SCHUR EISENSTEIN SERIES AND QUASIMODULAR FORMS

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Abstract. In this survey article, we summarize some results of [BY, Yu] and introduce Schur multiple Eisenstein series, which are defined by Schur-type sums over semi-standard Young tableaux. To study them, we use the Young tableaux Hopf algebra and the linearization map to the quasi-shuffle algebra. Furthermore, we explain the Jacobi–Trudi formula on the Young tableaux Hopf algebra. In the case where all entries are equal to 2, the Schur Eisenstein series are quasimodular forms and span the ring of quasimodular forms. The Schur–MacMahon q-series, appearing in the Fourier expansion of Schur multiple Eisenstein series, are also discussed. In joint work with H. Bachmann, we study their relation to the algebraic structure of quasimodular forms with integer coefficients.

1. Introduction

The purpose of this note is to explain the relation between Schur multiple Eisenstein series (SMES) and quasimodular forms. For $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ write $q = e^{2\pi i\tau}$, and for even $k \geq 2$ let

$$\mathbb{G}(k) = \mathbb{G}(k;\tau) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m\tau+n)^k}$$

be the *Eisenstein series*. Here, when $k = 2$ we use the Eisenstein summation. Let

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n \in \mathbb{Z}[[q]], \quad \sigma_{k-1}(n) = \sum_{d|n} d^{k-1},$$

be the normalized *Eisenstein series*, where B_k is the k-th Bernoulli number. For even $k \geq 4$, the E_k are modular forms. The Eisenstein series E_2 is not a modular form. Including E_2 leads to the notion of *quasimodular forms*, which appear throughout mathematics and physics, for instance, in enumerative geometry, the theory of partitions, and string theory. The notion was introduced by Kaneko and Zagier [KZ] in the context of counting ramified covers of elliptic curves. The ring of quasimodular forms is $\widetilde{\mathcal{M}} = \mathbb{Q}[\mathbb{G}(2), \mathbb{G}(4), \mathbb{G}(6)] \subset \mathbb{C}[[q]]$, which is the extension of the ring of modular forms $\mathcal{M} = \mathbb{Q}[\mathbb{G}(4), \mathbb{G}(6)] \subset \mathbb{C}[[q]]$ closed under the derivative $D = q\frac{d}{dq}$.

Set $G(2k) := (2\pi i)^{-2k}\mathbb{G}(2k) = -\frac{B_{2k}}{2\,(2k)!}E_{2k}(\tau) \in \mathbb{Q}[[q]]$. We denote the spaces of modular forms and quasimodular forms with rational coefficients by

$$\mathcal{M}_{\mathbb{Q}} = \mathbb{Q}[G(4), G(6)] \subset \mathbb{Q}[[q]] \quad \text{and} \quad \widetilde{\mathcal{M}_{\mathbb{Q}}} = \mathbb{Q}[G(2), G(4), G(6)] \subset \mathbb{Q}[[q]].$$

Multiple Eisenstein series give a connection between modular forms and multiple zeta values. These series were introduced by Gangl, Kaneko and Zagier [GKZ] and developed by Bachmann [Ba]. They are defined by replacing the sum over one lattice point with an ordered sum over several lattice points:

$$\mathbb{G}(s_1,\dots,s_r)=\mathbb{G}(s_1,\dots,s_r;\tau)=\sum_{\substack{0\prec\lambda_1\prec\cdots\prec\lambda_r\\ \lambda_i\in\mathbb{Z}\tau+\mathbb{Z}}}\frac{1}{\lambda_1^{s_1}\cdots\lambda_r^{s_r}}\quad(\tau\in\mathbb{H};s_i\in\mathbb{N}_{\geq 2},\ i=1,\dots,r),$$

In [Yu], Schur versions of these series are constructed. Instead of summing over ordered lattice points, we sum over semi-standard Young tableaux whose entries are lattice points.

A *partition* of $n\in\mathbb{N}$ is a tuple $\lambda=(\lambda_1,\dots,\lambda_l)$ of positive integers satisfying

$$\lambda_1\geq\cdots\geq\lambda_l\geq 1,\qquad |\lambda|:=\lambda_1+\cdots+\lambda_l=n.$$

Here, we also allow for the empty partition $\varnothing$, with $|\varnothing|=0$. When comparing two partitions, we extend them by adding zero parts at the end if necessary. Let λ and μ be partitions. Write $\mu\subseteq\lambda$ if $0\leq\mu_i\leq\lambda_i$ for all i. For $\mu\subseteq\lambda$, the *(skew) Young diagram* of shape λ/μ is

$$D(\lambda/\mu):=\left\{(i,j)\in\mathbb{Z}^2\,\middle|\,1\leq i\leq\mathrm{dep}(\lambda),\ \mu_i<j\leq\lambda_i\right\}.$$

Here we use the convention that $\mu_i=0$ if $i>\mathrm{dep}(\mu)$. If $\mu=\varnothing$, or equivalently, if all parts of μ are zero, then we write $D(\lambda):=D(\lambda/\varnothing)$ and call it a *non-skew Young diagram.* In general, when μ is nonempty, $D(\lambda/\mu)$ is called a *skew Young diagram.* We refer to λ/μ as a *skew shape* and to λ as a *non-skew shape.*

Let $\mathcal{A}$ be a set, and λ/μ be a skew shape. A Young tableau $\mathbf{h}$ of shape λ/μ with entries in $\mathcal{A}$ is a collection $(h_{i,j})_{(i,j)\in D(\lambda/\mu)}$. For such a tableau, we write $\mathrm{sh}(\mathbf{h})=\lambda/\mu$. Later, we write $\eta\subseteq\mathrm{sh}(\mathbf{h})$ to mean that there exists a partition η' with $\mu\subseteq\eta'\subseteq\lambda$ such that $\eta=\eta'/\mu$. The notation $\mathrm{YT}(\lambda/\mu,\mathcal{A})$ denotes the set of all *Young tableaux* with entries in $\mathcal{A}$:

$$\mathrm{YT}(\lambda/\mu,\mathcal{A})=\{(m_{i,j})_{(i,j)\in D(\lambda/\mu)}\mid m_{i,j}\in\mathcal{A}\}.$$

$\mathrm{YT}(\mathcal{A})$ denotes the set of all skew Young tableaux with entries in $\mathcal{A}$. Assume $\mathcal{A}$ is a totally ordered set with order $<$. A Young tableau $m=(m_{i,j})_{(i,j)\in D(\lambda/\mu)}\in\mathrm{YT}(\lambda/\mu,\mathcal{A})$ is a *semi-standard Young tableau* if $m_{i,j}\leq m_{i,j+1}$ and $m_{i,j}<m_{i+1,j}$ for every $(i,j)\in D(\lambda/\mu)$. We use $\mathrm{SSYT}(\lambda/\mu,\mathcal{A})$ to denote the set of all semi-standard Young tableaux with entries in $\mathcal{A}$.

To define Schur multiple Eisenstein series, set $\mathbb{Z}_M=\{-M,\dots,M\}$. For $M,N\geq 1$ and $\tau\in\mathbb{H}$, consider the set $X^\tau_{M,N}=\mathbb{Z}_M\tau+\mathbb{Z}_N$. This gives a finite ordered set $\mathcal{X}^\tau_{M,N}=(X^\tau_{M,N},\prec)$. Here we consider the order $m_1\tau+n_1\prec m_2\tau+n_2:\Leftrightarrow m_1<m_2\vee(m_1=m_2\wedge n_1<n_2)$. Next, we consider the subset of “positive” lattice points defined as

$$X^{\tau,>0}_{M,N}=\{\lambda\in X^\tau_{M,N}\mid 0\prec\lambda\}.$$

This gives another finite ordered set $\mathcal{X}^{\tau,>0}_{M,N}=(X^{\tau,>0}_{M,N},\prec)$.

Definition 1.1. The *(truncated) Schur multiple Eisenstein series* for $\mathbf{h}\in\mathrm{YT}(\lambda/\mu,\mathbb{N})$ is

$$\mathbb{G}_{M,N}(\mathbf{h};\tau)=\sum_{(m_{i,j}\tau+n_{i,j})\in\mathrm{SSYT}(\lambda/\mu,\mathcal{X}^{\tau,>0}_{M,N})}\ \prod_{(i,j)\in D(\lambda/\mu)}\frac{1}{(m_{i,j}\tau+n_{i,j})^{h_{i,j}}}.$$

Proposition 1.2 ([Yu])**.** *If $h_{i,j}\geq 2$ for all $(i,j)\in D(\lambda/\mu)$, then the following limit exists*

$$\mathbb{G}(\mathbf{h})=\mathbb{G}(\mathbf{h};\tau):=\lim_{M\to\infty}\lim_{N\to\infty}\mathbb{G}_{M,N}(\mathbf{h};\tau).$$

A single-column Schur multiple Eisenstein series gives a multiple Eisenstein series.

2. The Young tableaux algebra

Recall the algebraic setup of [Yu]. Let $\mathcal{A}$ be a countable alphabet and k a commutative ring. By $\mathsf{k}\boxed{\mathcal{A}}$ we denote the commutative k-algebra generated by connected skew Young tableaux with entries in $\mathcal{A}$, modulo the relation that identifies a disconnected tableau with the product of its connected components. Further, we allow the Young tableaux to have entries in $\mathsf{k}\mathcal{A}$ and then extend them k-linearly in each entry. For example, let $a, b, c, d \in \mathcal{A}$, $\alpha, \beta \in \mathsf{k}$, and $X = \alpha a + \beta b$. Then we write

$$\ytableaushort{Xc,d} = \alpha\ \ytableaushort{ac,d} + \beta\ \ytableaushort{bc,d}.$$

The product $*$ places two tableaux as disconnected components of a single shape. We equip $\mathsf{k}\boxed{\mathcal{A}}$ with the following *cutting coproduct* Δ_{cut}. Then we have

Definition 2.1 (Cutting coproduct Δ_{cut})**.** Let w be a Young tableau with shape $\mathrm{sh}(w) = \lambda/\mu$. The coproduct Δ_{cut} is defined by

$$\Delta_{cut}(w) = \sum_{\mu \subseteq \eta \subseteq \lambda} w_{\eta/\mu} \otimes w_{\lambda/\eta},$$

where $w_{\eta/\mu}$ and $w_{\lambda/\eta}$ denote the restrictions of w to $D(\eta/\mu)$ and $D(\lambda/\eta)$, respectively. The counit $\epsilon : \mathsf{k}\boxed{\mathcal{A}} \to \mathsf{k}$ is defined by $\epsilon(\mathbf{1}) = 1_{\mathsf{k}}$ and $\epsilon(w) = 0_{\mathsf{k}}$ otherwise.

It cuts a tableau w of shape λ/μ into an upper-left part and a lower-right part along each intermediate shape. For example,

$$\Delta_{cut}\left(\ytableaushort{ab,c}\right) = 1 \otimes \ytableaushort{ab,c} + \ytableaushort{a} \otimes \ytableaushort{\none b,c} + \ytableaushort{ab} \otimes \ytableaushort{c} + \ytableaushort{a,c} \otimes \ytableaushort{b} + \ytableaushort{ab,c} \otimes 1.$$

Theorem 2.2 ([Yu])**.** *Δ_{cut} is well-defined on $\mathsf{k}\boxed{\mathcal{A}}$. Moreover, $(\mathsf{k}\boxed{\mathcal{A}}, *, \Delta_{cut})$ is a bialgebra.*

Since both $*$ and Δ_{cut} are graded by the number of boxes of Young tableaux, we have

Theorem 2.3 ([Yu])**.** *$(\mathsf{k}\boxed{\mathcal{A}}, *, \Delta_{cut}, S)$ is a connected commutative graded Hopf algebra.*

$\mathsf{k}\boxed{\mathcal{A}}$ is graded by the number of boxes. The antipode S can be computed recursively. For example,

$$S\left(\ytableaushort{ab,c}\right) = -\ytableaushort{ab,c} + \ytableaushort{ab} * \ytableaushort{c} + \ytableaushort{a,c} * \ytableaushort{b} - \ytableaushort{a} * \ytableaushort{b} * \ytableaushort{c}.$$

Compare $\mathsf{k}\boxed{\mathcal{A}}$ with the quasi-shuffle algebra [Ho, HI]. Let $\diamond$ be a commutative and associative product on the k-vector space $\mathsf{k}\mathcal{A}$. The quasi-shuffle product $*_\diamond$ on $\mathsf{k}\langle\mathcal{A}\rangle$ is given by $1 *_\diamond w = w *_\diamond 1 = w$, and

$$aw *_\diamond bv = a(w *_\diamond bv) + b(aw *_\diamond v) + (a \diamond b)(w *_\diamond v),$$

where $a, b \in \mathcal{A}$ and w, v are words.

Recall how a Schur multiple Eisenstein series expands into multiple Eisenstein series. The semi-standard condition only constrains the relative order of the summation variables. Summing over all orderings compatible with this condition gives an expansion of a Schur multiple Eisenstein series as a sum of multiple Eisenstein series. Consider the tableau $\mathbf{h} = \ytableaushort{ab,c}$ of shape $(2, 1)$, whose entries a, b, c stand for exponents. The semi-standard condition requires

$m_a \le m_b$ along the row and $m_a < m_c$ along the column. Enumerating the orderings of m_a, m_b, m_c compatible with these constraints gives

$$\mathbb{G}\left(\,\ytableaushort{ab,c}\,\right) = \mathbb{G}(a+b,\, c) + \mathbb{G}(a,\, b+c) + \mathbb{G}(a,\, b,\, c) + \mathbb{G}(a,\, c,\, b).$$

The *linearization map* $L_\diamond$ formalizes this procedure. A *semi-standard decomposition* of $\mathbf{h}$ is a way of cutting its boxes into ordered groups $D_1, \ldots, D_r$. Filling each box of D_a with the number a produces a semi-standard tableau: the entries weakly increase along rows and strictly increase along columns. Each decomposition records one ordering of the boxes compatible with the semi-standard condition. Writing $\mathrm{SSD}(\mathrm{sh}(\mathbf{h}))$ for the set of all such decompositions and $|\mathbf{h}_{D_a}|_\diamond$ for the $\diamond$-product of the entries in D_a, we define

$$L_\diamond : \mathsf{k}\boxed{\mathcal{A}} \longrightarrow \mathsf{k}\langle \mathcal{A} \rangle, \qquad L_\diamond(\mathbf{h}) = \sum_{(D_1,\ldots,D_r)\in \mathrm{SSD}(\mathrm{sh}(\mathbf{h}))} |\mathbf{h}_{D_1}|_\diamond \cdots |\mathbf{h}_{D_r}|_\diamond .$$

For the tableau $\mathbf{h} = \ytableaushort{ab,c}$, the four semi-standard decompositions are represented by the labeled tableaux below. The label a marks the boxes of D_a:

$$\ytableaushort{11,2}\,, \quad \ytableaushort{12,2}\,, \quad \ytableaushort{12,3}\,, \quad \ytableaushort{13,2}\,,$$

and therefore $L_\diamond(\mathbf{h}) = (a \diamond b)\, c + a\, (b \diamond c) + a\, b\, c + a\, c\, b$. $L_\diamond$ is surjective, and the quotient

$$\mathsf{k}\boxed{\mathcal{A}}_\diamond := \mathsf{k}\boxed{\mathcal{A}} / \ker(L_\diamond)$$

carries the induced structure.

Theorem 2.4 ([Yu])**.** *$L_\diamond$ induces an isomorphism of Hopf algebras*

$$(\mathsf{k}\boxed{\mathcal{A}}_\diamond, *, \Delta_{cut}) \cong (\mathsf{k}\langle \mathcal{A} \rangle, *_\diamond, \Delta_{\mathrm{dec}}).$$

Let $\mathsf{k}\boxed{A}^{\mathrm{diag}}$ be the subspace of $\mathsf{k}\boxed{\mathcal{A}}$ spanned by the *diagonal constant* Young tableaux, in which all boxes on a common diagonal have the same entry. This space is closed under the product and coproduct and therefore $\mathsf{k}\boxed{A}^{\mathrm{diag}}$ is a Hopf subalgebra of $\mathsf{k}\boxed{\mathcal{A}}$.

Next, we introduce some statements on the subalgebra $\mathsf{k}\boxed{A}^{\mathrm{diag}}$. The first identity expands a single $\diamond$-power $h^{\diamond l}$, which corresponds to a depth-one value, as an alternating sum of hook-shaped tableaux. This allows us to write a Riemann zeta value as a combination of Schur multiple zeta values with constant entries. For $h \in \mathcal{A}$, let $\{h\}^{\lambda/\mu}$ denote the Young tableau of shape λ/μ whose entries are all equal to h. We also write $h^{\diamond n} := \underbrace{h \diamond \cdots \diamond h}_{n \text{ factors}}$.

Theorem 2.5 ([Yu])**.** *For $h \in \mathcal{A}$ and $l \in \mathbb{Z}_{\ge 1}$, we have*

$$L_\diamond\big(\, \{h^{\diamond l}\}^{(1)} \,\big) = \sum_{\substack{a+b=l,\\ a\ge 1,\, b\ge 0}} (-1)^b L_\diamond\Big(\{h\}^{(a,\{1\}^b)} \Big)\,.$$

The *Jacobi–Trudi function* $J : \mathsf{k}\boxed{A}^{\mathrm{diag}} \to \mathsf{k}\boxed{A}^{\mathrm{diag}}$ is defined by a determinant in which each entry of the classical Jacobi–Trudi matrix is replaced by the corresponding column tableau. Furthermore, J is shown to act trivially modulo $\ker L_\diamond$, which allows a diagonal Schur multiple zeta value to be written as a determinant of multiple zeta values. The hook identity also underlies the expansion of this determinant.

Definition 2.6. Let $\mathbf{h} = (h_{i,j}) \in \mathsf{k}\boxed{A}^{\mathrm{diag}}$ be a diagonal constant Young tableau, and write $h_{i,i+n} = a_n$ for $n \in \mathbb{Z}$. Assume that $\mathrm{sh}(\mathbf{h}) = \lambda/\mu$, where $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$. Let $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ and $\mu' = (\mu'_1, \ldots, \mu'_s)$ be the conjugates of λ and μ, where zero parts are allowed. The *Jacobi–Trudi function* J is the k-linear map $J : \mathsf{k}\boxed{A}^{\mathrm{diag}} \to \mathsf{k}\boxed{A}^{\mathrm{diag}}$ defined by

$$J(\mathbf{h}) = \det \left[\begin{array}{|c|} \hline a_{-\mu'_j+j-1} \\ \hline a_{-\mu'_j+j-2} \\ \hline \vdots \\ \hline a_{-\mu'_j+j-(\lambda'_i-\mu'_j-i+j)} \\ \hline \end{array} \right]_{1\le i,j\le s}.$$

Here, $a_{-\mu'_j+j-1} \cdots a_{-\mu'_j+j-(\lambda'_i-\mu'_j-i+j)} = 1$ if $\lambda'_i - \mu'_j - i + j = 0$ and 0 if $\lambda'_i - \mu'_j - i + j < 0$.

Theorem 2.7 ([Yu]). *The Jacobi–Trudi function J satisfies the following.*

(i) $J : \mathsf{k}\boxed{A}^{\mathrm{diag}} \to \mathsf{k}\boxed{A}^{\mathrm{diag}}$ *is a Hopf algebra homomorphism.*

(ii) For any $\mathbf{h} \in \mathsf{k}\boxed{A}^{\mathrm{diag}}$ *we have*

$$J(\mathbf{h}) - \mathbf{h} \in \ker L_\diamond.$$

Example 2.8. When $\mathrm{sh}(\mathbf{h}) = \lambda/\mu = (4,3,2)/(1,1)$, we have

$$L_\diamond\left(\begin{array}{cccc} \cline{2-4} & \multicolumn{1}{|c|}{a_1} & \multicolumn{1}{c|}{a_2} & \multicolumn{1}{c|}{a_3} \\ \cline{2-4} & \multicolumn{1}{|c|}{a_0} & \multicolumn{1}{c|}{a_1} & \\ \cline{1-3} \multicolumn{1}{|c|}{a_{-2}} & \multicolumn{1}{c|}{a_{-1}} & & \\ \cline{1-2} \end{array} \right) = \det \begin{pmatrix} a_{-2} & a_1a_0a_{-1}a_{-2} & a_2a_1a_0a_{-1}a_{-2} & a_3a_2a_1a_0a_{-1}a_{-2} \\ 1 & a_1a_0a_{-1} & a_2a_1a_0a_{-1} & a_3a_2a_1a_0a_{-1} \\ 0 & a_1 & a_2a_1 & a_3a_2a_1 \\ 0 & 0 & 1 & a_3 \end{pmatrix}.$$

Part (ii) states that J acts trivially modulo $\ker L_\diamond$, so the Jacobi–Trudi identity holds in $\mathsf{k}\boxed{\mathcal{A}}_\diamond$. Applying any algebra homomorphism out of $\mathsf{k}\boxed{\mathcal{A}}_\diamond$ then yields the corresponding determinant identity.

Next, we consider constant-entry tableaux, which form a special class of diagonal-constant Young tableaux. All entries of such a tableau are the same. The key tool is the Newton identity in the quasi-shuffle algebra $\mathsf{k}\langle\mathcal{A}\rangle$. It holds over any commutative ring. In characteristic zero, it integrates to the exponential identity of Hoffman–Ihara [HI]. For the linearized quotient of the Young tableaux Hopf algebra, we have

Theorem 2.9 ([Yu]). *Let* $\{a\}^{\lambda/\mu} \in \mathsf{k}\boxed{\mathcal{A}}_\diamond$ *and* $n = |\lambda| - |\mu|$. *Then*

$$\{a\}^{\lambda/\mu} \in \mathbb{Z}[\tfrac{1}{n!}]\big[\{a^{\circ l}\}^{(1)} \mid l \ge 1\big],$$

the subalgebra of $\mathsf{k}\boxed{\mathcal{A}}_\diamond$ *generated over* $\mathbb{Z}[\frac{1}{n!}]$ *by the depth-one tableaux under* $*$. *In particular, if* $\mathbb{Q} \subseteq \mathsf{k}$, *then* $\{a\}^{\lambda/\mu} \in \mathbb{Q}[\{a^{\circ l}\}^{(1)} \mid l \ge 1]$.

3. Schur multiple Eisenstein series

Let $\mathcal{H}^2_\diamond = \mathbb{Q}\boxed{\mathbb{N}_{\ge 2}}_\diamond$ be the quotient generated by connected Young tableaux with entries in $\mathbb{N}_{\ge 2}$.

Theorem 3.1 ([Yu]). *The Schur multiple Eisenstein series map*

$$\mathbb{G}(-) : \mathcal{H}^2_\diamond \longrightarrow \mathcal{O}(\mathbb{H}), \quad \mathbf{h} \longmapsto \mathbb{G}(\mathbf{h})$$

is an algebra homomorphism.

Corollary 3.2 (Jacobi–Trudi formula)**.** *Let $\mathbf{h} = (h_{i,j})_{(i,j)\in\mathrm{sh}(\mathbf{h})}$ be a diagonal constant Young tableau. Let* $\mathrm{sh}(\mathbf{h}) = \lambda/\mu$*, where $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$. Let $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ and $\mu' = (\mu'_1, \ldots, \mu'_s)$ be the conjugates of λ and μ. Then we have*

$$\mathbb{G}(\mathbf{h};\tau) = \det\left[\mathbb{G}(a_{-\mu'_j+j-1}, \ldots, a_{-\mu'_j+j-(\lambda'_i-\mu'_j-i+j)};\tau)\right]_{1\le i,j\le s}.$$

Here, $\mathbb{G}(a_{-\mu'_j+j-1}, \ldots, a_{-\mu'_j+j-(\lambda'_i-\mu'_j-i+j)};\tau) := 1$ *if* $\lambda'_i-\mu'_j-i+j = 0$ *and* 0 *if* $\lambda'_i-\mu'_j-i+j < 0$*.*

Suppose every entry equals $h \ge 2$. Corollary 3.2 expresses this Schur multiple Eisenstein series as a polynomial in standard multiple Eisenstein series of the form $\mathbb{G}(h, \ldots, h;\tau)$.

Corollary 3.3. *For $h \ge 2$, we have*

$$\exp\left(\sum_{i\ge1}\frac{(-1)^{i-1}}{i}\mathbb{G}(ih;\tau)X^i\right) = \sum_{n=0}^{\infty}\mathbb{G}(\underbrace{h, \ldots, h}_{n};\tau)X^n.$$

It rewrites the series $\mathbb{G}(h, \ldots, h;\tau)$ as a polynomial in depth-1 Eisenstein series $\mathbb{G}(ih;\tau)$.

Together, these two corollaries provide a complete reduction. First, Corollary 3.2 reduces a constant-entry Schur multiple Eisenstein series to a polynomial in $\mathbb{G}(h, \ldots, h;\tau)$. Then, Corollary 3.3 reduces those terms further to polynomials in $\mathbb{G}(ih;\tau)$. Since standard depth-1 Eisenstein series have well-known Fourier expansions, this reduction explicitly determines the Fourier expansion of the original Schur multiple Eisenstein series. To see how this reduction works, we consider the following example:

$$\mathbb{G}(2;\tau) = \mathbb{G}\left(\,\begin{ytableau} 2 \end{ytableau}\,;\tau\right),\ \mathbb{G}(4;\tau) = \mathbb{G}\left(\,\begin{ytableau} 2 & 2 \end{ytableau}\,;\tau\right) - \mathbb{G}\left(\begin{ytableau} 2 \\ 2 \end{ytableau}\,;\tau\right),$$

$$\mathbb{G}(6;\tau) = \mathbb{G}\left(\,\begin{ytableau} 2 & 2 & 2 \end{ytableau}\,;\tau\right) - \mathbb{G}\left(\begin{ytableau} 2 & 2 \\ 2 \end{ytableau};\tau\right) + \mathbb{G}\left(\begin{ytableau} 2 \\ 2 \\ 2 \end{ytableau}\,;\tau\right).$$

These formulas show that the relation is invertible. Thus $\mathbb{G}(2;\tau)$, $\mathbb{G}(4;\tau)$, and $\mathbb{G}(6;\tau)$ can be written as $\mathbb{Z}$-linear combinations of Schur multiple Eisenstein series with all entries equal to 2. Since they generate the ring of quasimodular forms, this means that all quasimodular forms can be expressed using these Schur multiple Eisenstein series. This idea extends to the following modularity results,

Corollary 3.4 (Modularity)**.** *Let $\mathbf{h} = \{h\}^{\lambda/\mu} \in \mathcal{H}^2_{\diamond}$, then*

(i) $\mathbb{G}(\mathbf{h};\tau) \in \mathbb{Q}[\mathbb{G}(hl;\tau) | l \ge 1]$.

(ii) If $h = 2$, $\mathbb{G}(\mathbf{h};\tau)$ is a quasimodular form, and ***every*** *quasimodular form can be written as a linear combination of these $\mathbb{G}(\mathbf{h};\tau)$.*

(iii) For even $h \ge 4$, $\mathbb{G}(\mathbf{h};\tau)$ is a modular form.

4. Schur Eisenstein series, Schur–MacMahon series and quasimodular forms

This section considers the Schur multiple Eisenstein series with all entries equal to 2, referred to as the *Schur Eisenstein series*. They are expressed using the normalized notation:

$$G_{\lambda/\mu} := (2\pi i)^{-2|\lambda/\mu|}\mathbb{G}(\{2\}^{\lambda/\mu};\tau), \qquad G_{2k} := (2\pi i)^{-2k}\mathbb{G}(2k;\tau).$$

Theorem 4.1 ([BY]). *$\{G_\lambda \mid \lambda_1 \leq 3\}$ is a $\mathbb{Q}$-basis of $\widetilde{\mathcal{M}_{\mathbb{Q}}}$.*

Example 4.2. For example, the Ramanujan Δ function can be written as

$$\Delta = \frac{1}{96}G_{(3,3)} - \frac{1}{96}G_{(3,2,1)} + \frac{1}{96}G_{(3,1,1,1)} - \frac{17}{720}G_{(2,2,2)} + \frac{49}{1440}G_{(2,2,1,1)} - \frac{2}{45}G_{(2,1,1,1,1)} + \frac{2}{45}G_{(1,1,1,1,1,1)}.$$

Let $\widetilde{\mathcal{M}_{\mathbb{Z}}} := \widetilde{\mathcal{M}_{\mathbb{Q}}} \cap \mathbb{Z}[[q]]$ be the space of quasimodular forms with integer coefficients. Note that $\widetilde{\mathcal{M}_{\mathbb{Z}}}$ does not equal $\mathbb{Z}[E_2, E_4, E_6]$. Indeed, Ramanujan's differential equation gives

$$DE_2 = \frac{E_2^2 - E_4}{12} = -24\sum_{n\geq 1} n\sigma_1(n)q^n \in \widetilde{\mathcal{M}_{\mathbb{Z}}}.$$

Thus DE_2 is an element of $\widetilde{\mathcal{M}_{\mathbb{Z}}}$ that is not contained in $\mathbb{Z}[E_2, E_4, E_6]$. So far, no explicit conjectural spanning set of $\widetilde{\mathcal{M}_{\mathbb{Z}}}$ has been proposed in the literature.

In [Mac], MacMahon introduced, for $r \geq 1$, a generalization of the generating series of the sum-of-divisors function. Andrews and Rose [AR] showed that these q-series are quasimodular, and Bachmann [Ba] gave explicit expressions in terms of Eisenstein series. For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, we define the **Schur–MacMahon q-series** by

$$g_\lambda := \sum_{M \in \mathrm{SSYT}(\lambda)} \prod_{(i,j)\in D(\lambda)} \frac{q^{m_{ij}}}{\left(1 - q^{m_{ij}}\right)^2},$$

where $D(\lambda)$ is the Young diagram of λ, and $\mathrm{SSYT}(\lambda)$ denotes the set of semistandard Young tableaux of shape λ with positive integer entries. For example, for $\lambda = (2,1)$ we have

$$g_{(2,1)} = \sum_{\substack{0 < m_2 \leq m_3 \\ \wedge \\ m_1}} \prod_{i=1}^{3} \frac{q^{m_i}}{(1 - q^{m_i})^2} = q^4 + 6q^5 + 21q^6 + 58q^7 + 129q^8 + 266q^9 + \ldots.$$

By including the empty partition, $g_\varnothing$ is set to 1. For $\lambda = (1)$, this yields the divisor generating series $g_{(1)} = \sum_{1\leq n} \sigma_1(n)q^n = \frac{1}{24}(1 - E_2)$. More generally, for a single column $\lambda = (\{1\}^r)$, this reduces to MacMahon's q-series. Moreover, it is a q-analogue of (Schur) multiple zeta values. In particular,

$$\lim_{q\to 1^-}(1-q)^{2r} g_{(\{1\}^r)} = \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^2 \cdots m_r^2} = \zeta(2,\ldots,2) = \frac{\pi^{2r}}{(2r+1)!}.$$

For $A \in \{\mathbb{Z}, \mathbb{Q}\}$, we set $\mathcal{G}_A := \mathrm{span}_A\{g_\lambda \mid \lambda \text{ is a partition}\}$. Our first result is the following.

Theorem 4.3 ([BY]). *For all partitions λ, the Schur–MacMahon series g_λ is a quasimodular form of mixed weight, with maximal weight $2|\lambda|$. In particular, $\mathcal{G}_{\mathbb{Q}} = \widetilde{\mathcal{M}_{\mathbb{Q}}}$.*

While Theorem 4.3 establishes that the Schur–MacMahon series span $\widetilde{\mathcal{M}_{\mathbb{Q}}}$ over $\mathbb{Q}$, there are many linear relations among them. A natural question is which subset forms a basis. We can prove that g_λ for partitions whose largest part is ≤ 3 give a basis.

Theorem 4.4 ([BY]). *$\{g_\lambda \mid \lambda_1 \leq 3\}$ is a $\mathbb{Q}$-basis of $\widetilde{\mathcal{M}_{\mathbb{Q}}}$.*

Example 4.5. For example, the Ramanujan Δ function can be written on this basis

$$\begin{aligned}\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24} &= g_{(1)} - 27g_{(2)} + 473g_{(1,1)} - 90g_{(3)} - 1890g_{(2,1)}\\ &- 103650g_{(1,1,1)} + 15120g_{(3,1)} + 17640g_{(2,2)} - 693000g_{(2,1,1)}\\ &+ 1998360g_{(1,1,1,1)} + 453600g_{(3,2)} - 453600g_{(3,1,1)} - 4324320g_{(2,2,1)}\\ &+ 8648640g_{(2,1,1,1)} - 12519360g_{(1,1,1,1,1)} + 1814400g_{(3,3)} - 1814400g_{(3,2,1)}\\ &+ 1814400g_{(3,1,1,1)} - 21409920g_{(2,2,2)} + 23224320g_{(2,2,1,1)}\\ &- 25038720g_{(2,1,1,1,1)} + 25038720g_{(1,1,1,1,1,1)}.\end{aligned}$$

Since the Fourier coefficients of g_λ are non-negative integers, we have $g_\lambda \in \widetilde{\mathcal{M}_{\mathbb{Z}}}$. Together with Theorem 4.3, this gives $\mathrm{span}_{\mathbb{Z}}\{g_\lambda\} \subseteq \widetilde{\mathcal{M}_{\mathbb{Z}}}$. Based on numerical experiments, we expect the following.

Conjecture 4.6 ([BY]). $\widetilde{\mathcal{M}_{\mathbb{Z}}} = \mathcal{G}_{\mathbb{Z}}$.

As further evidence for this conjecture, we consider how g_λ behaves under differentiation.

Theorem 4.7 ([BY]). *$\mathcal{G}_{\mathbb{Z}}$ is closed under $D = q\frac{d}{dq}$.*

As we saw, D does not preserve $\mathbb{Z}[E_2, E_4, E_6]$ as $DE_2 = \frac{E_2^2 - E_4}{12} \notin \mathbb{Z}[E_2, E_4, E_6]$, but Theorem 4.7 shows that $\mathcal{G}_{\mathbb{Z}}$ is preserved under D. For example, we have

$$Dg_{(1)} = \frac{E_4 - E_2^2}{288} = \sum_{1\le n} n\sigma_1(n)q^n = g_{(1)} + 3g_{(2)} - 7g_{(1,1)}.$$

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