Congruences and differential operators on modular forms

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Abstract

This article surveys Kurokawa–Mizumoto congruences for vector-valued Siegel modular forms, focusing on a sufficient condition for a Klingen–Eisenstein lift to be congruent to another Siegel modular eigenform of degree two. The proof uses a pullback formula, differential operators, and a congruence criterion of Atobe–Chida–Ibukiyama–Katsurada–Yamauchi; we also recall the representation-theoretic meaning of the differential operators.

1 Introduction

Kurokawa–Mizumoto congruences are congruences between Klingen–Eisenstein lifts and Siegel cusp forms. Let k and ν be even integers, and let

$$f(z) = \sum_{n>0} a(n,f)\mathbf{e}(nz) \in S_{k+\nu}(\mathrm{Sp}_1(\mathbb{Z}))$$

be a normalized elliptic Hecke eigenform. The expected congruence says that a prime divisor of a suitable algebraic standard L-value of f should produce a Siegel cusp eigenform of degree two whose Hecke eigenvalues are congruent to those of the Klingen–Eisenstein lift $[f]^{(k+\nu,k)}$. For the Hecke operator $T(p)$, this takes the form

$$\lambda_G(T(p)) \equiv (1+p^{k-2})a(p,f) \pmod{\mathfrak{p}}.$$

Here the right hand side is the eigenvalue of the Klingen–Eisenstein lift, as computed by Arakawa [1].

The first examples were proved by Kurokawa [13] and Mizumoto [14] in scalar-valued cases. Satoh treated the first vector-valued case [15], and Katsurada–Mizumoto later gave general scalar-valued sufficient conditions [11]; following this tradition, we call congruences of this kind Kurokawa–Mizumoto congruences. The vector-valued criterion discussed below is the main result of [16]. Related vector-valued Klingen–Eisenstein congruences also appear in the work of Atobe–Chida–Ibukiyama–Katsurada–Yamauchi on Harder's conjecture [2].

First, we apply a suitable differential operator to a Siegel Eisenstein series and use a pullback formula. This produces a spectral expansion whose coefficients contain standard L-values. Second, we apply a key lemma of Atobe–Chida–Ibukiyama–Katsurada–Yamauchi [2, Lemma 6.10], which detects a Hecke congruence from a non-integral coefficient in such an expansion. The same differential operators have an intrinsic representation-theoretic meaning: they are governed by branching of lowest weight modules, Howe duality, and pluriharmonic polynomials; this point of view goes back to Ibukiyama [7, 8] and Ban [3].

2 Hecke congruences and Klingen–Eisenstein series

Let $\mathbb{H}_n$ be the Siegel upper half space of degree n and put $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$. If (ρ, V) is an irreducible algebraic representation of $\mathrm{GL}_n(\mathbb{C})$, we denote by $M_\rho(\Gamma_n)$ and $S_\rho(\Gamma_n)$ the spaces of V-valued Siegel modular forms and cusp forms of weight ρ. For a dominant integral weight $\mathbf{k}$, we also write $M_{\mathbf{k}}(\Gamma_n)$ and $S_{\mathbf{k}}(\Gamma_n)$.

Let $\mathcal{H}_n$ be the Hecke algebra associated with the pair $(\Gamma_n, \mathrm{GSp}_n(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}))$. For a Hecke eigenform F, let $\lambda_F(T)$ be the eigenvalue of $T \in \mathcal{H}_n$ on F.

Definition 2.1. Let F and G be Hecke eigenforms in $M_{\mathbf{k}}(\Gamma_n)$, and let $\mathfrak{p}$ be a prime ideal of the composite Hecke field $\mathbb{Q}(F,G)$. We say that F and G are Hecke congruent modulo $\mathfrak{p}$ if

$$\lambda_F(T) \equiv \lambda_G(T) \pmod{\mathfrak{p}} \qquad (T \in \mathcal{H}_n).$$

We write $F \equiv_{\mathrm{ev}} G \pmod{\mathfrak{p}}$.

Let $n \geq r > 0$. For

$$Z = \begin{pmatrix} Z_1 & Z_{12} \\ {}^tZ_{12} & Z_2 \end{pmatrix} \in \mathbb{H}_n, \qquad Z_1 \in \mathbb{H}_r,$$

write $\mathrm{pr}^n_r(Z) = Z_1$. Let

$$\Delta_{n,r} = \left\{ \begin{pmatrix} * & * \\ O_{n-r,n+r} & * \end{pmatrix} \in \Gamma_n \right\}.$$

If $F \in M_{\mathbf{k}'}(\Gamma_r)$ and $\mathbf{k}$ is the corresponding weight of degree n, the Klingen–Eisenstein series is

$$[F]^{\mathbf{k}}(Z,s) = \sum_{g \in \Delta_{n,r} \backslash \Gamma_n} \left(\frac{\det \Im Z}{\det \Im(\mathrm{pr}^n_r Z)} \right)^s F(\mathrm{pr}^n_r Z)|_{\rho_{n,\mathbf{k}}} g.$$

When it is holomorphic at $s = 0$, we put $[F]^{\mathbf{k}}(Z) = [F]^{\mathbf{k}}(Z,0)$. In the Kurokawa–Mizumoto setting, $r = 1$ and $n = 2$, and the lift of $f \in S_{k+\nu}(\Gamma_1)$ is denoted by $[f]^{(k+\nu,k)} \in M_{(k+\nu,k)}(\Gamma_2)$.

We shall also use the integral structure of Fourier coefficients. Fix a $\mathbb{Z}$-lattice $V(\mathbb{Z}) \subset V$ stable under $\mathrm{GL}_n(\mathbb{Z})$, and put $V(R) = V(\mathbb{Z}) \otimes_{\mathbb{Z}} R$ for a subring $R \subset \mathbb{C}$. A modular form F is said to have Fourier coefficients in R if all Fourier coefficients $a(F,T)$ lie in $V(R)$. If K is a number field, $\mathfrak{p}$ is a prime ideal of K, and $a = \sum_i a_i v_i \in V(K)$ with respect to a fixed integral basis, we put $\mathrm{ord}_{\mathfrak{p}}(a) = \min_i \mathrm{ord}_{\mathfrak{p}}(a_i)$.

3 A sufficient condition

We now state the vector-valued sufficient condition from [16, Theorem 1.1]. For the normalized eigenform $f \in S_{k+\nu}(\Gamma_1)$, put

$$\mathbb{L}(s,f,\mathrm{St}) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+k+\nu-1)\frac{L(s,f,\mathrm{St})}{(f,f)},$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ and (f,f) is the Petersson norm. For $m \geq 1$, define

$$Z(m,k) = \zeta(1-k) \prod_{j=1}^{\lfloor m/2 \rfloor} \zeta(1+2j-2k),$$

$$\mathcal{C}_{m,k}(f) = \frac{Z(m,k)}{Z(2,k)} \mathbb{L}(k-1,f,\mathrm{St}).$$

In particular, the coefficient appearing below is $\mathcal{C}_{4,k}(f) = Z(4,k)Z(2,k)^{-1}\mathbb{L}(k-1,f,\mathrm{St})$. The notation $H_2(\mathbb{Z})_{>0}$ denotes the set of half-integral positive definite matrices of degree two. For $A \in H_2(\mathbb{Z})_{>0}$, $a(A,[f]^{(k+\nu,k)})$ means the A-th Fourier coefficient of the Klingen–Eisenstein lift.

Theorem 3.1 (T. [16])**.** *Let k and ν be positive even integers with $k \geq 6$. Let $f_{1,1} = f, \ldots, f_{1,d_1}$ be a basis of $S_{k+\nu}(\Gamma_1)$ consisting of normalized Hecke eigenforms, and let $f_{2,1}, \ldots, f_{2,d_2}$ be a Hecke eigenbasis of $S_{(k+\nu,k)}(\Gamma_2)$. Let p be a rational prime and let $A \in H_2(\mathbb{Z})_{>0}$. Assume that*

(1) $\mathrm{ord}_p(\mathbb{L}(k-1,f,\mathrm{St})) = \alpha > 0$*;*

(2) $\mathrm{ord}_p(\mathcal{C}_{4,k}(f)a(A,[f]^{(k+\nu,k)})) = 0$;

(3) $p \geq 2(k+\nu) - 3$.

Then there exists a Hecke eigenform $G \in M_{(k+\nu,k)}(\Gamma_2)$, not a scalar multiple of $[f]^{(k+\nu,k)}$, such that

$$[f]^{(k+\nu,k)} \equiv_{\mathrm{ev}} G \pmod{\mathfrak{p}}$$

for some prime ideal $\mathfrak{p}$ above p in $\mathbb{Q}(G)$.

If, moreover, the remaining degree-one algebraic standard values are p-integral and p is prime to the relevant congruence modules in the sense of Mizumoto [14], then G may be chosen in $S_{(k+\nu,k)}(\Gamma_2)$ and the congruence holds modulo $\mathfrak{p}^\alpha$.

Condition (1) is the expected divisibility condition on the critical L-value. Condition (2) is a non-vanishing and integrality condition for the chosen Fourier coefficient of the Klingen–Eisenstein lift. In scalar-valued examples, Katsurada–Mizumoto [11] observed that such a condition cannot simply be ignored. Condition (3) is used to control denominators in the Eisenstein series and the differential operator; it can be slightly weakened in some cases, as explained in [16, Theorem 1.1].

4 The congruence lemma and the pullback construction

The following lemma is the linear-algebraic mechanism which converts a non-integral spectral coefficient into a Hecke congruence. It is quoted from [2, Lemma 6.10], and is used in [16, Section 5].

Lemma 4.1 (Atobe–Chida–Ibukiyama–Katsurada–Yamauchi)**.** *Let $F_1, \ldots, F_d$ be Hecke eigenforms in $M_{\mathbf{k}}(\Gamma_n)$ which are linearly independent over $\mathbb{C}$. Let K be the composite of their Hecke fields, let $\mathfrak{O}$ be the ring of integers of K, and let $\mathfrak{p}$ be a prime ideal of K. Suppose that*

$$G(Z) \in (M_{\rho_n}(\Gamma_n) \otimes V_{n,\mathbf{k}})(\mathfrak{O}_{(\mathfrak{p})})$$

is expressed as

$$G(Z) = \sum_{i=1}^{d} c_i F_i(Z), \qquad c_i \in V_{n,\mathbf{k}}.$$

If, for some $A \in H_n(\mathbb{Z})$,

$$\mathrm{ord}_{\mathfrak{p}}(c_1 a(A, F_1)) < 0,$$

then there exists $i \neq 1$ such that $F_i \equiv_{\mathrm{ev}} F_1 \pmod{\mathfrak{p}}$.

If no other F_i were congruent to F_1, then the eigenvalue differences $\lambda_{F_i}(T) - \lambda_{F_1}(T)$ would be $\mathfrak{p}$-adic units for suitable Hecke operators. Applying such operators to the integral form G would force the non-integral contribution of F_1 to disappear, contradicting the assumed negative valuation of a Fourier coefficient.

It remains to construct a modular form G to which the lemma applies. Let $E_{n,k}$ be the scalar Siegel Eisenstein series of degree n and weight k. We apply a vector-valued differential operator to $E_{4,k}$ and then restrict to $\mathbb{H}_2 \times \mathbb{H}_2$. Pullback formulas of this kind go back to Garrett [5] and to Böcherer–Satoh–Yamazaki [4]; the vector-valued formula used here is due to Ibukiyama and related work of Kozima [9, 12].

In the present setting the formula has the following schematic form:

$$\mathcal{E}(Z_1, Z_2) = \sum_{r=1}^{2} \gamma_r \sum_j \mathcal{C}_{4,k}(f_{r,j})[f_{r,j}]^{\rho_2}(Z_1)[f_{r,j}]^{\rho_2}(Z_2),$$

where $f_{1,j}$ runs through elliptic eigenforms, $f_{2,j}$ runs through degree-two cusp eigenforms, and γ_r is a rational factor which is a p-unit under the size condition on p; see [16, Proposition 5.3]. Taking the A-th Fourier coefficient in the second variable gives

$$g_A(Z_1) = \sum_{r=1}^{2} \gamma_r \sum_j \mathcal{C}_{4,k}(f_{r,j})[f_{r,j}]^{\rho_2}(Z_1)a(A, [f_{r,j}]^{\rho_2}).$$

The rationality and p-integrality of g_A follow from the integrality of the normalized Eisenstein series and the differential operator; this is the point where the denominator estimate in the theorem is used. In the summand corresponding to $f_{1,1} = f$, the relevant scalar is

$$\gamma_1 \mathcal{C}_{4,k}(f)a(A, [f]^{(k+\nu,k)}).$$

To apply the congruence lemma we have to multiply this scalar by the A-th Fourier coefficient of $[f]^{(k+\nu,k)}$. Using $\mathrm{ord}_p(\mathbb{L}(k-1, f, \mathrm{St})) = \alpha$ and $\mathrm{ord}_p(\mathcal{C}_{4,k}(f)a(A, [f]^{(k+\nu,k)})) = 0$, this product has negative p-adic valuation after the normalization used in [16, proof of Theorem 1.1]. Hence the congruence lemma produces an eigenform congruent to the Klingen–Eisenstein lift.

5 Differential operators

The differential operator used above is not merely an ad hoc expression in partial derivatives. It is characterized by a representation-theoretic branching problem. Let $n = n_1 + \cdots + n_d$. A differential operator

$$\mathbb{D} : C^\infty(\mathbb{H}_n) \longrightarrow C^\infty(\mathbb{H}_{n_1} \times \cdots \times \mathbb{H}_{n_d}) \otimes V$$

is required to satisfy the following compatibility property: if F is a scalar-valued modular form of weight $\det^k$, then the restriction of $\mathbb{D}F$ to $\mathbb{H}_{n_1} \times \cdots \times \mathbb{H}_{n_d}$ is a vector-valued modular form of the prescribed weight. This is often called Condition (A), following Ibukiyama [7].

The construction can be described at the infinite place. Let $G_n = \mathrm{Sp}_n(\mathbb{R})$ and let $\widetilde{G}_n = \mathrm{Mp}_n(\mathbb{R})$ be the metaplectic cover. If K_n is the stabilizer of iI_n, then $K_n \simeq U(n)$, and we write $\widetilde{K}_n$ for its inverse image. For an irreducible representation τ of $\widetilde{K}_n$, holomorphic automorphic forms of type τ can be identified with

$$\mathrm{Hom}_{(\mathfrak{g}_{n,\mathbb{C}}, \widetilde{K}_n)}\left(L(\tau), C^\infty_{\mathrm{mod}}(\widetilde{\Gamma}_n \backslash \widetilde{G}_n)\right),$$

where $L(\tau)$ is the lowest weight module with lowest $\widetilde{K}_n$-type τ. Thus a differential operator compatible with restriction corresponds to a homomorphism between lowest weight modules after restricting from $\widetilde{G}_n$ to $\widetilde{G}_{n_1} \times \cdots \times \widetilde{G}_{n_d}$; this is the viewpoint used by Ban [3] and in [16, Section 4].

Howe duality makes this Hom space explicit. Let $L_{n,k} = \mathbb{C}[M_{n,k}]$ be the polynomial ring in the entries of an $n \times k$ matrix X. The Weil representation gives an action of $(\mathfrak{g}_{n,\mathbb{C}}, \widetilde{K}_n)$, and right translation gives a commuting action of O_k. Howe duality decomposes $L_{n,k}$ into a multiplicity-free sum of lowest weight modules paired with irreducible O_k-representations [6]. The corresponding harmonic part is described by Kashiwara–Vergne [10].

A polynomial $P(X) \in L_{n,k}$ is pluriharmonic if

$$\sum_{s=1}^{k} \frac{\partial^2 P}{\partial X_{i,s} \partial X_{j,s}} = 0 \qquad (1 \leq i, j \leq n).$$

Let $\mathfrak{H}_{n,k}$ be the space of pluriharmonic polynomials. The invariant-theoretic description of Kashiwara–Vergne says that the relevant part of the polynomial ring is built from $\mathfrak{H}_{n,k}$ and the O_k-invariant polynomials [10]. Consequently, the symbol of a differential operator satisfying Condition (A) is characterized by two requirements: pluriharmonicity and covariance under the block diagonal action of $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_d}$. In this form, the characterization agrees with Ibukiyama's theorem on automorphic differential operators [7, 8].

6 Examples

We record two examples from [16, Section 7]. Let Δ_{16} be the normalized elliptic Hecke eigenform of weight 16.

First take $(k, \nu) = (14, 2)$. The relevant algebraic value is

$$\mathbb{L}(13, \Delta_{16}, \mathrm{St}) = \frac{2^{20} \cdot 3^4 \cdot 373}{7}.$$

The prime 373 satisfies the divisibility condition, and the Fourier coefficient condition can be checked directly. Therefore there exists a degree-two Siegel cusp eigenform $\Delta_{16,14}$ of weight $(16, 14)$ such that

$$\Delta_{16,14} \equiv_{\mathrm{ev}} [\Delta_{16}]^{(16,14)} \pmod{373}.$$

Next take $(k, \nu) = (8, 8)$. In this case

$$\mathbb{L}(7, \Delta_{16}, \mathrm{St}) = \frac{2^{15} \cdot 23^2}{11 \cdot 13}.$$

We obtain a congruence modulo the prime square:

$$\Delta_{16,8} \equiv_{\mathrm{ev}} [\Delta_{16}]^{(16,8)} \pmod{23^2}.$$

7 Hermitian analogues

The same strategy has also been carried out in Hermitian settings. The author has constructed differential operators for Hermitian modular forms [18] and a pullback formula for vector-valued Hermitian modular forms on $\mathrm{U}_{n,n}$ over CM fields [17]. Together with the congruence-lemma method used by Atobe–Chida–Ibukiyama–Katsurada–Yamauchi [2, Lemma 6.10], this gives congruences between Klingen–Eisenstein series and cusp forms on $\mathrm{U}_{n,n}$ [19].

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