A CONJECTURAL IDENTITY FOR THE RATIO OF PETERSSON NORMS OF DEGREE 2 SIEGEL CUSP FORMS AND ASSOCIATED HALF-INTEGRAL WEIGHT FORMS

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Abstract. This article is an expanded version of the talk I gave at the conference *Research for automorphic forms, automorphic representations and L-functions* held at RIMS, Kyoto, in January 2026. I discuss an explicit identity relating the Petersson norm of a degree 2 Siegel cusp form F to the Petersson norms of the half-integral weight cusp forms f_m obtained from the Fourier–Jacobi expansion of F. Assuming Xue's refined Gan–Gross–Prasad conjecture for Fourier–Jacobi periods on Sp_4, one obtains an exact formula for $\langle f_m, f_m\rangle/\langle F, F\rangle$ when m is odd and squarefree, in terms of central values of degree 10 Rankin–Selberg L-functions. I also compare this with the explicit identities arising from the refined Gan–Gross–Prasad formalism for Bessel periods on SO_5.

1. Introduction

Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Siegel cusp form of degree 2, full level, and even weight k. Assume that F is a Hecke eigenform. The purpose of this note is to explicate an identity for the ratio of Petersson norms

$$\frac{\langle f_m, f_m\rangle}{\langle F, F\rangle},$$

where f_m is a half-integral weight cusp form naturally associated to the mth Fourier–Jacobi coefficient of F. The identity is conditional on Xue's refined Gan–Gross–Prasad conjecture [12, 13] for Fourier–Jacobi periods on symplectic groups. Assuming that conjecture, we obtain a striking and rather elegant answer: the above ratio is a weighted average of central values of L-functions of the form $L(1/2, \pi_F \times \pi_f, \rho_5 \otimes \rho_2)$.

The result described here is part of ongoing joint work with Biplab Paul, Ameya Pitale, and Ralf Schmidt.

2. Siegel cusp forms and the forms f_m

Let

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

and for each commutative ring R, let $\mathrm{Sp}_4(R)$ be the group of matrices $g \in \mathrm{GL}_4(R)$ satisfying ${}^t g J g = J$. The Siegel upper half-space of degree 2 is

$$\mathbb{H}_2 = \{Z \in M_2(\mathbb{C}) : {}^t Z = Z,\ \mathrm{Im}(Z) > 0\},$$

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with the usual action of $\mathrm{Sp}_4(\mathbb{R})$ by fractional linear transformations. A Siegel cusp form of degree 2, full level and weight k is a holomorphic function F on $\mathbb{H}_2$ satisfying

$$F(\gamma Z) = \det(CZ + D)^k F(Z), \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}),$$

and vanishing at the cusps; we denote the space of such forms by $S_k(\mathrm{Sp}_4(\mathbb{Z}))$; see for example [1]. If F is a Hecke eigenform, its adelization generates an irreducible cuspidal automorphic representation π_F of $\mathrm{GSp}_4(\mathbb{A})$ with trivial central character.

The dual group of GSp_4 is $\mathrm{GSp}_4(\mathbb{C})$. We write ρ_4 for the 4-dimensional spin representation, ρ_5 for the 5-dimensional standard representation, and $\rho_{10} = \mathrm{Ad}$ for the adjoint representation.

Every $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ has a Fourier expansion

$$F(Z) = \sum_{S \in \Lambda_2} a(F, S) e^{2\pi i \mathrm{Tr}(SZ)},$$

where

$$\Lambda_2 = \left\{ S = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right) : a, b, c \in \mathbb{Z},\ S > 0 \right\}.$$

For $S = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$ we put $\mathrm{disc}(S) = -4\det(S) = b^2 - 4ac < 0$. The coefficients with $\mathrm{disc}(S)$ a fundamental discriminant play a distinguished role: they may be viewed as the independent "building blocks" for the Fourier coefficients of a Hecke eigenform, as two such coefficients with different discriminants cannot be related by Hecke operators.

One way to understand the Fourier coefficients of Siegel cusp forms is to view them through the lens of Fourier coefficients of classical half-integral weight forms. Write $Z = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$ and let $\phi_m(\tau, z)$ be the mth Fourier–Jacobi coefficient of F. Let f_m be the sum of the theta components of ϕ_m with the rescaling $\tau \mapsto 4m\tau$. Equivalently, one may define directly

$$f_m(\tau) = \sum_{n \geq 1} a_m(n) e^{2\pi i n \tau},$$

where

$$a_m(n) = \sum_{\substack{r \,(\mathrm{mod}\, 2m) \\ r^2 \equiv -n \,(\mathrm{mod}\, 4m)}} a\left(F, \begin{pmatrix} \frac{n+r^2}{4m} & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix}\right).$$

The function f_m is a holomorphic cusp form of half-integral weight. More precisely, $f_m \in S_{k-\frac{1}{2}}(\Gamma_0(4m))$ and, when m is odd and squarefree, it lies in the Kohnen plus-space [4, Theorem 5.6]. Note also that the coefficients of f_m are explicit linear combinations of the Fourier coefficients of F.

This construction of f_m from F is particularly useful for understanding fundamental coefficients. Indeed, if $S \in \Lambda_2$ has fundamental discriminant and $a(F, S) \neq 0$, then after replacing S by an $\mathrm{SL}_2(\mathbb{Z})$-equivalent matrix one can arrange that

$$S' = \left(\begin{smallmatrix} a & b/2 \\ b/2 & p \end{smallmatrix}\right)$$

with p an odd prime. Since Fourier coefficients are invariant under $\mathrm{SL}_2(\mathbb{Z})$-equivalence, $a(F, S) = a(F, S')$, and now $a(F, S')$ appears as a Fourier coefficient of f_p. In this sense every fundamental Fourier coefficient of F is captured by the family $\{f_p\}_p$. This point of view underlies several results on non-vanishing and related questions for fundamental coefficients; see, [11, 8].

3. The main conjectural identity

Because the forms f_m are built canonically from F, it is natural to ask whether their Petersson norms are related. For Saito–Kurokawa lifts there is a classical answer. If F is a Saito–Kurokawa lift and a Hecke eigenform, and $m = 1$, then Kohnen and Skoruppa [9] proved

$$\frac{\langle f_1, f_1 \rangle}{\langle F, F \rangle} = \frac{24\pi^k}{\Gamma(k)L(3/2, f)}, \tag{1}$$

where $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ is the classical Hecke eigenform on the upper half-plane whose Saito–Kurokawa lift is F. Analogous formulas in this Saito–Kurokawa case for $\frac{\langle f_m, f_m \rangle}{\langle F, F \rangle}$ can likely be derived from (1) for all squarefree m, using existing expressions (e.g., [4, Theorem 6.2]) for the mth Fourier–Jacobi coefficient in terms of the first.

The natural problem is to understand what replaces (1) when F is *not* a Saito–Kurokawa lift. We prove the following conditional identity which expresses the above ratio as a weighted average of central values of degree 10 L-functions.

Theorem 3.1. *Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Hecke eigenform, and assume that k is even and that F is not a Saito–Kurokawa lift. Let m be an odd squarefree integer. Assume the refined Gan–Gross–Prasad conjecture for Fourier–Jacobi periods on Sp_4. Then*

$$\frac{\langle f_m, f_m \rangle}{\langle F, F \rangle} = C_k \sum_{\substack{s,q \geq 1 \\ sq = m}} r_{s,q} \sum_{f \in \mathcal{B}^{\mathrm{new}}_{2k-2}(q)} \frac{L(\frac{1}{2}, \pi_F \times \pi_f, \rho_5 \otimes \rho_2)}{L(1, \pi_F, \mathrm{Ad})L(1, \pi_f, \mathrm{Ad})}. \tag{2}$$

Here $\mathcal{B}^{\mathrm{new}}_{2k-2}(q)$ is an orthogonal Hecke basis of holomorphic newforms of weight $2k-2$ and level $\Gamma_0(q)$, π_f is the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ attached to f, C_k is an explicit constant depending only on k, and $r_{s,q}$ is an explicit product of local factors at the primes dividing m.

Remark 3.2. Using the usual measures to define the Petersson norms, the constant C_k is a nonzero multiple of $(4\pi)^k/(\Gamma(k)(2k-2))$. The weights $r_{s,q} \asymp m^{-1+o(1)}$ arise only from ramified local calculations; at unramified places the normalized local periods are equal to 1.

Remark 3.3. In the case of Saito–Kurokawa lifts, $L(s, \pi_F, \mathrm{Ad})$ has a pole at $s = 1$, and an analogue of (2) for $m = 1$ would reduce to a single nonzero term corresponding to the unique f (which lifts to F) for which $L(\frac{1}{2}, \pi_F \times \pi_f, \rho_5 \otimes \rho_2)$ also has a pole. Simplifying the resulting expression gives us (1) up to a constant.

Remark 3.4. As a consequence of Theorem 3.1, we obtain an essentially optimal upper bound for $\frac{\langle f_m, f_m \rangle}{\langle F, F \rangle}$ under GRH.

4. Fourier–Jacobi periods and the proof strategy

The key input is Xue's refined Gan–Gross–Prasad conjecture for Fourier–Jacobi periods on symplectic groups [12]. In the case relevant here we take $G = \mathrm{Sp}_4$. Let $H \subset G$ be the Heisenberg subgroup

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

and embed SL_2 into G by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If $\widetilde{\mathrm{SL}}_2$ denotes the metaplectic double cover, we set $R = \mathrm{SL}_2 \cdot H$ and $\widetilde{R} = \widetilde{\mathrm{SL}}_2 \ltimes H$. For a non-trivial additive character ψ of $\mathbb{Q}\backslash\mathbb{A}$ and $m \in \mathbb{Q}^\times$, and denote $\psi^m(x) = \psi(mx)$. Let ω_{ψ^m} be the Weil representation of $\widetilde{R}(\mathbb{A})$ on the space of Schwartz functions $\mathcal{S}(\mathbb{A})$, and define the theta function

$$\Theta_{\psi^m}(r, \phi) = \sum_{x \in \mathbb{Q}} \omega_{\psi^m}(r)\phi(x), \qquad r \in \widetilde{R}(\mathbb{A}),\ \phi \in \mathcal{S}(\mathbb{A}).$$

Now let (π, V_π) be a cuspidal automorphic representation of $\mathrm{Sp}_4(\mathbb{A})$ and (σ, V_σ) a genuine cuspidal automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. For $\Psi \in V_\pi$, $\Lambda \in V_\sigma$, and $\phi \in \mathcal{S}(\mathbb{A})$ one defines the global Fourier–Jacobi period

$$\mathcal{FJ}(\Psi, \Lambda, \phi; \psi^m) = \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \Psi(ng)\Lambda(g)\Theta_{\psi^m}(ng, \phi)\, dn\, dg.$$

For factorizable data there are corresponding local linear forms $\alpha_v(\Psi_v, \Lambda_v, \phi_v; \psi_v^m)$, and after dividing by suitable local norms and local zeta factors one obtains normalized local periods $\alpha_v^\#(\Psi_v, \Lambda_v, \phi_v; \psi_v^m)$ that are equal to 1 for almost all v.

Xue's conjecture is an exact analogue of the Ichino–Ikeda formalism [7]. In our setting it predicts that, under temperedness hypotheses,

$$\frac{|\mathcal{FJ}(\Psi, \Lambda, \phi; \psi^m)|^2}{\langle \Psi, \Psi\rangle \langle \Lambda, \Lambda\rangle \langle \phi, \phi\rangle} = C \cdot \frac{L_{\psi^m}(\frac{1}{2}, \pi \times \sigma)}{L(1, \pi, \mathrm{Ad}) L_{\psi^m}(1, \sigma, \mathrm{Ad})} \prod_v \alpha_v^\#(\Psi_v, \Lambda_v, \phi_v; \psi_v^m). \tag{3}$$

In our setting, we let $\Psi = \Phi_F$ be the adelization of the Siegel cusp form F, and we choose σ and the test data so that the left-hand side becomes a multiple of

$$\frac{|\langle f_m, h\rangle|^2}{\langle F, F\rangle \langle h, h\rangle}$$

for a half-integral weight cusp form $h \in S_{k-\frac{1}{2}}(\Gamma_0(4m))$. In other words, the global Fourier–Jacobi period admits a classical interpretation in terms of Petersson inner products.

At this point one sums over an orthogonal Hecke basis of the relevant half-integral weight space and applies Parseval. The Shimura–Waldspurger correspondence then converts the representation σ into an automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A})$, with f a newform of weight $2k-2$ and level dividing m. This produces the inner sum in (2).

The technically hardest part of the proof is local. At each prime $p \mid m$ one must compute the normalized local factors $\alpha_p^\#(\Psi_p, \Lambda_p, \phi_p; \psi_p^m)$ explicitly. After several reductions, one arrives at integrals on the Jacobi group involving matrix coefficients associated to induced representations of the Jacobi group. The local integral breaks naturally into many valuation regions and the final weights $r_{s,q}$ in (2) are precisely the products of these ramified local contributions.

5. Comparison with Bessel periods

There is another refined Gan–Gross–Prasad story for degree 2 Siegel modular forms, namely the one based on Bessel periods. Let π be an automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$, let $K/\mathbb{Q}$ be an imaginary quadratic field, and let Λ be a character of $K^\times\backslash\mathbb{A}_K^\times$. One can then form a Bessel period $B(\varphi, \Lambda)$ for vectors φ in the space of π. A refined conjecture of Yifeng Liu expresses the square of this period in terms of central values of L-functions and explicit local integrals [10].

For holomorphic Siegel cusp forms of degree 2, the local factors in Liu's conjecture were computed in joint work with Dickson, Pitale, and Schmidt [3]. Furusawa and Morimoto proved Liu's refined conjecture first for special Bessel periods and later, in the full $(\mathrm{SO}_5, \mathrm{SO}_2)$ setting relevant here, for arbitrary tempered representations [5, 6]. Combining these works one obtains the following explicit identity.

Theorem 5.1. *Let $F \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a Hecke eigenform and assume that F is not a Saito–Kurokawa lift, let $d < 0$ be a fundamental discriminant, put $K = \mathbb{Q}(\sqrt{d})$, and let Λ be a character of Cl_K. For the corresponding positive definite binary quadratic forms $\{S_c\}_{c\in \mathrm{Cl}_K}$ one has*

$$\frac{\left|\sum_{c\in\mathrm{Cl}_K} \Lambda(c) a(F, S_c)\right|^2}{\langle F, F\rangle} = 2^{2k-4} w(K)^2 |d|^{k-1} \frac{L(1/2, \pi_F \times \mathcal{AI}(\Lambda))}{L(1, \pi_F, \mathrm{Ad})}. \tag{4}$$

This theorem and the conjectural identity (2) are related in spirit. Both equate squares of natural periods attached to Siegel cusp forms to central L-values, and both lead to arithmetic and analytic applications.

However, the Bessel-period picture for degree 2 relies on the accidental isomorphism $\mathrm{PGSp}_4 \simeq \mathrm{SO}_5$, so it is special to genus 2. By contrast, the Fourier–Jacobi framework exists for Sp_{2n} for every n, and this is the main reason the conjectural identity discussed here should have higher-rank analogues. The obstacle is not the absence of a formal conjecture, but rather the need for new ramified local calculations.

6. Local computations at the ramified primes

The coefficients $r_{s,q}$ in (2) come entirely from the primes p dividing m. Since m is assumed odd and squarefree, each such prime is odd and we are in the case $v_p(m) = 1$. Fix one such p, write $E = \mathbb{Q}_p$, let $\mathfrak{o}$ be its ring of integers, ϖ a uniformizer, and $q = \#(\mathfrak{o}/\varpi\mathfrak{o})$. Let $\pi = \pi_{F,p}$ be the unramified local representation of $\mathrm{GSp}_4(E)$ generated by the spherical vector v_0, and let

$$\Phi_0(g) = \frac{\langle \pi(g)v_0, v_0\rangle}{\langle v_0, v_0\rangle}$$

be its normalized spherical matrix coefficient. On the metaplectic side one considers a genuine representation σ of $\widetilde{\mathrm{SL}}_2(E)$ together with the standard test function $\phi^{(m)} = \mathbf{1}_{m^{-1}\mathfrak{o}}$ in the Weil representation. The basic local quantity is obtained by integrating the matrix coefficients of π, σ, and the Weil representation over $\mathrm{SL}_2(E) \ltimes H(E)$ and then summing over an orthogonal basis of $\Gamma^0(\varpi)$-fixed vectors in σ. After multiplying by the explicit local zeta and L-factors from Xue's conjecture, one gets a normalized number $\alpha_p^{\#}$; at the places not dividing m this normalized factor is equal to 1 [12]. Thus the only new local input in Theorem 3.1 is the exact evaluation of $\alpha_p^{\#}$ for $p \mid m$.

The first key step is to replace the pair $\sigma \otimes \omega_{\psi^m}$ by a representation of the Jacobi group $J(E) = \mathrm{SL}_2(E) \ltimes H(E)$. If σ is an unramified principal series of $\widetilde{\mathrm{SL}}_2(E)$, then $\sigma \otimes \omega_{\psi^m}$ identifies with a Jacobi principal series $B^J_{\eta,m}$; if σ is special, it identifies with the corresponding special subrepresentation $\sigma_{\xi,m} \subset B^J_{\eta,m}$ studied by Berndt and Schmidt [2]. In the unramified case the space of $K^{J,0}(\varpi)$-fixed vectors is two-dimensional, with a convenient orthogonal basis f_1, f_w; in the special case it is one-dimensional and generated by the spherical vector f_0. A useful consequence is that the sum over $\Gamma^0(\varpi)$-fixed vectors collapse to a sum of Jacobi-group integrals of the form

$$\alpha(\pi,\sigma;m) = I(\Phi_0,\Phi^J) := \int_{J(E)} \Phi_0(g)\Phi^J(g)\,dg, \tag{5}$$

where Φ^J is the normalized matrix coefficient of a relevant vector on the Jacobi side. This reduction is the point at which the local problem becomes completely explicit.

To evaluate (5), one uses the Iwasawa decomposition on $\mathrm{SL}_2(E)$ together with the support properties of the vectors in the Jacobi principal series. Using the Cartan decomposition for $\mathrm{GSp}_4(E)$ and Macdonald's formula, one expresses Φ_0 in terms of the Satake parameters α,β of π and the parameter $\delta = \eta(\varpi)$ coming from the Jacobi principal series. This leads to a sum of integrals over many p-adic pieces. After summing these pieces and simplifying, one obtains a closed rational expression for the normalized local factors.

7. Concluding remarks

The ratio $\langle f_m, f_m\rangle/\langle F,F\rangle$ is a natural invariant from both the classical and adelic points of view. Classically it measures the growth of Petersson norms of the half-integral weight forms extracted from the Fourier–Jacobi expansion of F. Adelically it is governed by the refined Gan–Gross–Prasad formalism for Fourier–Jacobi periods. The resulting identity fits well with the philosophy that periods should detect central values of L-functions.

What makes this especially attractive is the prospect of extending the picture to higher degree Siegel modular forms. The Bessel-period method has already led to strong theorems in degree 2, but it does not naturally go beyond that setting. The Fourier–Jacobi method, on the other hand, is available for all symplectic groups. Once the necessary local computations are carried out, one expects a new family of explicit norm identities for $S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$, with consequences for Fourier coefficients, non-vanishing, and related analytic problems.

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