Numerical semigroups and the distribution of prime numbers

Takao Komatsu
Institute of Mathematics, Henan Academy of Sciences
and
Department of Mathematics, Institute of Science Tokyo

1 Introduction

Let $A = \{a_1, a_2, \dots, a_k\}$ $(k \geq 2)$ be a set of integers satisfying $a_j \geq 2$ $(1 \leq j \leq k)$ and $\gcd(a_1, a_2, \dots, a_k) = 1$. Let $\mathbb{N}_{(0)}$ denote the set of non-negative integers.

For a non-negative integer ℓ, the ℓ-numerical semigroup[1], denoted by $S_\ell(A)$, is the set of integers $n \in \mathbb{N}_{(0)}$ for which the Diophantine equation

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = n, \qquad x_1, x_2, \dots, x_k \in \mathbb{N}_{(0)},$$

admits more than ℓ representations in non-negative integers. The complement $G_\ell(A) := \mathbb{N}_{(0)} \setminus S_\ell(A)$ is called the set of ℓ-gaps. Thus $G_\ell(A)$ consists of those integers n for which the above equation has at most ℓ solutions. The largest element of $G_\ell(A)$ is called the ℓ-Frobenius number and is denoted by $g_{\ell,A}$. The cardinality of $G_\ell(A)$ is called the ℓ-genus (or ℓ-Sylvester number) and is denoted by $n_{\ell,A}$. When $\ell = 0$, the set $S(A) = S_0(A)$ coincides with the classical numerical semigroup generated by A. In this case $g_A = g_{0,A}$ and $n_A = n_{0,A}$ correspond to the classical Frobenius number and genus.

In general, determining explicit formulas for the Frobenius number and the genus is notoriously difficult when $k \geq 3$; indeed, no general closed formula is known in this case ([4]). For $k = 2$, let $A := \{a, b\}$. Sylvester [20] proved that $g_{0,a,b} = ab - a - b$ and further showed that for any non-negative integer $n \leq g_{0,a,b}$, exactly one of n and $g_{0,a,b} - n$ belongs to $S_0(a, b)$. Consequently, exactly half of the integers in the interval $[0, g_{0,a,b}]$ can be expressed as $ax + by$ with $(x, y) \in \mathbb{N}^2_{(0)}$. More generally, for any pair of relatively prime integers (a, b), we have (see, e.g., [13, Corollary 2])

$$g_{\ell,a,b} = (\ell + 1)ab - a - b\,,$$
$$n_{\ell,a,b} = \frac{1}{2}\big((2\ell + 1)ab - a - b + 1\big)\,.$$

More general formulas can be found in [12, 13].

As noted earlier, there are exactly $n_{0,a,b} = (a - 1)(b - 1)/2$ integers in the set $G_0(a, b)$. Let $\mathcal{P}$ denote the set of prime numbers. For any given non-negative integer ℓ, define

$$\pi_{\ell,a,b} = \#\{p \in \mathcal{P} : p \in S_\ell(a, b), p \leq g_{\ell,a,b}\}\,.$$

[1] In a series of recent papers ([12, 13, 14, 15, 16]), we call it the p-numerical semigroup. However, this paper focuses on the discussion of prime numbers, and since it is common to use p to represent prime numbers, we will use ℓ instead of p here.

In 2020, Ramírez Alfonsín and Skałba [18] proved that for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$\pi_{0,a,b} > C(\varepsilon)\frac{g_{0,a,b}}{(\log g_{0,a,b})^{2+\varepsilon}} \tag{1}$$

provided that $g_{0,a,b}$ is sufficiently large. Based on (1) and numerical experiments, they proposed the following conjecture [18, Conjecture 2].

Conjecture 1. *Let $2 < a < b$ be two relatively prime integers. Then $\pi_{0,a,b} > 0$.*

Dai, Ding, and Wang [5] confirmed Conjecture 1. More recently, Chen and Zhu [2] proved that

$$\pi_{0,a,b} > 0.005 \cdot \frac{g_{0,a,b}}{\log g_{0,a,b}}.$$

Ramírez Alfonsín and Skałba also posed another conjecture [18, Conjecture 3].

Conjecture 2. *Let $2 < a < b$ be two relatively prime integers. Then*

$$\pi_{0,a,b} \sim \frac{\pi(g_{0,a,b})}{2} \quad (as\, a \to \infty),$$

where $\pi(x)$ denotes the number of primes not exceeding x.

Ding [7] proved that Conjecture 2 holds for almost all cases. Subsequently, Ding, Zhai, and Zhao [10] confirmed it unconditionally. More generally, Ding and Komatsu [8] proved that

$$\pi_{\ell,a,b} \sim \frac{1}{2(\ell+1)}, \pi(g_{\ell,a,b}) \quad \text{as } a \to \infty, \tag{2}$$

generalizing Conjecture 2. Huang and Zhu [11] further extended the result of Ding, Zhai, and Zhao to the distribution of prime powers. Very recently, Chen and Zhu [2] also proved that for any fixed integer $a \geq 3$ with $\gcd(a, b) = 1$,

$$\pi_{0,a,b} = \left(\frac{a-2}{2(a-1)} + o(1)\right)\frac{g_{0,a,b}}{\log g_{0,a,b}} \quad (as\, b \to \infty).$$

We shall generalize the results of Chen and Zhu [2].

Theorem 1. *Let ℓ be any non-negative integer. For a fixed integer $a \geq 3$ with $\gcd(a, b) = 1$,*

$$\pi_{\ell,a,b} = \left(\frac{a-2}{2(\ell a + a - 1)} + o(1)\right)\pi\big(g_{\ell,a,b}\big) \quad (as\ b \to \infty)\,.$$

Theorem 2. *Let ℓ be a non-negative integer and let $a < b$ be two relatively prime integers with $a \geq e^{\ell+1}$. Then*

$$\pi_{\ell,a,b} > 0.005 \cdot \frac{1}{\ell+1}\frac{g_{\ell,a,b}}{\log g_{\ell,a,b}}.$$

Chen and Zhu also studied the number of primes in $G_0(a,b)$. For any non-negative integer ℓ, define

$$\pi^*_{\ell,a,b} = \#\{p \in \mathcal{P} : p \in G_\ell(a,b)\}.$$

Chen and Zhu [3] proved that for a fixed integer $a \geq 3$ with $\gcd(a,b) = 1$,

$$\pi^*_{0,a,b} > \frac{a}{2(a-1)} \frac{g_{0,a,b}}{\log g_{0,a,b}} \quad (as\, b \to \infty),$$

and

$$\pi^*_{0,a,b} > 0.04\pi(g_{0,a,b}).$$

They also proposed the following conjecture [3, Conjecture 1.4].

Conjecture 3. *For $b > a \geq 1$ with $\gcd(a,b) = 1$, we have*

$$\pi^*_{0,a,b} \geq \frac{1}{2}\pi(g_{0,a,b}),$$

with equality if and only if one of the following holds:

$$(1)\, a = 1; \quad (2)\, (a,b) = (2,3); \quad (3)\, (a,b) = (2,5); \quad (4)\, (a,b) = (3,5).$$

Along these lines, we obtain the following results.

Theorem 3. *Let ℓ be a non-negative integer. For a fixed integer $a \geq 3$ and sufficiently large b with $\gcd(a,b) = 1$,*

$$\pi^*_{\ell,a,b} > \frac{(2\ell+1)a}{2(\ell a + a - 1)} \frac{g_{\ell,a,b}}{\log g_{\ell,a,b}}.$$

Theorem 4. *Let ℓ, a and b be non-negative integers with $\ell < a < b$ and $\gcd(a,b) = 1$. Then*

$$\pi^*_{\ell,a,b} > \frac{\ell + 0.02}{\ell + 1} \frac{g_{\ell,a,b}}{\log g_{\ell,a,b}}.$$

2 Proofs

In this section, we establish the main results of the paper. Throughout, we assume that $A = \{a,b\}$ with $\gcd(a,b) = 1$.

The first lemma is quoted from Ding and Komatsu [8, Lemma 2].

Lemma 1. *Suppose that the number of solutions to*

$$n = ax + by, \quad (x,y) \in \mathbb{N}^2_{(0)}$$

is exactly $\ell + 1$. Then we have

$$n = \ell ab + ax_0 + by_0$$

for some $0 \leq x_0 \leq b-1, 0 \leq y_0 \leq a-1$.

Combining Lemma 1 with Sylvester's result mentioned earlier yields the following characterization.

Lemma 2. *Let a and b be relatively prime positive integers. Then a non-negative integer $n \le g_{\ell,a,b}$ has exactly $\ell+1$ representations of the form $n = ax + by$ with $(x, y) \in \mathbb{N}_{(0)}^2$ if and only if $g_{\ell,a,b} - n$ cannot be expressed as $ax + by$ with $x, y \in \mathbb{N}_{(0)}$.*

Let $\pi(x; k, l)$ denote the number of primes $p \le x$ with $p \equiv l \pmod{k}$, and let $\varphi(m)$ be Euler's totient function. The third lemma is the Siegel–Walfisz Theorem (see [6, Chapter 22]).

Lemma 3. *(Siegel–Walfisz Theorem) Let A be a positive real number and let m, l be coprime integers with $1 \le m \le (\log x)^A$. Then there exists a positive constant D depending only on A such that*

$$\pi(x; m, l) = \frac{1}{\varphi(m)} \int_2^x \frac{1}{\log t}\,\mathrm{d}t + O\left(x \exp\left(-D\sqrt{\log x}\right)\right),$$

uniformly in m.

The following lemma is quoted from Bennett, Martin, O'Bryant, and Rechnitzer [1, (1.15)].

Lemma 4. *For $x \ge 1474279333$, we have*

$$\left|\pi(x) - \mathrm{Li}(x)\right| < 0.0008375 \frac{x}{\log^2 x},$$

where

$$\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}.$$

As an application of Lemma 4, obtain the following result.

Lemma 5. *Let ℓ be a positive integer and $x \ge \max\{\ell e^{2(\ell+1)}, 7 \cdot 10^5\}$. Then*

$$\pi(x) - \pi(cx) \ge 0.11604925 \cdot \frac{1}{\ell+1} \frac{x}{\log x},$$

where $c = 1 - \frac{1}{8(\ell+1)}$.

We require the following lemma from Montgomery and Vaughan [17, Theorem 2].

Lemma 6. *Let k, l be integers and let x, y be positive real numbers with $1 \le k < y$. Then*

$$\pi(x+y; k, l) - \pi(x; k, l) < \frac{2y}{\varphi(k) \log(y/k)}.$$

We also need the following lemma from Chen and Zhu [2, Lemma 2.8].

Lemma 7. *Let m, k be two relatively prime integers. Then for any positive real number x and any integer l,*

$$\sum_{\substack{1 \le n \le x \\ \gcd(nk+l, m) = 1}} 1 = \frac{\varphi(m)}{m} x + \theta_{x,m,k,l} 2^{\omega(m)},$$

where $\theta_{x,m,k,l}$ is a real number with $|\theta_{x,m,k,l}| \le 1$ and $\omega(m)$ denotes the number of distinct prime divisors of m.

The following lemma is an application of Lemma 7.

Lemma 8. *Let $b > a \ge 10$ be two relatively prime integers. Then we have*

$$\begin{aligned}
&\#\{p \in \mathcal{P} : cg < p \le g, p \notin S\} \\
&\qquad < \frac{1}{4(\ell+1)\varphi(a)} \left(\sum_{\substack{1 \le y \le a/8+1 \\ \gcd(y,a)=1}} 1 \right) \left(1 - \frac{\log\big(8(\ell+1)a\big)}{\log g} \right)^{-1} \cdot \frac{g}{\log g} \\
&\qquad < \frac{1}{4(\ell+1)} \left(\frac{1}{8} + \frac{1}{a} + \frac{2^{\omega(a)}}{\varphi(a)} \right) \left(1 - \frac{\log\big(8(\ell+1)a\big)}{\log g} \right)^{-1} \cdot \frac{g}{\log g},
\end{aligned}$$

where $\ell \ge 1$ and $c = 1 - \frac{1}{8(\ell+1)}$.

We also need the following lemma from Bennett, Martin, O'Bryant, and Rechnitzer [1, Corollary 1.6].

Lemma 9. *Let m, l be two relatively prime integers with $1 \le m \le 1200$. Then for all $x \ge 50m^2$, we have*

$$\frac{x}{\varphi(m)\log x} < \pi(x; m, l) < \frac{x}{\varphi(m)\log x} \left(1 + \frac{5}{2\log x} \right).$$

The following lemma, quoted from Chen and Zhu [3, Corollary 2.3], is a variant of Lemma 9.

Lemma 10. *Let m, l be two relatively prime integers. Then there exists $X(m)$ such that for all $x \ge X(m)$,*

$$\frac{x}{\varphi(m)\log x} < \pi(x; m, l) < \frac{x}{\varphi(m)\log x} \left(1 + \frac{5}{2\log x} \right).$$

The final lemma is quoted from Rosser and Schoenfeld [19, Corollary 1].

Lemma 11. *For $x \ge 17$ we have $\pi(x) > x/\log x$.*

2.1 Proof of Theorem 1

By Lemmas 1 and 3, for $\ell \ge 1$ we obtain

$$\begin{aligned}
\pi_{\ell,a,b} &= \sum_{\substack{\ell ab < p < g \\ p = \ell ab + ax + by \\ x,y \in \mathbb{Z}_{\ge 0}}} 1 = \sum_{\substack{y=1 \\ \gcd(y,a)=1}}^{a-1} \sum_{\substack{p \equiv by \pmod a \\ \ell ab + by \le p < g}} 1 \\
&= \int_2^g \frac{\mathrm{d}t}{\log t} - \frac{1}{\varphi(a)} \sum_{\substack{y=1 \\ \gcd(y,a)=1}}^{a-1} \int_2^{\ell ab + by} \frac{\mathrm{d}t}{\log t} + O\big(g e^{-D\sqrt{\log g}}\big).
\end{aligned}$$

For $1 \le y \le a-1$, integration by parts yields

$$\int_2^{\ell ab+by} \frac{\mathrm{d}t}{\log t} = \frac{\ell ab+by}{\log(\ell ab+by)} - \frac{2}{\log 2} + \int_2^{\ell ab+by} \frac{\mathrm{d}t}{(\log t)^2}. \tag{3}$$

Observe that

$$\frac{\ell ab+by}{\log(\ell ab+by)} = \frac{\ell ab+by}{\log g}\frac{\log g}{\log(\ell ab+by)}$$

and

$$\frac{\log g}{\log(\ell ab+by)} = \frac{\log b + \log\big((\ell+1)a - \frac{a}{b} - 1\big)}{\log b + \log(\ell a + y)} \to 1 \quad (b\to\infty)\,.$$

Consequently,

$$\frac{\ell ab+by}{\log(\ell ab+by)} \sim \frac{\ell ab+by}{\log g} \quad (b\to\infty)\,. \tag{4}$$

We now estimate the remaining integral. Splitting the range of integration gives

$$\begin{aligned}\int_2^{\ell ab+by} \frac{\mathrm{d}t}{(\log t)^2} &= \int_2^{\sqrt{b}} \frac{\mathrm{d}t}{(\log t)^2} + \int_{\sqrt{b}}^{\ell ab+by} \frac{\mathrm{d}t}{(\log t)^2} \\ &< \frac{\sqrt{b}}{(\log 2)^2} + \frac{\ell ab+by}{\log g}\frac{8\log g}{(\log b)^2}.\end{aligned}$$

Since

$$\frac{8\log g}{(\log b)^2} = \frac{8\log b + 8\log\big((\ell+1)a - \frac{a}{b} - 1\big)}{(\log b)^2} \to 0 \quad (b\to\infty)\,,$$

we deduce

$$\int_2^{\ell ab+by} \frac{\mathrm{d}t}{(\log t)^2} = \frac{\ell ab+by}{\log g}o(1) \quad (b\to\infty)\,. \tag{5}$$

Combining (3), (4), and (5), we obtain

$$\int_2^{\ell ab+by} \frac{\mathrm{d}t}{\log t} = \frac{\ell ab+by}{\log g}\big(1+o(1)\big) \quad (b\to\infty)\,.$$

Substituting this asymptotic into the sum over y yields

$$\begin{aligned}\frac{1}{\varphi(a)} \sum_{\substack{y=1\\ \gcd(y,a)=1}}^{a-1} \int_2^{\ell ab+by} \frac{\mathrm{d}t}{\log t} &= \frac{1+o(1)}{\varphi(a)\log g}\left(\ell ab\varphi(a) + b\cdot\frac{a\varphi(a)}{2}\right) \\ &= \frac{1+o(1)}{\log g}\left(\ell ab + \frac{ab}{2}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\pi_{\ell,a,b} &= \int_2^g \frac{\mathrm{d}t}{\log t} - \frac{1+o(1)}{\log g}\frac{(2\ell+1)ab}{2} + O(ge^{-D\sqrt{\log g}}) \\ &= \left(1 - \frac{(2\ell+1)a}{2(\ell a+a-1)} + o(1)\right)\pi(g)\,.\end{aligned}$$

This completes the proof of Theorem 1.

2.2 Proof of Theorem 2

Let $c = 1 - \frac{1}{8(\ell+1)}$. Since $\ell \geq 1$ and $b > a > e^{\ell+1} > 7$, we have

$$cg = (\ell+1)ab - a - b - \frac{(\ell+1)ab - a - b}{8(\ell+1)} > \ell ab + \left(\frac{7}{8} - \frac{1}{a} - \frac{1}{b}\right) ab > \ell ab\,.$$

Thus $cg > \ell ab$, and by Lemma 1 we obtain

$$\begin{aligned}\pi_{\ell,a,b} &= \#\left\{p \in \mathcal{P} : \ell ab < p \leq g, p = \ell ab + ax + by, (x,y) \in \mathbb{N}_{(0)}^2\right\} \\ &\geq \pi(g) - \pi(cg) - \#\{p \in \mathcal{P} : cg < p \leq g, p \notin S\}.\end{aligned}$$

We now consider seven cases.

Case 1. $a \geq \max\{e^{(\ell+1)}, 6 \cdot 10^4 + 1\}$. Applying Lemmas 8 and 5 yields

$$\pi_{\ell,a,b} > 0.0059 \cdot \frac{1}{\ell+1} \frac{g}{\log g}.$$

Case 2. $\max\{e^{\ell+1}, 631\} \leq a \leq 6 \cdot 10^4$. By Lemmas 8 and 5, together with a computer-assisted calculation,

$$\pi_{\ell,a,b} > 0.0061 \cdot \frac{1}{\ell+1} \frac{g}{\log g}.$$

Case 3. $\max\{e^{\ell+1}, 149\} \leq a \leq 630$ and $g \geq 3 \cdot 10^6$. Using Lemmas 8 and 5 along with a computer verification gives

$$\pi_{\ell,a,b} > 0.0106 \cdot \frac{1}{\ell+1} \frac{g}{\log g}.$$

Case 4. $\max\{e^{\ell+1}, 16\} \leq a \leq 148$ and $g \geq 7 \cdot 10^5$. Again employing Lemmas 8 and 5 with computational assistance, we obtain

$$\pi_{\ell,a,b} > 0.0066 \cdot \frac{1}{\ell+1} \frac{g}{\log g}.$$

Case 5. Either $149 \leq a \leq 630$ with $g < 3 \times 10^6$, or $16 \leq a \leq 148$ with $g < 7 \times 10^5$. In this range, Lemma 1 together with a computer search yields

$$\pi_{\ell,a,b} \geq 0.006 \cdot \frac{1}{\ell+1} \frac{g}{\log g}.$$

Case 6. $e^{\ell+1} \leq a \leq 15$, that is, $\ell = 1$, and $b \geq 50a^2$. By Lemma 1 and Lemma 9, with computational support, we have

$$\pi_{\ell,a,b} \cdot \frac{\log g}{g} > 0.04658379\ldots \cdot \frac{1}{\ell+1}$$

with equality attained at $a = 13$.

Case 7. $e^{\ell+1} \leq a \leq 15$ (so $\ell = 1$) and $b < 50a^2$ with $\gcd(a,b) = 1$. Applying Lemma 1 together with a computer-assisted enumeration gives

$$\pi_{\ell,a,b} > 0.1 \cdot \frac{1}{\ell+1} \frac{g}{\log g}.$$

Having established the desired lower bound in each case, we conclude the proof of Theorem 2.

2.3 Proof of Theorem 3

Recall that the ℓ-Apéry set $\mathrm{Ap}_\ell(a,b) = \{m_0^{(\ell)}, m_1^{(\ell)}, \ldots, m_{a-1}^{(\ell)}\}$ is characterised by the properties

$$\text{(i)}\, m_i^{(\ell)} \equiv i \pmod a \quad \text{(ii)}\, m_i^{(\ell)} \in S_\ell(a,b) \quad \text{(iii)}\, m_i^{(\ell)} - a \in G_\ell(a,b)$$

([13, pp.58-59]). For the numerical semigroup generated by a and b, it is known that $\mathrm{Ap}_\ell(a,b) = \{b(\ell a + v) : 0 \le v \le a-1\}$. Consequently,

$$\begin{aligned}\pi^*_{\ell,a,b} &= \sum_{v=0}^{a-1} \#\left\{p \in \mathcal{P} : p < b(\ell a + v),\, p \equiv bv \pmod a\right\} \\ &\ge \pi(\ell ab - 1; a, 0) + \sum_{\substack{v=1 \\ \gcd(v,a)=1}}^{a-2} \pi\big(b(\ell a+v); a, bv\big) + \pi\big(g; a, b(a-1)\big)\,.\end{aligned}$$

By Lemma 10, there exists $X(m)$ such that for all $x \ge X(m)$,

$$\pi(x; m, l) > \frac{x}{\varphi(m)\log x}.$$

Thus, provided $\ell ab \ge X(a)$, we obtain

$$\begin{aligned}\pi^*_{\ell,a,b} &\ge 0 + \sum_{\substack{v=1 \\ \gcd(v,a)=1}}^{a-2} \frac{1}{\varphi(a)} \frac{b(\ell a+v)}{\log\big(b(\ell a+v)\big)} + \frac{1}{\varphi(a)}\frac{g}{\log g} \\ &\ge \frac{b}{\varphi(a)\log g} \sum_{\substack{v=1 \\ \gcd(v,a)=1}}^{a-1} (\ell a + v) - \frac{a}{\varphi(a)\log g}\,.\end{aligned}$$

Using the well-known identity

$$\sum_{\substack{v=1 \\ \gcd(v,a)=1}}^{a-1} v = \frac{a\varphi(a)}{2}\,,$$

we deduce

$$\begin{aligned}\pi^*_{\ell,a,b} &\ge \frac{b}{\varphi(a)\log g}\left(\ell a\varphi(a) + \frac{a\varphi(a)}{2}\right) - \frac{a}{\varphi(a)\log g} \\ &> \frac{(2\ell+1)a}{2(\ell a + a - 1)}\frac{g}{\log g}.\end{aligned}$$

This completes the proof of Theorem 3.

2.4 Proof of Theorem 4

We first treat the case $a = 2$. Since $\gcd(a,b) = 1$, b is an odd integer greater than 2. Suppose p is a prime in $S_1(2,b)$ with $p \le g_{1,2,b}$. By Lemma 1, there exists a non-negative

integer x such that $p = 2b + 2x + b$ with $1 \le x \le b-1$. However, this contradicts $p \le g_{1,2,b} = 4b - 2 - b$. Hence $\pi^*_{1,2,b} = \pi(g_{1,2,b})$. If $g_{1,2,b} \ge 17$, Lemma 11 gives

$$\pi^*_{1,2,b} = \pi(g_{1,2,b}) > \frac{g_{1,2,b}}{\log g_{1,2,b}}.$$

If $g_{1,2,b} = 3b - 2 < 17$, then $b = 3$ or $b = 5$. Direct computation yields

$$\pi^*_{1,2,3} = \#\{2,3,5,7\} > \frac{7}{\log 7} = \frac{g_{1,2,3}}{\log g_{1,2,3}},$$

and

$$\pi^*_{1,2,5} = \#\{2,3,5,7,11,13\} > \frac{13}{\log 13} = \frac{g_{1,2,5}}{\log g_{1,2,5}}.$$

Thus Theorem 4 holds for $a = 2$. In what follows, we assume $a \ge 3$.

Let ϑ be a positive number with $\vartheta < 1$, and set $\theta = (\ell+\vartheta)/(\ell+1)$ so that $\theta g > \ell ab + a$. Then

$$a < \theta g - \ell ab = \vartheta ab - \theta a - \theta b < \vartheta ab$$

and consequently $0 < \vartheta - \theta/b - \theta/a < \vartheta < 1$. By Lemma 1,

$$\begin{aligned}\pi^*_{\ell,a,b} &\ge \pi(\theta g) - \#\{p \in \mathcal{P} : \ell ab < p \le \theta g, p \in S\} \\ &= \pi(\theta g) - \#\{p \in \mathcal{P} : \ell ab < p \le \theta g, p = \ell ab + ax + by, (x,y) \in \mathbb{N}^2_{(0)}\}. \end{aligned} \tag{6}$$

If $y = 0$, then $p = a(\ell b + x)$, which is composite for $\ell \ge 1$ and $b > a > 2$. Hence only terms with $y \ge 1$ contribute, and we obtain

$$\begin{aligned}&\#\{p \in \mathcal{P} : \ell ab < p \le \theta g, p = \ell ab + ax + by, (x,y) \in \mathbb{N}^2_{(0)}\} \\ &\le \sum_{\substack{1 \le y \le (\theta g - \ell ab)/b \\ \gcd(y,a)=1}} \big(\pi(\theta g; a, by) - \pi(\ell ab; a, by)\big). \end{aligned} \tag{7}$$

Since $\theta g - \ell ab < \vartheta ab$, combining (6) and (7) gives

$$\pi^*_{\ell,a,b} \ge \pi(\theta g) - \sum_{\substack{1 \le y \le \vartheta a \\ \gcd(y,a)=1}} \big(\pi(\theta g; a, by) - \pi(\ell ab; a, by)\big). \tag{8}$$

Because $\theta g - \ell ab > a$, Lemma 6 yields

$$\begin{aligned}\pi(\theta g; a, by) - \pi(\ell ab; a, by) &< \frac{2(\theta g - \ell ab)}{\varphi(a) \log\big((\theta g - \ell ab)/a\big)} \\ &= \frac{2}{\varphi(a)} \frac{\log g}{\log\big((\theta g - \ell ab)/a\big)} \frac{\theta g - \ell ab}{g} \frac{g}{\log g}. \end{aligned} \tag{9}$$

Recalling $g = (\ell+1)ab - a - b$ and $\theta = (\ell+\vartheta)/(\ell+1)$, we compute

$$\frac{\theta g - \ell ab}{g} = \frac{\vartheta ab - \theta a - \theta b}{(\ell+1)ab - a - b} = \frac{\vartheta - \theta/b - \theta/a}{\ell + 1 - 1/b - 1/a}. \tag{10}$$

Furthermore,

$$\frac{\log\big((\theta g-\ell ab)/a\big)}{\log g}=1+\frac{\log\big((\theta-\ell ab)/g\big)}{\log g}-\frac{\log a}{\log g}$$
$$>1-\frac{\log a}{\log g}-\frac{\log(\ell+1)}{\log g}-\frac{\log(\vartheta-\theta/a-\theta/b)^{-1}}{\log g}:=H(\ell,a,b,\vartheta). \tag{11}$$

If $\theta g>17$, Lemma 11 gives

$$\pi(\theta g)>\frac{\theta g}{\log(\theta g)}>\theta\frac{g}{\log g}. \tag{12}$$

Substituting (9), (10), (11), and (12) into (8) yields

$$\pi^*_{\ell,a,b}\geq\pi(\theta g)-\sum_{\substack{1\leq y\leq\vartheta a\\ \gcd(y,a)=1}}\big(\pi(\theta g;a,by)-\pi(\ell ab;a,by)\big)$$
$$>\left(\theta-\frac{2}{\varphi(a)}\left(\sum_{\substack{1\leq y\leq\vartheta a\\ \gcd(y,a)=1}}1\right)H(\ell,a,b,\vartheta)^{-1}\frac{\vartheta-\theta/b-\theta/a}{\ell+1-1/b-1/a}\right)\frac{g}{\log g}.$$

We now consider four separate cases to complete the estimate.

Case 1. $a>\max\{\ell,6\cdot10^4\}$. Applying Lemma 7 yields

$$\pi^*_{\ell,a,b}>\frac{\ell+0.02258}{\ell+1}\frac{g}{\log g}.$$

Case 2. $\max\{\ell,650\}<a\leq6\cdot10^4$. We obtain

$$\pi^*_{\ell,a,b}>\frac{\ell+0.02088778\ldots}{\ell+1}\frac{g}{\log g},$$

with equality attained when $\ell=669$ and $a=670$.

Case 3. $\max\{\ell,15\}<a\leq650$. If $b\leq1000$ with $\gcd(a,b)=1$, by Lemmas 2 and 11,

$$\pi^*_{\ell,a,b}>\frac{\ell+0.021}{\ell+1}\frac{g}{\log g}.$$

If $b\geq1001$ with $\gcd(a,b)=1$, we have

$$\pi^*_{\ell,a,b}>\frac{\ell+0.02315834\ldots}{\ell+1}\frac{g}{\log g},$$

with equality attained when $\ell=563$ and $a=564$.

Case 4. $\max\{\ell,2\}<a\leq15$. If $b\leq4$ with $\gcd(a,b)=1$, then the only admissible pair is $(a,b)=(3,4)$, with $\ell=1$ or $\ell=2$. A direct verification gives

$$\pi^*_{1,3,4}>\frac{g_{1,3,4}}{\log g_{1,3,4}}\quad\text{and}\quad\pi^*_{2,3,4}>\frac{g_{2,3,4}}{\log g_{2,3,4}}.$$

If $b \geq 5$, by Lemmas 11 and 1,

$$\pi^*_{\ell,a,b} > \frac{\ell + 0.066}{\ell + 1} \frac{g}{\log g}.$$

Having established the required lower bound in all cases, we conclude the proof of Theorem 4.

3 Concluding remarks and future directions

The results presented in this paper reveal an intriguing interplay between numerical semigroups and certain aspects of prime number distribution.

Several natural questions remain open. In particular, it would be of interest to investigate whether analogous phenomena occur for numerical semigroups generated by more than two integers.

Consider, for instance, the triple

$$A = \left\{ \frac{a^n - b^n}{a - b}, \frac{a^{n+1} - b^{n+1}}{a - b}, \frac{a^{n+2} - b^{n+2}}{a - b} \right\}$$

with $a > b$ and $\gcd(a, b) = 1$, it is conjectured that

$$\pi_{A,\ell} \sim \frac{\pi\big(g_\ell(A)\big)}{2} \quad (n \to \infty)\,,$$

independently of ℓ. Let $\nu_n = \nu_n(a, b) = (a^n - b^n)/(a - b)$ $(a > b)$. Define $q = \big\lfloor \nu_n/(a+b) \big\rfloor$ and $r = \nu_n - (a+b)q$. Then, following [15], for $0 \leq \ell \leq \big\lfloor \nu_n/(a+b) \big\rfloor$, we have the following explicit formulas for the Frobenius number: When $r\nu_{n+1} > ab\nu_n$,

$$g_\ell(A) = (r-1)\nu_{n+1} - \nu_n + \left(\ell + \left\lfloor \frac{\nu_n}{a+b} \right\rfloor \right) \nu_{n+2}\,;$$

when $r\nu_{n+1} < ab\nu_n$,

$$g_\ell(A) = (a+b-1)\nu_{n+1} - \nu_{n+2} - \nu_n + \left(\ell + \left\lfloor \frac{\nu_n}{a+b} \right\rfloor \right) \nu_{n+2}\,.$$

We conclude by examining an extremal case. For an integer $a \geq 3$, consider the set $A = \{a, a+1, \ldots, 2a-1\}$. As shown in [14],

$$g_0(A) = a - 1 \quad \text{and} \quad S_0(A) = \{2a, 2a+1, 2a+2, \ldots\}\,,$$

and for $1 \leq \ell \leq (a-1)/2$,

$$g_\ell(A) = 2a + 2\ell - 1 \quad \text{and} \quad S_\ell(A) = \{2a + 2\ell, 2a + 2\ell + 1, 2a + 2\ell + 2, \ldots\}\,.$$

Since no prime p satisfies $p \in S_\ell(A)$ and $p \leq g_\ell(A)$, we have $\pi_{\ell,A} = 0$ for all $0 \leq \ell \leq (a-1)/2$. Consequently, any result analogous to the case $A = \{a, b\}$ fails to hold in this setting.

The extended version together with Yuchen Ding and Honghu Liu is published in [9].

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Institute of Mathematics
Henan Academy of Sciences
Zhengzhou 450046
CHINA
and
Department of Mathematics
Institute of Science Tokyo
Tokyo 152-8551
JAPAN
E-mail address: komatsu@zstu.edu.cn; komatsu.t.al@m.titech.ac.jp