathbb{Z})$, and define

$$S_X(f,\alpha,\beta) = \sum_n \lambda_f(n) e(\alpha n^\beta)\phi\Big(\frac{n}{X}\Big),$$

where $\alpha \in \mathbb{R}^\times$ and $0 < \beta < 1$. Their results show a sharp distinction between the resonant case $\beta = 1/2$ and the non-resonant cases. In particular, if $\beta \neq 1/2$, then

$$S_X(f,\alpha,\beta) \ll X^{\beta+\varepsilon} + X^{1/2-\beta/4+\varepsilon}.$$

When $\beta = 1/2$, a main term appears if $|\alpha|$ is close to $2\sqrt{q}$ for some $q \in \mathbb{Z}^+$. More precisely, if

$$\big||\alpha| - 2\sqrt{q}\big| < X^{-1/2},$$

then

$$S_X\Big(f,\alpha,\frac{1}{2}\Big) = cq^{-1/4}\lambda_f(q)X^{3/4} + O(X^{3/8+\varepsilon}),$$

where c is an explicit constant depending on the smooth weight and the sign of α. If no such q exists, then

$$S_X\Big(f,\alpha,\frac{1}{2}\Big) \ll X^{3/8+\varepsilon}.$$

Thus resonance occurs only at precise "frequencies".

Ren and Ye [16] extended the analysis to a Maass form f for $\mathrm{SL}_3(\mathbb{Z})$ with Fourier coefficients $A_f(m,n)$. In this setting one obtains the bounds

$$S_X(f,\alpha,\beta) \ll X^{3\beta/2}\log X \qquad (\beta \neq 1/3),$$

while the resonant exponent is $\beta = 1/3$. At $\alpha = 3q^{1/3}$ one obtains a main term of size

$$cX^{2/3} + O(X^{1/3+\varepsilon}),$$

and away from such resonant values one has

$$S_X\Big(f,\alpha,\frac{1}{3}\Big) \ll X^{1/3+\varepsilon}.$$

This breaks the $X^{3/4}$ barrier in the GL_3 setting and identifies the new resonance frequency $e(3(qn)^{1/3})$.

The same pattern continues for higher rank (Ren and Ye [17]). For a full-level cusp form on $\mathrm{GL}_m(\mathbb{Z})$, resonance occurs when

$$\beta = \frac{1}{m}, \qquad \alpha \approx \pm mq^{1/m},$$

with a main term of size

$$X^{1/2+1/(2m)}.$$

Czarnecki [2] extended these results to Rankin–Selberg products of Maass forms on $\mathrm{SL}_m(\mathbb{Z})$ and $\mathrm{SL}_{m'}(\mathbb{Z})$. In that case resonance appears at

$$\beta = \frac{1}{mm'}, \qquad \alpha \approx \pm mm'q^{1/(mm')},$$

with main terms of size

$$X^{1/2+1/(2mm')}.$$

More recently, Gillespie [5] extended such bounds and main-term formulas to isobaric sums

$$\pi_1 \boxplus \cdots \boxplus \pi_k$$

of self-dual automorphic representations π_j of $\mathrm{GL}_{n_j}(\mathbb{A}_{\mathbb{Q}})$, with $m = n_1+\cdots+n_k$. These results also apply to self-dual automorphic representations over certain Galois extensions of $\mathbb{Q}$ when automorphic induction to $\mathbb{Q}$ is available. Such induction was proved by Arthur and Clozel [1] for towers of cyclic extensions of prime degrees and is expected in greater generality by Langlands functoriality.

The lesson is that the expected $O(X^{1/2+\varepsilon})$ square-root cancellation is obstructed, for fixed automorphic forms, by genuine main terms. Any approach toward Hypothesis S must therefore distinguish carefully between non-resonant automorphic sums and resonant automorphic sums.

5. Voronoi summation and asymptotic expansions

The proofs of the resonance results rely on Voronoi summation formulas and on asymptotic expansions of their integral kernels. Voronoi's original summation formula involving quadratic forms appeared in 1904. Wilton [20] later generalized it to holomorphic cusp forms. If f is a holomorphic cusp form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ with Fourier coefficients $\lambda_f(n)$, then a typical Voronoi formula has the shape

$$\sum_{n\geq 1} \lambda_f(n) e\Big(\frac{nx}{c}\Big)\phi(n) = \sum_{r\geq 1} \lambda_f(r) e\Big(-\frac{r\bar{x}}{c}\Big)\Phi(r),$$

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where $x\bar{x} \equiv 1 \pmod{c}$ and

$$\Phi(y) = \frac{2\pi i^k}{c} \int_0^\infty \phi(t) J_{k-1}\Big(\frac{4\pi\sqrt{ty}}{c}\Big)\, dt.$$

The transformation exchanges the original additive twist with a dual additive twist and a Bessel integral transform. The resonant main term is ultimately produced by stationary phase in this transformed expression.

For higher rank groups, Miller and Schmid [11] established Voronoi summation for full-level automorphic forms on GL_3 over $\mathbb{Q}$, and Miller–Schmid [12] and Goldfeld–Li [3, 4] developed the general GL_n theory. In these formulas the additive character on the dual side is replaced by Kloosterman or hyper-Kloosterman sums.

Ichino and Templier [6] gave an adelic treatment of the general Voronoi summation formula for irreducible cuspidal automorphic representations of GL_n over arbitrary number fields. Gillespie [5] provides a simplified version without additive twists.

For applications to resonance one needs not only the summation formula but also an asymptotic expansion of its integral kernel. In the classical case such asymptotics go back to work of Ivić [8]. For the GL_3 Miller–Schmid formula, the needed asymptotic expansion was proved by Ren and Ye [15]. The corresponding GL_n expansion was later developed in the same line of work (Ren and Ye [18]).

The connection with the functional equation is central. For a finite-part automorphic L-function $L(s,\pi)$ attached to a cuspidal representation π of GL_n over $\mathbb{Q}$, the functional equation may be written in the form

$$L(s,\pi) = \tau(\pi) Q_\pi^{-s} \frac{\prod_{j=1}^n \Gamma_{\mathbb{R}}(1-s+\overline{\mu_\pi(j)})}{\prod_{j=1}^n \Gamma_{\mathbb{R}}(s+\mu_\pi(j))} L(1-s,\widetilde{\pi}), \tag{5.1}$$

where $Q_\pi > 0$ is the conductor, $\tau(\pi)\tau(\widetilde{\pi}) = Q_\pi$, $\widetilde{\pi}$ is the contragredient representation, and

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big).$$

The ratio of gamma factors in (5.1) leads to Mellin–Barnes integrals of the form

$$\int_L \frac{\prod_{j=1}^n \Gamma_{\mathbb{R}}(1-s+\overline{\mu_\pi(j)})}{\prod_{j=1}^n \Gamma_{\mathbb{R}}(s+\mu_\pi(j))} x^s\, ds,$$

where the contour L separates the relevant poles. Czarnecki [2] observed that such integrals can be expressed as the Meijer G-functions,

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\middle| x\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j - s)\prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{j=m+1}^q \Gamma(1-b_j+s)\prod_{j=n+1}^p \Gamma(a_j - s)} x^s\, ds.$$

Known asymptotic expansions of the Meijer G-functions, due for example to Luke [9, 10], then give asymptotics for the Voronoi kernel. Czarnecki [2] applied this to Rankin–Selberg products, and Gillespie [5] applied the same framework to the Ichino–Templier Voronoi formula [6] in the setting of isobaric sums of automorphic representations.

6. Changing forms

The resonance results discussed above concern fixed modular forms or fixed automorphic representations. A natural way to try to overcome resonance is to let the cusp form vary. The guiding idea is that while an individual form may resonate with a special phase, resonance should be much harder to sustain uniformly across a family of forms.

Let H_k be an orthonormal basis consisting of Hecke eigenforms of weight k for $\mathrm{SL}_2(\mathbb{Z})$, and let $h(t) = \operatorname{sech}(t)$. For

$$S_X(f,\alpha,\beta) = \sum_n \lambda_f(n) e(\alpha n^\beta)\phi\Big(\frac{n}{X}\Big),$$

consider the weighted second moment

$$\sum_{2|k} h\Big(\frac{k-K}{L}\Big) \sum_{f\in H_k} |S_X(f,\alpha,\beta)|^2. \tag{6.1}$$

The following results by Ye [21] are representative estimates in this direction.

Theorem 6.1. *Let $K^\varepsilon \le L \le K^{1-\varepsilon}$, $\alpha \in \mathbb{R}^\times$, and $0<\beta<1$. Then*

$$\begin{aligned}
&\sum_{2|k} h\Big(\frac{k-K}{L}\Big) \sum_{f\in H_k} |S_X(f,\alpha,\beta)|^2 \\
&\ll KLX^{1+\varepsilon}, && \text{if } KL \ge X^{1+\varepsilon};\\
&\ll KLX^{1+\varepsilon} + \Big(\frac{L}{K}\Big)^{1/2} X^{2+\varepsilon}, && \text{if } K \ge X^{(1+\beta)/2+\varepsilon};\\
&\ll KLX^{1+\varepsilon} + X^{2-\beta/2+\varepsilon}, && \text{if } KL \ge X^{\beta+\varepsilon} \text{ and } X^\beta \ge \frac{K^{1+\varepsilon}}{L}.
\end{aligned}$$

Since the trivial bound for $S_X(f,\alpha,\beta)$ is $O(X^{1+\varepsilon})$, the first estimate in Theorem 6.1 does not by itself give a nontrivial bound for each individual f. It does, however, show square-root cancellation on average, which supports the belief that allowing $k\to\infty$ may remove the fixed-form resonance obstruction.

One also obtains individual consequences from the last two averaged estimates in Theorem 6.1.

Theorem 6.2. *Let $\alpha\in\mathbb{R}^\times$ and $0<\beta<1$. For all $f\in H_k$ with $K-L\le k\le K+L$,*

$$\begin{aligned}
S_X(f,\alpha,\beta) &\ll (KL)^{1/2}X^{1/2+\varepsilon} + \Big(\frac{L}{K}\Big)^{1/4} X^{1+\varepsilon}, && \text{if } K \ge X^{(1+\beta)/2+\varepsilon};\\
&\ll (KL)^{1/2}X^{1/2+\varepsilon} + X^{1-\beta/4+\varepsilon}, && \text{if } KL \ge X^{\beta+\varepsilon} \text{ and } X^\beta \ge \frac{K^{1+\varepsilon}}{L}.
\end{aligned}$$

The best bound obtainable from these cases is of the shape

$$S_X(f,\alpha,\beta) \ll X^{5/6+\varepsilon}$$

for certain ranges of β and $k \asymp X^\mu$ with $0<\mu<1$. Such estimates appear to be the first nontrivial bounds for $S_X(f,\alpha,\beta)$ in settings where the form f is allowed to move.

We briefly recall the proof mechanism. Expanding the square in (6.1) gives a sum over k, m, n and f. Applying the Petersson formula to the f-sum yields a diagonal term and an off-diagonal term:

$$\sum_f \lambda_f(m)\lambda_f(n) = \delta(m,n) + 2\pi i^k \sum_{c\ge1} \frac{S(m,n;c)}{c} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

where

$$S(m,n;c) = \sideset{}{^*}\sum_{z\,(\mathrm{mod}\,c)} e\Big(\frac{mz+n\bar{z}}{c}\Big).$$

The diagonal term gives the first term in Theorem 6.1. For the off-diagonal term, one uses an asymptotic expansion for the k-sum involving Bessel functions, after which the c-sum is effectively restricted to

$$1 \le c \le \frac{X}{K^{1-\varepsilon}L}.$$

If $KL \ge X^{1+\varepsilon}$ this range is empty, and there is no off-diagonal contribution.

When off-diagonal terms remain, the Kloosterman sum is unfolded and Poisson summation is applied in both m and n. One obtains double oscillatory integrals. A two-dimensional Riemann–Lebesgue lemma and first-derivative estimates show that all but finitely many dual frequencies are negligible; in fact it suffices to treat at most nine terms with $|m| \le 1$ and $|n| \le 1$. These remaining integrals are bounded by a two-dimensional second-derivative test, completing the proof of the averaged estimates.

7. Concluding remarks

Hypothesis S remains a compelling problem because it connects exponential sums over primes, low-lying zeros, and deep cancellation phenomena in families of automorphic L-functions. Bounds beyond $X^{3/4}$ should be possible and would have significant applications. At the same time, automorphic resonance shows that one must be careful: for fixed automorphic forms, the exponent $3/4$ may be forced by a genuine main term.

The developments surveyed here suggest three complementary directions. First, one may seek sharper prime exponential-sum estimates, perhaps through improved zero-density estimates or other analytic input. Second, one may classify resonant and non-resonant phases for automorphic coefficients using Voronoi summation and asymptotic analysis. Third, one may study varying families of forms, where averaging over the family may weaken or eliminate the resonance obstruction.

It is worth emphasizing, however, that the principal unresolved difficulties appear to lie in the GL(1) and GL(2) settings, where the resonance phenomenon makes the exponent $3/4$ a genuine barrier in the current methods. For cusp forms and automorphic representations on higher-rank groups, the results surveyed here already yield estimates that break the $X^{3/4}$ barrier. Another interesting direction for future work is to explore applications of these stronger bounds for higher-rank automorphic forms, where the improvement beyond the $3/4$ barrier may lead to new results in analytic number theory.

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