Shalika functions on $\mathrm{GL}_4(\mathbb{R})$

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1 Whittaker and Shalika functions on $\mathrm{GL}_{2m}(\mathbb{R})$

The aim of this manuscript is to announce explicit formulas for Shalika functions on $\mathrm{GL}_4(\mathbb{R})$ for certain generalized principal series representations by solving a system of partial differential equations. As an application, we compute the archimedean part of the Friedberg-Jacquet integral.

Let $n = 2m$ and $G = \mathrm{GL}_n(\mathbb{R})$. Define a maximal unipotent subgroup U and Shalika subgroup S of G by

$$U = \{(u_{i,j}) \in G \mid u_{i,i} = 1, u_{i,j} = 0 \ (i > j)\},$$
$$S = \left\{ s(h, X) = \begin{pmatrix} h & \\ & h \end{pmatrix} \begin{pmatrix} 1_m & X \\ & 1_m \end{pmatrix} \middle| \begin{array}{l} h \in \mathrm{GL}_m(\mathbb{R}) \\ X \in \mathrm{M}_m(\mathbb{R}) \end{array} \right\}.$$

Let $\psi(x) = \exp(2\pi i x)$ $(x \in \mathbb{R})$. For $R = U, S$, define a character ψ_R of R by

$$\psi_U((u_{i,j})) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n}),$$
$$\psi_S(s(h, X)) = \psi(\mathrm{tr}(X)).$$

We set

$$C^\infty(\psi_R) = \{f \in C^\infty(G, \mathbb{C}) \mid f(rg) = \psi_R(r)f(g), \forall (r, g) \in R \times G\}$$

on which G acts by right translation. For an irreducible Casselman-Wallach representation π of G, we set

$$I_R(\pi) = \mathrm{Hom}_G(\pi, C^\infty(\psi_R)).$$

The following is known for the dimension of the intertwining space $I_R(\pi)$.

- $R = U$ (Whittaker model)
  - $\dim I_U(\pi) \le 1$ ([8], [7], [10])
  - $\dim I_U(\pi) = 1$ (π is generic) $\iff$ π is isomorphic to irreducible generalized principal series representation induced from a parabolic subgroup corresponding to the partition $(2, \ldots, 2, 1 \ldots, 1)$ of $n = 2m$ ([9]).
- $R = S$ (Shalika model)

- $\dim I_S(\pi) \le 1$ ([1]),
- For generic π, $\dim I_S(\pi) = 1 \iff \pi$ is symplectic ([2]).

We set

$$\begin{aligned} \mathrm{Wh}(\pi,\psi) &= \{\Phi(f) \mid \Phi \in I_U(\pi), f \in \pi\} \quad \text{(space of Whittaker functions)},\\ \mathrm{Sh}(\pi,\psi) &= \{\Phi(f) \mid \Phi \in I_S(\pi), f \in \pi\} \quad \text{(space of Shalika functions)}. \end{aligned}$$

Let $K = \mathrm{O}(n)$. For a K-type (τ, V_τ) of π, we have a K-homomorphism $\varphi : V_\tau \hookrightarrow \pi \xrightarrow{\Phi} \mathrm{Wh}(\pi,\psi)$ or $\mathrm{Sh}(\pi,\psi)$, where $\Phi \in I_R(\pi)$. By definition, we have

$$\varphi(v)(rgk) = \psi_R(r)\varphi(\tau(k)v)(g), \quad \forall (r,g,k) \in R \times G \times K.$$

Let

$$A = \begin{cases} \{\mathrm{diag}(a_1,\dots,a_{2m}) \mid a_i > 0\} & \text{if } R = U,\\ \{\mathrm{diag}(a_1,\dots,a_m,1,\dots,1) \mid a_i > 0\} & \text{if } R = S. \end{cases}$$

Since we have a decomposition $G = RAK$, Whittaker/Shalika function $\varphi(v)$ is determined by $\varphi(v)|_A$ (radial part of $\varphi(v)$). By using $(\mathfrak{g}_\mathbb{C}, K)$-module structure of π, we give a system of partial differential equations for $\{\varphi(v)(a) \mid v \in V_\tau\}$ $(a \in A)$

2 Representation theory of $G = \mathrm{GL}_4(\mathbb{R})$ and $K = \mathrm{O}(4)$

We briefly recall representation theory of $G = \mathrm{GL}_4(\mathbb{R})$ and $K = \mathrm{O}(4)$. See [6] for details.

Let us briefly explain a way of construction of irreducible representation of K. Let $(\tau_{\mathrm{st}}, V_{\mathrm{st}} = \mathrm{M}_{4,1}(\mathbb{C}))$ be the standard representation of K: $\tau_{\mathrm{st}}(h)v = hv$ $(h \in K,\ v \in V_{\mathrm{st}})$, where hv is the ordinary product of matrices h and v. Let ξ_i $(1 \le i \le 4)$ be the matrix unit in V_{st} with 1 at $(i,1)$-th entry and 0 at other entries. For $1 \le i,j \le 4$, we set $\xi_{ij} = \xi_i \wedge \xi_j \in V_{\mathrm{st}} \wedge_\mathbb{C} V_{\mathrm{st}}$. We define the graded $\mathbb{C}$-algebra $\mathcal{R} = \bigoplus_{\lambda_1 \ge \lambda_2 \ge 0} \mathcal{R}_{(\lambda_1,\lambda_2)}$ by

$$\mathcal{R} = \mathrm{Sym}(V_{\mathrm{st}}) \otimes_\mathbb{C} \mathrm{Sym}(V_{\mathrm{st}} \wedge_\mathbb{C} V_{\mathrm{st}}), \qquad \mathcal{R}_\lambda = \mathrm{Sym}^{\lambda_1-\lambda_2}(V_{\mathrm{st}}) \otimes_\mathbb{C} \mathrm{Sym}^{\lambda_2}(V_{\mathrm{st}} \wedge_\mathbb{C} V_{\mathrm{st}}).$$

Here $\mathrm{Sym}(V) = \bigoplus_{m \ge 0} \mathrm{Sym}^m(V)$ is the symmetric algebra on V with the usual grading. We regard $\mathcal{R}$ as a K-module via the action $\mathcal{T}$ which is induced from τ_{st}. Then $\mathcal{R}_{(\lambda_1,\lambda_2)}$ is a K-submodule of $\mathcal{R}$. We note that $\{\xi_i \mid 1 \le i \le 4\} \cup \{\xi_{jk} \mid 1 \le j < k \le 4\}$ is a system of generators of $\mathcal{R}$ as a $\mathbb{C}$-algebra.

Let $I_\mathcal{R}$ be the ideal of $\mathcal{R}$ generated by $\{(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 + (\xi_4)^2, \xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23}\} \cup \{\xi_1\xi_{i1} + \xi_2\xi_{i2} + \xi_3\xi_{i3} + \xi_4\xi_{i4} \mid 1 \le i \le 4\} \cup \{\xi_{i1}\xi_{j1} + \xi_{i2}\xi_{j2} + \xi_{i3}\xi_{j3} + \xi_{i4}\xi_{j4} \mid 1 \le i \le j \le 4\} \cup \{\xi_i\xi_{jk} - \xi_j\xi_{ik} + \xi_k\xi_{ij} \mid 1 \le i < j < k \le 4\}$. We can show that the ideal $I_\mathcal{R}$ is K-invariant. The action of K on $\mathcal{R}/I_\mathcal{R}$ induced from $\mathcal{T}$ is denoted by $\widehat{\mathcal{T}}$. Let $\mathrm{q}_\mathcal{R} \colon \mathcal{R} \ni r \mapsto r + I_\mathcal{R} \in \mathcal{R}/I_\mathcal{R}$ be the natural surjection. Let $\lambda = (\lambda_1,\lambda_2,\lambda_3) \in \mathbb{Z}^2 \times \{0,1\}$ with $\lambda_1 \ge \lambda_2 \ge 0$. We define a representation $(\tau_\lambda, V_\lambda)$ of K by

$$\tau_\lambda(k) = (\det k)^{\lambda_3}\widehat{\mathcal{T}}(k) \quad (k \in K), \qquad V_\lambda = \mathrm{q}_\mathcal{R}(\mathcal{R}_{(\lambda_1,\lambda_2)}).$$

Let

$$\Lambda_K = \{\lambda = (\lambda_1,\lambda_2,\lambda_3) \in \mathbb{Z}^2 \times \{0,1\} \mid \lambda_1 \ge \lambda_2 \ge 0,\ \lambda_2\lambda_3 = 0\}.$$

Then we can show that the correspondence $\lambda \leftrightarrow \tau_\lambda$ gives a bijection between Λ_K and the set of equivalence classes of irreducible representations of K ([6, Propsition 2.5]).

Let S_λ be the set of $l = (l_1, l_2, l_3, l_4, l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}) \in (\mathbb{Z}_{\geq 0})^{10}$ satisfying

$$l_1 + l_2 + l_3 + l_4 = \lambda_1 - \lambda_2, \qquad l_{12} + l_{13} + l_{14} + l_{23} + l_{24} + l_{34} = \lambda_2.$$

For $l = (l_1, l_2, l_3, l_4, l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}) \in S_\lambda$, we set

$$u_l = \mathrm{q}_{\mathcal{R}}\Bigg(\prod_{1 \leq i \leq 4} (\xi_i)^{l_i} \prod_{1 \leq j < k \leq 4} (\xi_{jk})^{l_{jk}} \Bigg).$$

We note that $\{u_l\}_{l \in S_\lambda}$ forms a system of generators of V_λ as a $\mathbb{C}$-vector space. It is convenient to set $u_l = 0$ if $l \notin (\mathbb{Z}_{\geq 0})^{10}$. We set $\mathbf{0} = (0,0,0,0,0,0,0,0,0,0)$, $e_1 = (1,0,0,0,0,0,0,0,0,0)$, $e_{12} = e_{21} = (0,0,0,0,1,0,0,0,0,0)$, etc. For $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda_K$, we have the following relations.

- When $\lambda_1 - \lambda_2 \geq 2$, for $l \in S_{\lambda-(2,0,0)}$, we have

$$u_{l+2e_1} + u_{l+2e_2} + u_{l+2e_3} + u_{l+2e_4} = 0.$$

- When $\lambda_1 > \lambda_2 > 0$, for $l \in S_{\lambda-(2,1,0)}$, we have

$$\begin{aligned} &\textstyle\sum_{1 \leq j \leq 4,\, j \neq i} \mathrm{sgn}(j-i) u_{l+e_j+e_{ij}} = 0 \quad (1 \leq i \leq 4), \\ &u_{l+e_i+e_{jk}} - u_{l+e_j+e_{ik}} + u_{l+e_k+e_{ij}} = 0 \quad (1 \leq i < j < k \leq 4). \end{aligned}$$

- When $\lambda_2 \geq 2$, for $l \in S_{\lambda-(2,2,0)}$, we have

$$\begin{aligned} &\textstyle\sum_{1 \leq k \leq 4,\, k \notin \{i,j\}} \mathrm{sgn}((k-i)(k-j)) u_{l+e_{ik}+e_{jk}} = 0 \quad (1 \leq i, j \leq 4), \\ &u_{l+e_{12}+e_{34}} - u_{l+e_{13}+e_{24}} + u_{l+e_{14}+e_{23}} = 0. \end{aligned}$$

We recall the definition of generalized principal series representations of G. We specify certain representations of $G_1 = \mathrm{GL}(1,\mathbb{R})$ and $G_2 = \mathrm{GL}(2,\mathbb{R})$ as follows:

- For $\nu \in \mathbb{C}$ and $\delta \in \{0,1\}$, we define a character $\chi_{(\nu,\delta)}$ of G_1 by $\chi_{(\nu,\delta)}(t) = \mathrm{sgn}(t)^\delta |t|^\nu$ $(t \in G_1)$.
- For $\nu \in \mathbb{C}$ and $\kappa \in \mathbb{Z}_{\geq 2}$, let $D_{(\nu,\kappa)}$ be an irreducible Hilbert representation of G_2 such that $D_{(\nu,\kappa)}(t1_2) = t^{2\nu}$ $(t \in \mathbb{R}_+)$ and $D_{(\nu,\kappa)} \simeq D_\kappa^+ \oplus D_\kappa^-$ as $(\mathfrak{sl}(2,\mathbb{R}), \mathrm{SO}(2))$-modules, where $D_\kappa^\pm$ are the discrete series representations of $\mathrm{SL}(2,\mathbb{R})$ with the minimal $\mathrm{SO}(2)$-type: $\mathrm{SO}(2) \ni \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \mapsto e^{\pm\sqrt{-1}\kappa\theta} \in \mathbb{C}^\times$.

For $\mathbf{n} \in \{(1,1,1,1), (2,1,1), (2,2)\}$, we associate the block upper triangular parabolic subgroup $P_\mathbf{n} = N_\mathbf{n} M_\mathbf{n}$ of G in the usual manner. Here $N_\mathbf{n}$ and $M_\mathbf{n}$ are the unipotent radical and the Levi part of $P_\mathbf{n}$, respectively. Because of Vogan's characterization, any irreducible admissible generic representation π of G is infinitesimally equivalent to some (normalized induction) $\mathrm{Ind}_{P_\mathbf{n}}^G(\sigma)$, which is induced from one of the following representations:

- Case 1 ($\mathbf{n} = (1, 1, 1, 1)$):
  $\sigma = \chi_{(\nu_1,\delta_1)} \boxtimes \chi_{(\nu_2,\delta_2)} \boxtimes \chi_{(\nu_3,\delta_3)} \boxtimes \chi_{(\nu_4,\delta_4)}$ where $\nu_1, \nu_2, \nu_3, \nu_4 \in \mathbb{C}$ and $\delta_1, \delta_2, \delta_3, \delta_4 \in \{0, 1\}$ with $\delta_1 \geq \delta_2 \geq \delta_3 \geq \delta_4$.
- Case 2 ($\mathbf{n} = (2, 1, 1)$):
  $\sigma = D_{(\nu_1,\kappa_1)} \boxtimes \chi_{(\nu_2,\delta_2)} \boxtimes \chi_{(\nu_3,\delta_3)}$ where $\nu_1, \nu_2, \nu_3 \in \mathbb{C}$, $\kappa_1 \in \mathbb{Z}_{\geq 2}$ and $\delta_2, \delta_3 \in \{0, 1\}$ with $\delta_2 \geq \delta_3$.
- Case 3 ($\mathbf{n} = (2, 2)$):
  $\sigma = D_{(\nu_1,\kappa_1)} \boxtimes D_{(\nu_2,\kappa_2)}$ where $\nu_1, \nu_2 \in \mathbb{C}$ and $\kappa_1, \kappa_2 \in \mathbb{Z}_{\geq 2}$ with $\kappa_1 \geq \kappa_2$.

We call $\mathrm{Ind}_{P_{\mathbf{n}}}^{G}(\sigma)$ a generalized principal series representation of G. To discuss three kinds of the generalized principal series representations simultaneously, we set $\kappa_1, \kappa_2 \in \mathbb{Z}$ and $\delta_1, \delta_2, \delta_3 \in \{0, 1\}$ for undefined cases:

- (Case 1) $\kappa_1 := \delta_1 - \delta_4$, $\kappa_2 := \delta_2 - \delta_3$.
- (Case 2) $\delta_1 \equiv \kappa_1 \pmod 2$, $\kappa_2 := \delta_2 - \delta_3$.
- (Case 3) $\delta_1 \equiv \kappa_1 \pmod 2$, $\delta_2 \equiv \kappa_2 \pmod 2$, $\delta_3 := 0$.

Then we know that $\tau_{(\kappa_1,\kappa_2,\delta_3)}$ is the minimal K-type of $\mathrm{Ind}_{P_{\mathbf{n}}}^{G}(\sigma)$.

According to [2], among the above generic representations, the following representations have Shalika model:

- (Case 1)
  (i) $\delta_1 = \delta_2 = \delta_3 = \delta_4$ and $\nu_{\sigma(1)} + \nu_{\sigma(3)} = \nu_{\sigma(2)} + \nu_{\sigma(4)} = 0$ for some $\sigma \in \mathfrak{S}_4$.
  (ii) $\delta_1 = \delta_2 = 1$, $\delta_3 = \delta_4 = 0$ and $\nu_1 + \nu_2 = \nu_3 + \nu_4$.
- (Case 2) $\kappa_1 \in 2\mathbb{Z}_{\geq 1}$, $\delta_2 = \delta_3$, $\nu_1 = 0$ and $\nu_2 + \nu_3 = 0$.
- (Case 3)
  (i) $\kappa_1 = \kappa_2$ and $\nu_1 + \nu_2 = 0$,
  (ii) $\kappa_1 - \kappa_2 \in 2\mathbb{Z}_{\geq 0}$ and $\nu_1 = \nu_2 = 0$.

3 Partial differential equations

For the generator $\{u_l \mid l = (l_1, \ldots, l_{34}) \in S_{(\kappa_1,\kappa_2,\delta_3)}\}$ of $V_{(\kappa_1,\kappa_2,\delta_3)}$, and a K-homomorphism $\varphi : V_{(\kappa_1,\kappa_2,\delta_3)} \to \mathrm{Sh}(\pi, \psi)$, we can derive a system of partial differential equations for $\{\varphi(u_l)|_A \mid l \in S_{(\kappa_1,\kappa_2,\delta_3)}\}$. As in Whittaker functions [6], the system of partial differential equations consists of 4 Capelli equations and 10 Dirac-Schmid equations. For $a = \mathrm{diag}(a_1, a_2, 1, 1) \in A$, we set $\partial_i = a_i \frac{\partial}{\partial a_i}$ $(i = 1, 2)$.

Proposition 1. *(Dirac-Schmid equations) Retain the notation. The functions* $\{\varphi_l = \varphi(u_l) \mid l \in S_{(\kappa_1,\kappa_2,\delta_3)}\}$ *satisfy the following.*

(i) *Assume that* $\kappa_1 > \kappa_2$. *For* $l = (l_1, l_2, l_3, l_4, l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}) \in S_{(\kappa_1-1,\kappa_2,\delta_3)}$,

$$\begin{aligned}
&\{\partial_1 - \nu_1 - \tfrac{\kappa_1}{2} - 1 + \tfrac{a_1^2}{a_1^2-a_2^2}(l_2+1)\}\varphi_{l+e_1} + (2\pi\sqrt{-1})a_1\varphi_{l+e_3} \\
&+ \tfrac{a_1^2}{a_1^2-a_2^2}(-l_1\varphi_{l-e_1+2e_2} - l_{13}\varphi_{l-e_{13}+e_2+e_{23}} - l_{14}\varphi_{l-e_{14}+e_2+e_{24}} \\
&\qquad + l_{23}\varphi_{l-e_{23}+e_2+e_{13}} + l_{24}\varphi_{l-e_{24}+e_2+e_{14}}) \\
&+ \tfrac{a_1a_2}{a_1^2-a_2^2}(-l_3\varphi_{l-e_3+e_2+e_4} + l_4\varphi_{l-e_4+e_2+e_3} - l_{13}\varphi_{l+e_2-e_{13}+e_{14}} \\
&\qquad + l_{14}\varphi_{l+e_2-e_{14}+e_{13}} - l_{23}\varphi_{l+e_2-e_{23}+e_{24}} + l_{24}\varphi_{l+e_2-e_{24}+e_{23}}) = 0,
\end{aligned}$$

$$\begin{aligned}
&\{\partial_2 - \nu_1 - \tfrac{\kappa_1}{2} - 1 - \tfrac{a_2^2}{a_1^2-a_2^2}(l_1+1)\}\varphi_{l+e_2} + (2\pi\sqrt{-1})a_2\varphi_{l+e_4} \\
&+ \tfrac{a_2^2}{a_1^2-a_2^2}(l_2\varphi_{l-e_2+2e_1} - l_{13}\varphi_{l+e_1-e_{13}+e_{23}} - l_{14}\varphi_{l+e_1-e_{14}+e_{24}} \\
&\qquad + l_{23}\varphi_{l+e_1-e_{23}+e_{13}} + l_{24}\varphi_{l+e_1-e_{24}+e_{14}}) \\
&+ \tfrac{a_1a_2}{a_1^2-a_2^2}(-l_3\varphi_{l-e_3+e_1+e_4} + l_4\varphi_{l-e_4+e_1+e_3} - l_{13}\varphi_{l+e_1-e_{13}+e_{14}} \\
&\qquad + l_{14}\varphi_{l+e_1-e_{14}+e_{13}} - l_{23}\varphi_{l+e_1-e_{23}+e_{24}} + l_{24}\varphi_{l+e_1-e_{24}+e_{23}}) = 0,
\end{aligned}$$

$$\begin{aligned}
&\{-\partial_1 - \nu_1 + \tfrac{\kappa_1}{2} + 1 - \tfrac{a_1^2}{a_1^2-a_2^2}(l_4+1)\}\varphi_{l+e_3} + (2\pi\sqrt{-1})a_1\varphi_{l+e_1} \\
&+ \tfrac{a_1^2}{a_1^2-a_2^2}(l_3\varphi_{l-e_3+2e_4} + l_{13}\varphi_{l+e_4-e_{13}+e_{14}} - l_{14}\varphi_{l+e_4-e_{14}+e_{13}} \\
&\qquad + l_{23}\varphi_{l+e_4-e_{23}+e_{24}} - l_{24}\varphi_{l+e_4-e_{24}+e_{23}}) \\
&+ \tfrac{a_1a_2}{a_1^2-a_2^2}(l_1\varphi_{l-e_1+e_2+e_4} - l_2\varphi_{l-e_2+e_1+e_4} + l_{13}\varphi_{l+e_4-e_{13}+e_{23}} \\
&\qquad + l_{14}\varphi_{l+e_4-e_{14}+e_{24}} - l_{23}\varphi_{l+e_4-e_{23}+e_{13}} - l_{24}\varphi_{l+e_4-e_{24}+e_{14}}) = 0,
\end{aligned}$$

$$\begin{aligned}
&\{-\partial_2 - \nu_1 + \tfrac{\kappa_1}{2} + 1 + \tfrac{a_2^2}{a_1^2-a_2^2}(l_3+1)\}\varphi_{l+e_4} + (2\pi\sqrt{-1})a_2\varphi_{l+e_2} \\
&+ \tfrac{a_2^2}{a_1^2-a_2^2}(-l_4\varphi_{l-e_4+2e_3} + l_{13}\varphi_{l+e_3-e_{13}+e_{14}} - l_{14}\varphi_{l+e_3-e_{14}+e_{13}} \\
&\qquad + l_{23}\varphi_{l+e_3-e_{23}+e_{24}} + l_{24}\varphi_{l+e_3-e_{24}+e_{23}} \\
&+ \tfrac{a_1a_2}{a_1^2-a_2^2}(l_1\varphi_{l-e_1+e_2+e_3} - l_2\varphi_{l-e_2+e_1+e_3} + l_{13}\varphi_{l+e_3-e_{13}+e_{23}} \\
&\qquad + l_{14}\varphi_{l+e_3-e_{14}+e_{24}} - l_{23}\varphi_{l+e_3-e_{23}+e_{13}} - l_{24}\varphi_{l+e_3-e_{24}+e_{14}}) = 0.
\end{aligned}$$

(ii) *Assume that* $\kappa_1 \geq \kappa_2 \geq 1$. *For* $l = (l_1, l_2, l_3, l_4, l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}) \in S_{(\kappa_1-1,\kappa_2-1,\delta_3)}$, *we have*

$$(\partial_1 + \partial_2 - \tfrac{\kappa_1+\kappa_2}{2} - 1 - \nu_1 - \nu_2)\varphi_{l+e_{12}} - (2\pi\sqrt{-1})a_1\varphi_{l+e_{23}} + (2\pi\sqrt{-1})a_2\varphi_{l+e_{14}} = 0,$$

$$(\partial_1 + \partial_2 - \tfrac{\kappa_1+\kappa_2}{2} - 1 + \nu_1 + \nu_2)\varphi_{l+e_{34}} - (2\pi\sqrt{-1})a_1\varphi_{l+e_{14}} + (2\pi\sqrt{-1})a_2\varphi_{l+e_{23}} = 0,$$

$$\begin{aligned}
&(l_1 + l_2 - l_3 - l_4 - 2\nu_1 - 2\nu_2)(a_1\varphi_{l+e_{13}} + a_2\varphi_{l+e_{24}}) \\
&- 2l_1a_1\varphi_{l-e_1+e_4+e_{34}} + 2l_2a_2\varphi_{l-e_2+e_3+e_{34}} - 2l_3a_1\varphi_{l-e_3+e_2+e_{12}} + 2l_4a_2\varphi_{l-e_4+e_1+e_{12}} = 0,
\end{aligned}$$

$$
\begin{aligned}
&(l_1 + l_2 - l_3 - l_4 + 2\nu_1 + 2\nu_2)(a_1\varphi_{l+e_{24}} + a_2\varphi_{l+e_{13}}) \\
&\ + 2l_1a_1\varphi_{l-e_1+e_4+e_{12}} - 2l_2a_2\varphi_{l-e_2+e_3+e_{12}} + 2l_3a_1\varphi_{l-e_3+e_2+e_{34}} - 2l_4a_2\varphi_{l-e_4+e_1+e_{34}} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\{(\partial_1 - \partial_2)(\partial_1 + \partial_2 - \tfrac{\kappa_1+\kappa_2}{2} - \nu_1 - \nu_2 - 1) - (2\pi)^2(a_1^2 - a_2^2)\}\varphi_{l+e_{12}} \\
&+ (2\pi\sqrt{-1})(l_1a_1\varphi_{l-e_1+e_3+e_{12}} - l_2a_2\varphi_{l-e_2+e_4+e_{12}} - l_3a_1\varphi_{l-e_3+e_1+e_{12}} + l_4a_2\varphi_{l-e_4+e_2+e_{12}}) \\
&+ (2\pi\sqrt{-1})(\tfrac{\kappa_1+\kappa_2}{2} - 1 - l_{12} - l_{34} + \nu_1 + \nu_2)(a_2\varphi_{l+e_{14}} + a_1\varphi_{l+e_{23}}) \\
&+ (2\pi\sqrt{-1})\{l_{14}(a_1\varphi_{l-e_{14}+e_{12}+e_{34}} + a_2\varphi_{l-e_{14}+2e_{12}}) \\
&\qquad + l_{23}(a_1\varphi_{l-e_{23}+2e_{12}} + a_2\varphi_{l-e_{23}+e_{12}+e_{34}})\} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\{(\partial_1 - \partial_2)(\partial_1 + \partial_2 - \tfrac{\kappa_1+\kappa_2}{2} + \nu_1 + \nu_2 - 1) - (2\pi)^2(a_1^2 - a_2^2)\}\varphi_{l+e_{34}} \\
&+ (2\pi\sqrt{-1})(l_1a_1\varphi_{l-e_1+e_3+e_{34}} - l_2a_2\varphi_{l-e_2+e_4+e_{34}} - l_3a_1\varphi_{l-e_3+e_1+e_{34}} + l_4a_2\varphi_{l-e_4+e_2+e_{34}}) \\
&+ (2\pi\sqrt{-1})(\tfrac{\kappa_1+\kappa_2}{2} - 1 - l_{12} - l_{34} + \nu_1 + \nu_2)(a_1\varphi_{l+e_{14}} + a_2\varphi_{l+e_{23}}) \\
&+ (2\pi\sqrt{-1})\{l_{14}(a_1\varphi_{l-e_{14}+2e_{34}} + a_2\varphi_{l-e_{14}+e_{12}+e_{34}}) \\
&\qquad + l_{23}(a_1\varphi_{l-e_{23}+e_{12}+e_{34}} + a_2\varphi_{l-e_{23}+2e_{34}})\} = 0.
\end{aligned}
$$

Proposition 2. *Retain the notation.*

$$
\{\Delta_2 + (2\pi\sqrt{-1})(a_1\mathfrak{K}_{13} + a_2\mathfrak{K}_{24}) - \tfrac{2a_1^2a_2^2}{(a_1^2-a_2^2)^2}(\mathfrak{K}_{12}^2 + \mathfrak{K}_{34}^2) - \tfrac{2a_1a_2(a_1^2+a_2^2)}{(a_1^2-a_2^2)^2}\mathfrak{K}_{12}\mathfrak{K}_{34}\}\varphi_l = \gamma_2\varphi_l,
$$

and

$$
\begin{aligned}
&\Delta_4 + (2\pi\sqrt{-1})a_1\Big\{-(\partial_2 - \tfrac{1}{2})(\partial_2 - \tfrac{3}{2}) - \tfrac{a_2^2}{a_1^2-a_2^2}\partial_1 + \tfrac{a_1^2}{a_1^2-a_2^2}\partial_2 - \tfrac{3}{2} + (2\pi)^2a_2^2\Big\}\mathfrak{K}_{13} \\
&+ (2\pi\sqrt{-1})a_2\Big\{-(\partial_1 - \tfrac{1}{2})(\partial_1 - \tfrac{3}{2}) - \tfrac{a_2^2}{a_1^2-a_2^2}\partial_1 + \tfrac{a_1^2}{a_1^2-a_2^2}\partial_2 - \tfrac{3}{2} + (2\pi)^2a_1^2\Big\}\mathfrak{K}_{24} \\
&+ (2\pi\sqrt{-1})\tfrac{a_1^2a_2}{a_1^2-a_2^2}(\partial_1 + \partial_2 - 2)(\mathfrak{K}_{12}\mathfrak{K}_{14} + \mathfrak{K}_{34}\mathfrak{K}_{23}) \\
&+ (2\pi\sqrt{-1})\tfrac{a_1a_2^2}{a_1^2-a_2^2}(\partial_1 + \partial_2 - 2)(\mathfrak{K}_{12}\mathfrak{K}_{23} + \mathfrak{K}_{34}\mathfrak{K}_{14}) \\
&+ \tfrac{a_1^2a_2^2}{(a_1^2-a_2^2)^2}\Big\{-2(\partial_1 - \tfrac{1}{2})(\partial_2 - \tfrac{1}{2}) - \tfrac{2a_2^2}{a_1^2-a_2^2}\partial_1 + \tfrac{2a_1^2}{a_1^2-a_2^2}\partial_2 - 3 \\
&\qquad + \tfrac{4(a_1^2+a_2^2)^2}{(a_1^2-a_2^2)^2} - (2\pi)^2(a_1^2 + a_2^2)\Big\}(\mathfrak{K}_{12}^2 + \mathfrak{K}_{34}^2) \\
&+ \tfrac{a_1a_2}{a_1^2-a_2^2}\Big\{-\tfrac{2(a_1^2+a_2^2)}{a_1^2-a_2^2}(\partial_1 - \tfrac{1}{2})(\partial_2 - \tfrac{1}{2}) + \Big(-\tfrac{2a_2^2(a_1^2+a_2^2)}{(a_1^2-a_2^2)^2} + 1\Big)(\partial_1 - \tfrac{1}{2}) \\
&\qquad + \Big(\tfrac{2a_1^2(a_1^2+a_2^2)}{(a_1^2-a_2^2)^2} - 1\Big)(\partial_2 - \tfrac{1}{2}) + \tfrac{16a_1^2a_2^2(a_1^2+a_2^2)}{(a_1^2-a_2^2)^4} - (2\pi)^2\tfrac{(a_1^2+a_2^2)^2}{a_1^2-a_2^2}\Big\}\mathfrak{K}_{12}\mathfrak{K}_{34} \\
&- (2\pi\sqrt{-1})\tfrac{a_1^2a_2^3}{(a_1^2-a_2^2)^2}(\mathfrak{K}_{12}^2\mathfrak{K}_{24} + \mathfrak{K}_{34}^2\mathfrak{K}_{24}) - (2\pi\sqrt{-1})\tfrac{a_1^3a_2^2}{(a_1^2-a_2^2)^2}(\mathfrak{K}_{12}^2\mathfrak{K}_{13} + \mathfrak{K}_{34}^2\mathfrak{K}_{13}) \\
&- (2\pi\sqrt{-1})\tfrac{a_1^2a_2(a_1^2+a_2^2)}{(a_1^2-a_2^2)^2}\mathfrak{K}_{12}\mathfrak{K}_{34}\mathfrak{K}_{13} - (2\pi\sqrt{-1})\tfrac{a_1a_2^2(a_1^2+a_2^2)}{(a_1^2-a_2^2)^2}\mathfrak{K}_{12}\mathfrak{K}_{34}\mathfrak{K}_{24} \\
&+ \tfrac{a_1^4a_2^4}{(a_1^2-a_2^2)^4}(\mathfrak{K}_{12}^4 + \mathfrak{K}_{34}^4) + \tfrac{2a_1^3a_2^3(a_1^2+a_2^2)}{(a_1^2-a_2^2)^2}(\mathfrak{K}_{12}^3\mathfrak{K}_{34} + \mathfrak{K}_{12}\mathfrak{K}_{34}^3) + \tfrac{a_1^2a_2^2(a_1^4+4a_1^2a_2^2+a_2^4)}{(a_1^2-a_2^2)^4}\mathfrak{K}_{12}^2\mathfrak{K}_{34}^2 = \gamma_4\varphi_l,
\end{aligned}
$$

where

$$
\Delta_2 = -\partial_1^2 - \partial_2^2 + 2(\partial_1 + \partial_2) - \tfrac{5}{2} - \tfrac{a_1^2+a_2^2}{a_1^2-a_2^2}(\partial_1 - \partial_2) + (2\pi)^2(a_1^2 + a_2^2),
$$

$$\Delta_4 = (\partial_1-\tfrac{1}{2})(\partial_1-\tfrac{3}{2})(\partial_2-\tfrac{1}{2})(\partial_2-\tfrac{3}{2}) - \tfrac{a_1^2}{a_1^2-a_2^2}(\partial_1-\partial_2)(\partial_1-\tfrac{1}{2})(\partial_2-\tfrac{3}{2})$$
$$- \tfrac{a_2^2}{a_1^2-a_2^2}(\partial_1-\partial_2)(\partial_1-\tfrac{3}{2})(\partial_2-\tfrac{1}{2}) - \tfrac{a_1^2a_2^2}{(a_1^2-a_2^2)^2}(\partial_1-\partial_2)^2 + \tfrac{2a_1^2a_2^2(a_1^2+a_2^2)}{(a_1^2-a_2^2)^3}(\partial_1-\partial_2)$$
$$+ (2\pi)^2\Big\{-a_1^2(\partial_2-\tfrac{3}{2})^2 - a_2^2(\partial_1-\tfrac{3}{2})^2 - \tfrac{2a_1^2a_2^2}{a_1^2-a_2^2}(\partial_1-\partial_2)\Big\} + (2\pi)^4a_1^2a_2^2,$$

$$\gamma_2 = \begin{cases} \nu_1\nu_2+\nu_1\nu_3+\nu_1\nu_4+\nu_2\nu_3+\nu_2\nu_4+\nu_3\nu_4 & \textit{case 1}, \\ -\nu_2^2-\frac{1}{4}(\kappa_1-1)^2 & \textit{case 2}, \\ -\nu_1^2-\nu_2^2-\frac{1}{4}((\kappa_1-1)^2+(\kappa_2-1)^2) & \textit{case 3}, \end{cases}$$
$$\gamma_4 = \begin{cases} \nu_1\nu_2\nu_3\nu_4 & \textit{case 1}, \\ \frac{1}{4}\nu_2^2(\kappa_1-1)^2 & \textit{case 2}, \\ (\nu_1^2-\frac{1}{4}(\kappa_1-1)^2)(\nu_2^2-\frac{1}{4}(\kappa_2-1)^2) & \textit{case 3}, \end{cases}$$

and

$$\begin{aligned}
\mathfrak{K}_{12}\varphi_l &= l_1\varphi_{l-e_1+e_2} + l_2\varphi_{l-e_2+e_1} \\
&\quad + l_{13}\varphi_{l-e_{13}+e_{23}} + l_{14}\varphi_{l-e_{14}+e_{24}} + l_{23}\varphi_{l-e_{23}+e_{13}} + l_{24}\varphi_{l-e_{24}+e_{14}}, \\
\mathfrak{K}_{23}\varphi_l &= l_2\varphi_{l-e_2+e_3} + l_3\varphi_{l-e_3+e_2} \\
&\quad + l_{12}\varphi_{l-e_{12}+e_{13}} + l_{13}\varphi_{l-e_{13}+e_{12}} + l_{24}\varphi_{l-e_{24}+e_{34}} + l_{34}\varphi_{l-e_{34}+e_{24}}, \\
\mathfrak{K}_{34}\varphi_l &= l_3\varphi_{l-e_3+e_4} + l_4\varphi_{l-e_4+e_3} \\
&\quad + l_{13}\varphi_{l-e_{13}+e_{14}} + l_{14}\varphi_{l-e_{14}+e_{13}} + l_{23}\varphi_{l-e_{23}+e_{24}} + l_{24}\varphi_{l-e_{24}+e_{23}}, \\
\mathfrak{K}_{13}\varphi_l &= l_1\varphi_{l-e_1+e_3} + l_3\varphi_{l-e_3+e_1} \\
&\quad - l_{12}\varphi_{l-e_{12}+e_{23}} + l_{14}\varphi_{l-e_{14}+e_{34}} + l_{23}\varphi_{l-e_{23}+e_{12}} - l_{34}\varphi_{l-e_{34}+e_{14}}, \\
\mathfrak{K}_{24}\varphi_l &= l_2\varphi_{l-e_2+e_4} - l_4\varphi_{l-e_4+e_2} \\
&\quad + l_{12}\varphi_{l-e_{12}+e_{14}} - l_{14}\varphi_{l-e_{14}+e_{12}} - l_{23}\varphi_{l-e_{23}+e_{34}} + l_{34}\varphi_{l-e_{34}+e_{23}}, \\
\mathfrak{K}_{14}\varphi_l &= -l_1\varphi_{l-e_1+e_4} - l_4\varphi_{l-e_4+e_1} \\
&\quad + l_{12}\varphi_{l-e_{12}+e_{24}} + l_{13}\varphi_{l-e_{13}+e_{34}} + l_{24}\varphi_{l-e_{24}+e_{12}} + l_{34}\varphi_{l-e_{34}+e_{13}}
\end{aligned}$$

4 Explicit formulas

We have explicit formulas for minimal K-type Shalika functions (of moderate growth), in cases (1-i) and (3-i). Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ ($s \in \mathbb{C}$), and $(a)_n = a(a+1)\cdots(a+n-1)$ ($a \in \mathbb{C}, n \in \mathbb{N}$).

Theorem 3. *Retain the notation.*

- *(Case 1-i) We have*

$$\varphi(u_{\mathbf{0}})(\mathrm{diag}(a_1,a_2,1,1))$$
$$= \frac{(a_1a_2)^{3/2}}{\sqrt{a_1^2+a_2^2}} \cdot \frac{1}{(4\pi\sqrt{-1})^2} \int_{-\sqrt{-1}\infty}^{\sqrt{-1}\infty}\int_{-\sqrt{-1}\infty}^{\sqrt{-1}\infty} V_{\mathbf{0}}(s_1,s_2)\,(a_1^2+a_2^2)^{-s_1}\Big(\frac{a_1a_2}{a_1^2+a_2^2}\Big)^{-s_2}\,ds_1ds_2,$$

where $V_{\mathbf{0}}(s_1, s_2)$ *is given by*

$$\frac{1}{(4\pi\sqrt{-1})^3}\int_{-\sqrt{-1}\infty}^{\sqrt{-1}\infty}\int_{-\sqrt{-1}\infty}^{\sqrt{-1}\infty}\int_{-\sqrt{-1}\infty}^{\sqrt{-1}\infty}\Gamma_{\mathbb{R}}(s_1-2s_2+1)\Gamma_{\mathbb{R}}(s_1-t_1)\Gamma_{\mathbb{R}}(1-t_1)$$
$$\times\Gamma_{\mathbb{R}}(s_2-t_2)\Gamma_{\mathbb{R}}(s_2-t_3)\Gamma_{\mathbb{R}}(t_1)\Gamma_{\mathbb{R}}(t_1-t_2-t_3)$$
$$\times\Gamma_{\mathbb{R}}(t_2+\nu_1)\Gamma_{\mathbb{R}}(t_2-\nu_1)\Gamma_{\mathbb{R}}(t_3+\nu_3)\Gamma_{\mathbb{R}}(t_3-\nu_3)\,dt_1dt_2dt_3.$$

- *(Case 3-i) Let* $l=(0,0,0,0,l_{12},l_{13},l_{14},l_{23},l_{24},l_{34})\in S_{(\kappa_1,\kappa_1,0)}$. *We have*

$$\varphi(u_l)(\mathrm{diag}(a_1,a_2,1,1))=\frac{1}{(a_1+a_2)^{l_{13}+l_{14}+l_{23}+l_{24}}}$$
$$\times\frac{1}{(4\pi\sqrt{-1})^2}\int_{-\sqrt{-1}\infty}^{\sqrt{-1}\infty}\int_{-\sqrt{-1}\infty}^{\sqrt{-1}\infty}V_l(s_1,s_2)\,(a_1^2+a_2^2)^{-s_1}\Big(\frac{a_1a_2}{a_1^2+a_2^2}\Big)^{-s_2}ds_1ds_2,$$

where

$$V_l(s_1,s_2)=\frac{(-1)^{l_{12}+l_{14}}}{(2\pi\sqrt{-1})^{l_{14}+l_{23}}}\sum_{j=0}^{l_{13}+l_{24}}\binom{l_{13}+l_{24}}{j}(-1)^j$$
$$\times(s_1+\kappa_1+l_{13}+l_{24}+1)_{l_{14}+l_{23}}$$
$$\times\Gamma_{\mathbb{R}}(s_1+\kappa_1+l_{13}+l_{24}+1)\Gamma_{\mathbb{R}}(s_1-2s_2)\Gamma_{\mathbb{R}}(s_2+\tfrac{\kappa_1}{2}+\nu_1+j+1)$$
$$\times\Gamma_{\mathbb{R}}(s_2+\tfrac{\kappa_1}{2}-\nu_1-j+l_{13}+l_{24}+1)$$

if $l_{12}+l_{14}+l_{23}+l_{34}\in 2\mathbb{Z}$, *and*

$$V_l(s_1,s_2)=0$$

if $l_{12}+l_{14}+l_{23}+l_{34}\notin 2\mathbb{Z}$.

5 Friedberg-Jacquet integrals

For $F\in\mathrm{Sh}(\pi,\psi)$ and a character χ of $\mathbb{R}^\times$, the archimedean Friedberg-Jacquet integral is

$$Z_{FJ}(s,F,\chi)=\int_{\mathrm{GL}_m(\mathbb{R})}F(\begin{pmatrix} g & 0_m\\ 0_m & 1_m\end{pmatrix})\chi(\det g)|\det g|^{s-1/2}\,dg$$

Friedberg-Jacquet [5], Aizenbud-Gourevitch-Jacquet [1] proved that $Z_{FJ}(s,F,\chi)$ admits meromorphic continuation and is a holomorphic multiple of the standard L-factor $L(s,\pi\otimes\chi)$.

For cohomological representations over $\mathbb{R}$, Chen-Jiang-Lin-Tian [3] constructed archimedean nonvanishing cohomological test vectors. Refined period relations and the Blasius-Deligne conjecture in the symplectic type case were further studied by Jiang-Liu-Sun-Tian [4].

On the other hand, as an application of the formulas in Theorem 3, we can give test vectors for Friedberg-Jacquet integrals. Our result covers a non-cohomological case.

Theorem 4. *Retain the notation in Theorem 3.*

- *(Case 1-i) If $\delta_i = 0$, then we have*

$$Z_{FJ}(s, \varphi(u_{\mathbf{0}}), 1) = C \prod_{1 \leq i \leq 4} \Gamma_{\mathbb{R}}(s + \nu_i)$$

with a nonzero constant C.

- *(Case 3-i) Let*

$$F = \begin{cases} \varphi(u_{(0,0,0,0,l_{12},0,0,0,0,l_{34})}) & \text{if } \kappa_1 \text{ is even,} \\ \sum_{(p,q) \in I} R(E_{p,q} + E_{q,p}) \varphi(u_{(0,0,0,0,l_{12},0,0,0,0,l_{34}) + e_{pq}}) & \text{if } \kappa_1 \text{ is odd,} \end{cases}$$

where $I = \{(1,3), (1,4), (2,3), (2,4)\}$ and $E_{p,q} \in \mathfrak{g}$ is matrix unit, and R is right differential. Then we have

$$Z_{FJ}(s, F, 1) = C' \Gamma_{\mathbb{C}}(s + \tfrac{\kappa_1 - 1}{2} + \nu_1) \Gamma_{\mathbb{C}}(s + \tfrac{\kappa_1 - 1}{2} + \nu_2)$$

with a nonzero constant C'.

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